AN ESSAY

ON

The Co-ordination of the Laws of Distribution

BY

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To GUSTAV F. STEFFEN.

MY DEAR STEFFEN,

Whatever the fate of this essay may be, you will not be ashamed of having your name associated with it, and your own work will place you far above the possibility of being injured by it. I therefore make bold, without asking your leave, to dedicate it to you in token of sincere friendship and admiration, and in the conviction that your approval is likely to give to it one of its best chances of life.

Yours very sincerely,

PHILIP H. WICKSTEED.

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PREFATORY NOTE.

RECENT economic studies have displayed a marked tendency towards a complete recasting of the Theory of Distribution; but I am not aware that any satisfactory attempt has been made to state what may be called the new theory of Distribution in its entirety; and still less have its relations to the old theory been defined. Where does it confute it, and where does it systematise, develop, and force into consciousness what it already implied?

In this essay I have tried, without any claim to originality, to mark out the field which such an enquiry must cover, and at the same time to offer the rigorous demonstration of some of the fundamental propositions of the new theory. I shall also indicate some of the directions in which I think it may be fruitfully applied to practical investigations, and in which it will react on other branches of theory.

I address myself only to experts; but even so it seems far from superfluous to explain briefly the scope and significance of the mathematical method as conceived and applied in this investigation. The wider question of its general limits and capacities in economic studies I leave untouched.

In investigating the laws of distribution we are in a field of enquiry in which experience shews that there is great danger of our making extremely definite assumptions and arriving at extremely definite conclusions half unconsciously. These assumptions and conclusions are sometimes capable of being stated in mathematical language without becoming one whit more definite than they were before, but with the desirable result that they are made more consciously and therefore more warily. For example, the bare statement that "the product is a function of the factors of production" when thrown into this mathematical form at once challenges our direct attention, and makes us aware of the decisive character of the unspoken assumption upon which the very conception of an objective basis for the Laws of Distribution rests. But further, we shall find in the course of our investigations that the usual statement of the Law of Rent assumes the following proposition:—

The Product being a function of the factors of production we have

$$P = f(a, b, c, \ldots\ldots).$$

and the form of the function is invariably such that if we have

$$\Pi = f(\alpha, \beta, \gamma \ldots\ldots)$$

we shall also have

$$\nu\Pi = f(\nu\alpha, \nu\beta, \nu\gamma \ldots\ldots)$$

My contention is that the proposition stated in this mathematical form is not more definite or more bold than it is in the form in which it is generally assumed by the economists, but that its mathematical form forces its definiteness and its boldness upon us, makes us realise what we are doing in assuming it and therefore gives us pause. I use the mathematical form of statement, then, in the first instance, as a safeguard against unconscious assumptions, and as a re-agent that will *precipitate* the assumptions held in solution in the verbiage of our ordinary disquisitions.

And in the second place, when a statement is put into mathematical form it becomes easy to see whether certain other statements, also put into mathematical form, are or are not involved in it. To shrink from the mathematical manipulation of mathematical symbols because those mathematical symbols can only represent economic facts imperfectly, appears to me to be tantamount to saying that looseness of deduction is likely to correct the effect of rashness of assumption. The mathematical economist, on the contrary, declares that by using the specialised language and logic of mathematics, whenever it is applicable, he eliminates a source of error. He does not suppose that the certainty of his method confers upon his premises any greater certainty than they have on their own merits, but he does maintain that his conclusions have all the certainty that his premises had. *He has lost nothing on his way* from premises to conclusion, however complex and abstract his mathematical reasoning may have been. The whole question is whether he imports into his premises or exports out of his conclusions anything that is not definitely and exactly contained in them. If he has, he deserves (and will receive) no quarter. But it is easy to detect his fraud.

Thus to narrow and define the area of error is a great gain.

It is true, of course, that every transformation of the mathematical expression of an economic fact or hypothesis must be ideally capable of direct expression in economic terms, and therefore every step of the mathematical argument is ideally capable of being translated into a logically cogent economic argument. Moreover to effect this translation will usually be in the highest degree instructive, throwing all manner of side lights upon the subject of discussion, leading the student into the inner recesses of the hypotheses on which he is working, and not seldom revealing some unsuspected weakness or inconsistency in his premises. But nevertheless the argument that if proposition A is true then it follows that proposition B is also true is complete if the two propositions have been accurately expressed in mathematical terms and the one has been shewn mathematically to involve the other. The intricacy of the mathematical argument itself is not a source of uncertainty and should not cause us any the least misgiving.

THE LAW OF DISTRIBUTION.

1. Source of difficulty in co-ordinating the laws of distribution.

2. A method of co-ordination suggested by the analogy of the law of exchange value.

3. The general law of distribution stated.

4. Is this law consonant with experience?

5. How is it related to existing economic theory?
   The law of rent in its "second form"
   (*a*) is built upon certain assumptions and implications that require more distinct formulating than they have yet received,
   (*b*) does not really give any direct formula for rent at all,
   (*c*) but assumes our law for all factors of production except land,
   (*d*) and suggests by analogy that the same law holds for land also,
   (*e*) a result which analysis shows to be rigorously involved in the assumptions from which we start.

6. Co-ordination of the laws of distribution.

7. Reflections.

8. Supplementary considerations as to the form of certain functions assumed or implied in the current discussions of the law of rent.

1. In investigating the laws of distribution it has been usual to take each of the great factors of production such as Land, Capital and Labour, severally, to enquire into the special circumstances under which that factor co-operates in production, the special considerations which act upon the persons that have control of it, and the special nature of the service that it renders, and from all these considerations to deduce a special law regulating the share of the product that will fall in distribution to that particular factor.

Now as long as this method is pursued it seems impossible to co-ordinate the laws of distribution and ascertain whether or not the shares which the theory assigns to the several factors cover the product and are covered by it. For in order that this may be possible it seems essential that all the laws should be expressed in common terms. As long as the law of rent, for example, is based on the objective standard of fertility of land, while the law of interest is based on the subjective standard of estimate of the future as compared with the present, it is difficult even to conceive any calculus by which the share of land and the share of capital could be added together and an investigation then instituted as to whether the residual share will coincide with what the theory assigns as the share of wages. But it is obvious that such a co-ordination must be within the purview of economic theory. The very term "distribution of the product" is a tacit acknowledgement of the obligation to co-ordinate the laws of distribution.

And accordingly a marked tendency has of late been observable towards bringing the several investigations of the laws of distribution into closer relation with each other. The basis of those laws is being sought not in the special nature of the services rendered by the several factors but in the common fact of *service rendered*. If an objective measure of the service rendered by each factor in its marginal application can be discovered there will seem to be at any rate a possibility of co-ordinating the claims based thereon.

2. The modern investigations into the theory of value have already given us the lead we require. Indeed the law of

exchange value is itself the law of distribution of the general resources of society. When we have safeguarded this statement by all the explanations necessary to enable us to speak of communal desires and satisfactions, we may say that the total satisfaction (**S**) of a community is a function (F) of the commodities, services, etc. (A, B, C,......) which it commands; or **S** = F (A, B, C,......). And the exchange value of each commodity or service, if purchasable, is determined by the effect upon the total satisfaction of the community which the addition or the withdrawal of a small increment of it would have, *all the other variables remaining constant.* Thus the claim upon the community which the command of any commodity or service, K, enables a man to enforce, is determined by the ratio $\frac{dS}{dK}$ (which expresses the exchange value of a unit of that commodity or service), and the exchange value of the whole stock of it is $\frac{dS}{dK} \cdot K$. In fact $\frac{dS}{dK}$ is the *marginal efficiency* or *significance* of K as a producer of satisfaction.

3. This would perhaps not be a convenient form in which to express the law of exchange value for general purposes; but it is useful to us at present because it suggests a formula of Distribution. In fact we may regard the total satisfaction enjoyed by the community as a "product" and the several services and commodities as the factors of production. Each factor then receives a share of the product regulated by its marginal efficiency as a producer. Here is a general law of distribution put into our hand. Is it applicable to the internal or "domestic" distribution of the whole share that falls to each service or commodity, regarded in its turn as a product to be shared amongst the several factors that produced it? Let the special product to be distributed (P) be regarded as a function (F) of the various factors of production (A, B, C,.. .). Then the (marginal) significance of each factor is determined by *the effect upon the product of a small increment of that factor*, all the others remaining

constant. It is suggested that the ratio of participation in the product on which any factor, K, can insist (by threat of withdrawal), will be $\frac{dP}{dK}$ per unit, and its total share will be $\frac{dP}{dK} . K$.

The theorem when applied to distribution is in some respects more satisfactory than when applied to exchange value; for satisfaction can only be measured, or quantitatively expressed, in terms of one of the very commodities or services of which it is regarded as a function; and the claim made good against the community by those who command each commodity or service can only be met by the grant of some other commodity or service. "Satisfaction," then, cannot be regarded as an external something which is directly distributed amongst claimants to a share in it. It is the variables themselves, of which "Satisfaction" is a function, which are really distributed amongst their own representatives.

But in the case of the distribution of a product, we have something external to the claimants, something not themselves, which is actually sliced up and divided amongst them. We may think of this something eitner as a material product—such as steel rails,—or more generally as the industrial position of the concern over against the rest of the world, in which the value of the rails as well as their quantity, and in some cases subtler elements of vantage, would have to be considered. But in either case there is something to be divided which is external to the factors of production, and can be expressed in terms external to them. And this something is definitely increased or diminished by an increment or decrement of any one of the factors of production. Therefore $\frac{dF}{dK}$ is measurable in terms of a unit which whether easy or difficult to determine is objective with respect to K. I shall call this total industrial vantage of the concern the *product* and shall represent it by P.

It now seems theoretically possible to attack the question of co-ordination. Each factor being remunerated not in accordance with the *nature* of the service it renders, but in accordance with the (marginal) *rate* at which its unit is rendering such service, and a practical method of testing and estimating that rate having been discovered, it remains to enquire, whether, from the known properties of F, we can deduce the property $\frac{dP}{dA} . A + \frac{dP}{dB} . B + \frac{dP}{dC} . C + \ldots\ldots = P.$

For if it can be shown that the formula $\frac{dF}{dK} . K$ really defines the share of the product which will fall to any factor K, and if it can be further shewn that when each of the factors has received its share the whole product is exactly accounted for, we shall then have accomplished our task of co-ordinating the laws of distribution.

Reserving, for the present, the consideration of the latter point we will now ask how our proposed law

$$\frac{dP}{dK} . K = \text{share of K in the product}$$

bears the test of experience and reason, and how it is related to current economic theory.

4. The general law of distribution, then, which we have now advanced and which we shall proceed to discuss, amounts simply to this:—That the share in the product which falls to any factor, no matter what be the character of that factor or of the service which it renders, is determined by the amount per unit which the concern, as a whole, would find it pay to allow to that factor sooner than have a portion of it withdrawn from co-operation. So stated the theorem may seem self-evident. And so indeed it is. Everyone knows that if a man "is not worth his salt" he is discharged, that if an employer cannot profitably keep all his hands at work he dismisses some of them (unless actuated by motives other than those usually described as "economic"), that if a machine is expected to "eat its head off" it is not bought, that unless I expect a piece of land to pay its own rent I do not take it for industrial purposes, and

so on. It may seem that little is to be gained by putting such truisms into mathematical form. But I think it will be found otherwise on investigation. The law of value, too, resting as it does on the law of indifference and the phenomena of marginal utility, amounts to nothing in the world but the assertion that the purchaser will not give more than he must for an article, and will in no case give more for it than he thinks it is worth to him. This was of course well known to everyone, and is constantly assumed in every economic treatise of whatsoever date; but nevertheless its exact expression in mathematical language has made an epoch, and is making a revolution, in economic science. For it is one thing to be practically familiar with a principle and to assume it in simple cases as a matter of course, and it is another thing to grasp it so consciously and so firmly as never to lose hold of it or admit anything inconsistent with it, however remote from familiar experience and however complicated and abstract may be the regions of enquiry in which we need it as our clue. Thus too in the present instance. The law of distribution which we are to examine is too obvious and self-evident not to be constantly assumed by economic writers, but if they assume it in one sentence they commonly ignore or contradict it in the next. It has seldom been clearly or consciously formulated and firmly held through the remoter deductions of economic speculation. And it is only by a few recent writers that this has been done at all.

It is therefore well worth while to dwell on the practical experiences and the general considerations that seem to confirm the law we have advanced.

Let us recall once more the analogy of exchange values. The exchange value of any commodity or service K is determined by the effect of an increment or decrement of it upon the satisfaction of the community and is, therefore $\frac{dS}{dK}$; but any individual going into the market finds this rate fixed independently of his special tastes, desires and estimates. To him $\frac{dS}{dK}$ is an externally and independently fixed price (v), and he has to ask himself whether (S being his particular satisfaction) the relation holds $\frac{dS}{dK} > v$. If so he buys; if not, not. Equilibrium is reached for each individual when for him the relation holds $\frac{dS}{dK} = v = \frac{d\mathbf{S}}{dK}$.

In like manner the individual entrepreneur, if he contemplates taking on or discharging a workman, will ask himself whether that workman will be worth his wage or not, *i.e.*, whether he will increase the product, other factors remaining constant, at least to the extent of his wage; and he will take on more men as long as the last one earns at least as much as his wage, but no longer. The man, on his side, can insist on having as much as the marginal significance of his work, *i.e.*, as much as the difference to the product which the withdrawal of his work would make. Preserving a uniform notation, we may say that the market price of K is determined by the significance of an increment or decrement of K to the total communal product, which we will call **P**. Then, from the general point of view, any particular kind of labour, K, can insist on remuneration at the rate of $\frac{d\mathbf{P}}{dK}$ per unit; and from the individual point of view $\left(\text{the price of labour being fixed at } w = \frac{d\mathbf{P}}{dK}\right)$ the individual entrepreneur will go on feeding his land, capital, etc., with that particular kind of labour, until, in his particular concern the relation is established $\frac{dP}{dK} = w = \frac{d\mathbf{P}}{dK}$.

The formula is quite general. The unit of the particular kind of labour may be an hour of attention (of a given quality) to the management and direction of a business; or it may be the attendance on a committee of a man who cannot manage or direct anything, but whose name and presence strengthen the industrial position of the concern for some other reason. Nor need K be any kind of labour at all either nominal or real. It may be land of given capacities. For in that case, too, the question whether I shall take a few more acres under cultivation, with my present resources, or whether I shall add a few more feet to my frontage, is the question whether to me individually $\frac{dP}{dK}$ is or is not greater than the price of land; and that price itself is fixed by the ratio $\frac{dP}{dK}$ to the community at large. Or K may be pick-axes, or any other kind of machinery or tools. In a word the formula is strictly general, and it is found to be closely in harmony with the practice of daily life. Observe too that it involves no pooling of unlike elements, and no expression of those unlike elements under a conventional unit, such as the £ sterling, and no artificial grouping of the different factors of production. Each factor is expressed in its own unit and treated as having its independent influence, at the margin, on the increment or decrement of the product. We shall return to this point under section 6.

5. Turning now to the question of the relation of this law to current economic theory, we begin with what is usually regarded as the most successfully elaborated portion of the theory of distribution. I refer to the theory of rent, in what is generally called the "second form of statement." In the exposition of this law the product is regarded as a function of land and "capital," capital being regarded as embracing labour and all else that is needed to make the land productive.[1] All the constituents of this generalised "capital" are regarded as reduced to their expression in money. Land is taken as constant, and capital-plus-labour is added in successive doses, each dose (at any rate after a certain point) yielding a smaller return than the previous dose. That is to say, land being constant, the product is regarded as a function of capital-plus-labour, and the first differential coefficient of this function (within the limits usually considered), is taken to be positive, and the second differential co-efficient,

[1] To keep in mind its composite character I shall call it capital-plus-labour, though I shall represent it by a simple C, or *c*, in algebraic symbols.

(at least after a certain value of x has been reached) is regarded as being negative.

Graphically this means that the function is represented by an area the higher boundary of which, at any rate for the portion specially considered, is a declining curve. (Fig. I.)

This means that the return to the last dose of capital-plus-labour is smaller than the return to the previous doses. But it must be adequate, or the dose would not be administered; and if adequate for the last dose it is, by the law of indifference, adequate for all the rest; and therefore the return to the last dose fixes the rate which will satisfy capital-plus-labour, and the excess or "surplus" return to the earlier increments constitutes the amount that the land owner is in the position to claim as rent.

(*a*) Now let us note certain points as to the argument and the figure that illustrates it. In the first place we are dealing

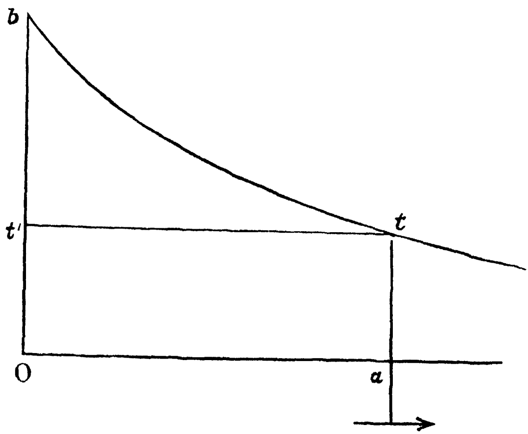


Fig. 1.

simply with *proportions* between land and capital-plus-labour, not *amounts* of them, and with *rates* of return to land and to capital-plus-labour not total amounts of return. Our figure is therefore equally applicable, without modification, whether we are dealing with a square acre or a square mile. We suppose the whole area between the ordinate, the abscissa, the intercept, and the curve, to show the total result per unit of land, on the supposition of successive doses of capital-plus-labour being applied: and these doses themselves are not so many units of capital-plus-labour, but so many units *per constant of land*. In other words the theory ignores all questions as to the suitable amount of land to come under a single management, and so forth. It assumes that, given any amount of land of uniform character, the proportionate application of capital-plus-labour upon it will yield a uniform result *per acre* whether the area cultivated be large or small. This may be expressed mathematically by saying that the product (P) being a function (F) of land (L) and capital-plus-labour (C), then if we have $\Pi = F(\Lambda, K)$, we shall also have $m\Pi = F(m\Lambda, mK)$.

Note too that the distribution between land and capital-plus-labour is also expressed in *rates*. The line *at* expresses the return *per unit* to capital-plus-labour, and the area *bt't* (supposing for convenience that we take the unit of land as our constant land basis) expresses the return *per unit* to land, on the supposition that the rate of allowance of capital-plus-labour to land is O*a* units of the former per unit of the latter.[1]

In the next place (and here I must beg the reader to have patience and to take me seriously) it should be observed that in order to obtain a definite area *bt't* the values of x must be carried back to the origin. Now what does this mean? We are dealing with *rates per unit* of "capital-plus-labour" to land, and as we approach the value of $x = o$ we are considering indefinitely small doses of "labour-plus-capital" per definite unit of land. In other words we are considering what will take place when the ratio of land to the "capital-plus-labour" applied to its cultivation is indefinitely great—say at the same rate as if the whole continent

[1] *i.e.* The dimensions of the area on the figure are PL^{-1}. The dimensions of the ordinates are PC^{-1} and of the abscissæ CL^{-1}. *cf.* p. 19.

of America had one hour's work per diem devoted to its cultivation. And this hour's work must be genuinely applied over the whole area, or the real ratio between land and "labour-plus-capital" will be something other than we have supposed. Now to treat such an inconceivable process as yielding a result at a definite rate, the measurement of which is essential to our theory of rent, seems absurd.

Well did the great French savant say "*les origines sont toujours obscures.*" My reason for plunging into their obscurities at the present moment is a very practical one.

The answer to the objection I have raised is obvious, viz., "In all representations of economic quantities by curves "we assume a continuity which is not physically possible. "Thus the area in our 'rent' curve is to be thought of as "actually composed of a number of rectangles. We increase "the allowance of capital-plus-labour to the unit of land by "definite and sensible doses, and though we are at liberty to "read our figure in any units we like, and to call £1 per "acre, ·0000382 of a penny per inch, if we choose, yet we "are not at liberty to add a dose of ·0000382 of a penny "*per acre*, which would not be sensible, and then "speak of the result as entering into our demonstration. "We must not confuse reading our ratio in infinitesimal "units, with altering it in infinitesimal degrees. The "figure is represented as continuous, but in interpreting it "back into actual fact we must remember that it is not "really so. Suppose we start with a definite unit of land, "say one acre, in our minds. We then measure off from "the origin on the abscissa the smallest amount of capital-"plus-labour which we can ideally conceive as physically "capable of being applied to an acre, and we represent the "result, which will be physically appreciable, as a thin rect-"angle on the capital-plus-labour base. The linear unit "in which we measure it is of course arbitrary. Now "the successive rectangles thus constructed constitute the "ultimate or atomic elements of the area. We represent "this area for convenience as bounded by a continuous curve, "but to drive us to the interpretation of something *less than* "*our atom* is to fasten upon an accident of graphic repre-

"sentation as though it were a fallacy of argument." This answer, when received with proper caution, is perfectly satisfactory; but we must follow it up a little further. We find our curve to be composed of atomic elements, the condition of the formation of which is that each basis should represent, in its totality, a change of ratio between labour-plus-capital and land which is physically significant. We have no grounds for asserting that the same addition of capital-plus-labour, per unit of land, will furnish this atomic basis at all portions of the curve, indeed it seems obvious that this will not be so. But in any case, all we care to assert is that every element of the area must have a significant basis. We may therefore carry the curve to the origin, if it is convenient to do so, under the proviso that it is really composed of atomic elements, and is only represented as continuous for convenience. Now from this it follows that we may also carry the curve as far to the right as we choose, and may, if convenient, regard it as carried to infinity. The only proviso is that the elements out of which it is actually composed must each be built upon a significant basis; and should there come a point after which *all* additions of capital-plus-labour are without significance (land being now in the vanishing ratio to capital-plus-labour, just as capital-plus-labour was in a vanishing ratio to land at the origin) the last atomic element of the curve will include the whole of this region. All this will become clearer when we work out some reciprocally related curves in a supplementary examination of their forms at the close of this essay. Meanwhile, I shall assume that I have established my right to regard the "rent" curve as carried to infinity whenever it is convenient to do so.

(*b*) Returning now to our figure, we note by aid of it that the supposed law of rent does not really amount to an independent law of rent at all. It tells us that the whole product being $F(x)$, and $F'(x)$ being the rate of remuneration per unit which satisfies capital-plus-labour, the whole amount which capital-plus-labour will draw out will be $x . F'(x)$, and the remaining $F(x) - x . F'(x)$ will be rent. Now this is simply a statement that when all the other factors of production have been payed off, the "surplus" or residuum can be claimed by the land-owner. In short, it gives no direct formula for rent at all. It is a "residual theory of rent." The full significance of this fact is seen when we follow the economists into their deductions from the supposed law of rent. Thus Walker attempts to formulate a "residual theory of wages," and in elaborating it he assumes that the share of land has been independently fixed by the law of rent. That is to say rent is what is left after taking out wages plus certain other charges, and wages are what is left after taking out rent plus these same other charges. The full beauty of this will come out if we throw it into the precise form of algebraic notation.

$S = x + y + z + u + \ldots\ldots$. S is known, the rest are all unknown. Let us for the sake of argument make the assumption—a very bold one—that independent laws have been discovered which enable us to determine z, u, etc., and that we have $z + u + \text{etc.} = s$. We are then offered two equations to determine x and y, viz.:

$$x = S - s - y$$

and

$$y = S - s - x$$

(c) But if the so-called law of rent does not really give a law of distribution with respect to land at all it assumes a very direct and precise law of distribution as holding for everything *except* land. And this is exactly the law already announced. Land being constant, and all the other factors being reduced to a common measure and called capital, or by us "capital-plus-labour," the product per unit of land is then treated as a function of capital-plus-labour. In the figure it is assumed that capital-plus-labour will be remunerated on the scale at per unit, and that the total share of capital-plus-labour will therefore be $at \, . \, Oa$, but it is obvious that at is $\frac{dp}{dc}$ and that $at \, . \, Oa$ is $\frac{dp}{dc} \, . \, c$, so that the share of c in the product is assumed to be

$\frac{dp}{dc} . c$, and this is the formal statement of our law (p. 10) as applied to c.[1]

[1] Let it be noted, once for all, that *in the figures* we are not really dealing with product as a function of the factors of production; but with ratio of product to a selected constant factor, as a function of the ratio of the other factors to it. Hence the return to the constant and the variable factors alike are *rates* of return, and when we say that p is the product, and $\frac{dp}{dc} . c$ the total return to c, we mean that the product is p per unit (for example) of land and that $\frac{dp}{dc} . c$ is the whole share of capital-plus-labour in the *product of a unit of land*. Of course behind all this lies *time*. We are dealing with the application of units of "capital-plus-labour" per unit of time, per unit of land, and with product per unit of time, per unit of land (*e g*, annual product of an acre). It would, I think, be intolerably cumbersome to preserve this strictly correct phraseology throughout our argument; but the neglect of such preciseness has a way of avenging itself and we must exercise ceaseless vigilance.

When dealing with an analytical formula not illustrated by curves it is often convenient to consider the whole product per unit of time (the total output per year for instance) as a function of the total factors of production.

To keep these distinctions clear in the mind it is necessary to have recourse to the theory of dimensions (cf my article on "Dimensions of Economic Quantities" in the Dictionary of Political Economy). If we take P as the dimension of product, L as that of land, and C as that of capital-plus-labour, we shall be dealing in our diagrams with areas that have the dimensions PL^{-1}. On the axis of x we shall measure c, of the dimensions CL^{-1}, *i.e.*, ratio of C to L. Our ordinates will have the dimensions $PL^{-1}\ c^{-1}$ or $PL^{-1}\ C^{-1}L$ or PC^{-1}.

In the text the small letters are used for ratios. Thus p is product per unit of L and has dimensions PL^{-1}, while c is capital-plus-labour per unit of land and has dimensions CL^{-1}. Care in the consistent use of this notation may perhaps safeguard us against the dangers of some occasional laxity in our language.

The dimension of time enters negatively into *all* the quantities we are discussing. "Land" is use of land per unit of time. Labour is hours of work per unit of time, etc. But the universality of this condition enables us to dispense with any special consideration of it. See some interesting and instructive remarks on this head in Marshall's "note on the meaning of the phrase 'A dose of Capital and Labour.'" *Principles*, vol. I., p 227 sqq., 2nd ed.

Here we have a real law of distribution assumed, though as yet only in an intricate form; because it is not applied to any special factor of production, separated from the rest, but only to an idealised amalgam, an essentially fluid "capital," which we have called "capital-plus-labour," and which includes, besides labour, such heterogeneous appliances as ploughs, artificial manures, and we know not what. Let us see then whether this very precise law which has been assumed in respect to the very vague factor "capital" may not be assumed with equal right in the case of the comparatively definite factor land.

(*d*) To ask the question is to answer it. Every single step of the argument by which the second form of the "law of rent" is usually established applies without modification, if we assume a fixed amount of capital-plus-labour and proceed to administer successive doses of land.

A figure exactly analogous to the one we have already employed will serve to illustrate this. Capital-plus-labour being constant it follows that, at any rate after a certain point, each successive dose of land will increase the yield less than the last dose did. For were this not the case it would be possible for a definite amount of capital-plus-labour to spread itself with advantage over an indefinite area of land. We shall therefore come to a point when a further increment of land would increase the total yield by less than its own rent at the current rate; *i.e.*, a point at which there is some more effective way of applying the next available dose of land than by bringing it under the cultivation of a "capital" already so well supplied as to be comparatively irresponsive to further increments. The last increment of land it is just worth while for "capital" to secure at current rates. And by hypothesis "land" is willing to cede this increment on these terms. But if these terms are adequate for the last increment, they are, by the law of indifference, adequate for all the rest, and we have (Fig. 2) land remunerated at the rate bt; and the total remuneration of land will be $bt.Ob$. But bt is $\frac{dp}{dl}$ and $bt.Ob$ is

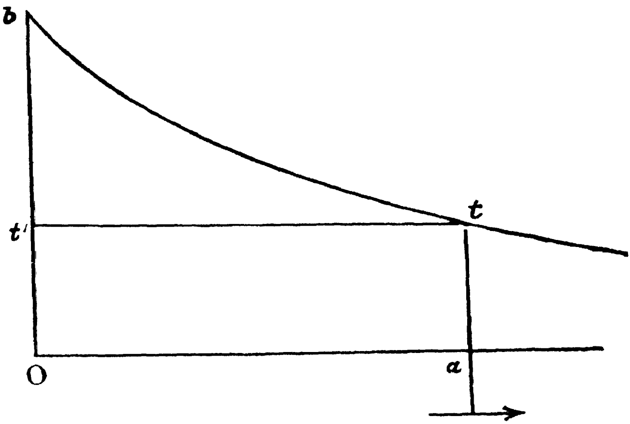


Fig. 1.

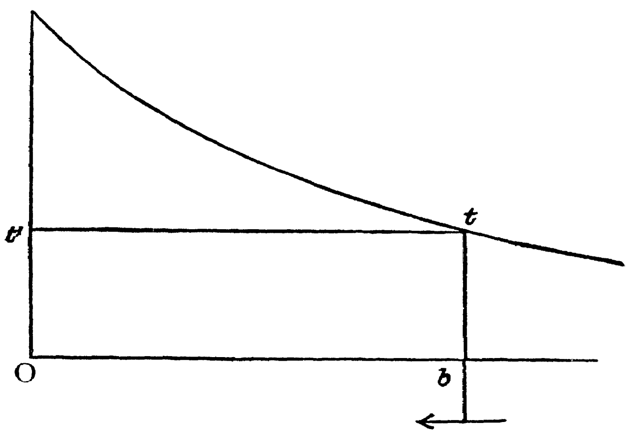


Fig. 2.

$\frac{dp}{dl} \cdot l$. Therefore the law of rent is:—rate of remuneration $\frac{dp}{dl}$, total remuneration $\frac{dp}{dl} \cdot l$.

We see then that the general law assumed for capital may with equal justification be assumed for land, and indeed it will be found that Marshall, for instance, falls into this line of argument at the beginning of his treatment of the theory of rent, and argues from the decreasing significance of doses of land, capital being constant, as freely as from the decreasing significance of doses of capital, land being constant.[1] Now since the high values of x upon the first figure (large allowance of capital-plus-labour to land) correspond to the low values of x on the second figure (small allowance of land to capital-plus-labour) it will be seen that there is no kind of contradiction between the two figures. Let us suppose ourselves to be increasing the rate of capital-plus-labour per unit of land. We shall then be decreasing the ratio of land per unit of capital-plus-labour. That is to say the intercept in the first figure will be moving away from the origin and the intercept on the second figure will be moving towards it, and both alike will give as the result increasing rent and decreasing return per unit to capital-plus-labour.[2]

[1] See his "Principles," vol. I. p. 206, sqq. 2nd ed.

[2] This will suggest sundry reflections, that cannot be developed here, as to the propriety of calling rent a "surplus," on the strength of a picture having been made of it as a curve-bound area before one had been made of it as a rectangle; and then calling everything else of which it is found convenient to make a picture with a curvilinear boundary "rent" also; *e.g.*, "rent of ability" "consumer's rent," etc. It is also of the utmost importance to note that the law of diminishing returns is two-edged Take land and capital-plus-labour in any workable ratio. Then if land remains constant and capital-plus-labour increases we shall have increasing returns per unit of land and decreasing returns per unit of capital-plus-labour. But if capital-plus-labour is constant and land increases, we shall have increasing returns per unit of the former, and decreasing returns per unit of the latter. For instance, if a farmer (who is farming to the

But it is not enough to show that the usual assumption of the law with respect to everything but land is not incompatible with its assumption with regard to land also. We must show further that the latter assumption is absolutely involved in the former, is implicitly formulated by it, covers the whole of the so-called "surplus," and is covered by it. In other words we must show that, on the hypothesis of the first figure, the area blt' is neither more nor less than $\frac{dp}{dl}$.

Before entering upon this demonstration it will be convenient to explain the notation that will be employed in it.

P will be taken to express a concrete product.

best advantage under given conditions of the land and labour market) were enabled to increase either his application of capital-plus-labour, or his land, the other factor remaining constant, the result would be, in either case, a diminishing return per unit of the increased and an increasing return per unit of the stationary factor. And observe that we are not speaking of the share in the product which falls to a unit of each factor, but of the *gross* yield "per capita," "per horas," "per jugera," or what not. Increasing this rate with reference to the one is decreasing it with reference to the other. To form an estimate, then, of the social significance of any change of ratio between two factors (or groups of factors) we must know with reference to which of the two we *desire* to increase the returns. And if it is capital-plus-labour that we "back," as against land, is it the element of labour or of capital in this complex factor that we are interested in? To ask these questions is to see how very confused and remote a bearing upon the well-being of a nation the law of "diminishing returns in agriculture" as usually stated has. For it tells us of a diminishing return to a complex unit in only one factor of which we are interested; and for anything we know that factor may be securing an increasing return, which is disguised by the diminishing returns to another factor with which it is "pooled." What we really want is to separate out labour and dose it with land-plus-capital, if possible to satiety. But when we try to imagine this being done we shall see that appliances cannot be indefinitely multiplied and utilised without bringing men close together and so cramping them for land. The "population" question, in its purely industrial aspects, will then be found ultimately to turn on a balance between the significance to each man of other free men regarded as appliances and the significance to him of the space those other men occupy. Is their room or their company the more important?

p_l a product-per-unit-of-land.

p_c a product-per-unit-of-capital + labour.

L and C will be taken to express respectively Land and "Capital + Labour" considered as amounts, each expressed in its appropriate unit.

c will be taken to signify a ratio of C to L.

l „ „ „ a ratio of L to C.

The Greek letters Π, Λ, K, π, λ, κ will be taken to signify specific values of the magnitudes represented by the corresponding English letters.

In the notation of *functions* this distinction between Greek and English letters will not be observed, but

F and f will be appropriated to functions which are constructed on the assumption that L is constant while,

Φ and ϕ will represent similar functions which are constructed with C constant.

Other letters are left available as we may need them.

To begin with then, we take

$$P = \Psi\,(L, C)$$

the form of Ψ being such that if we have

$$\Pi = \Psi\,(\Lambda, K)$$

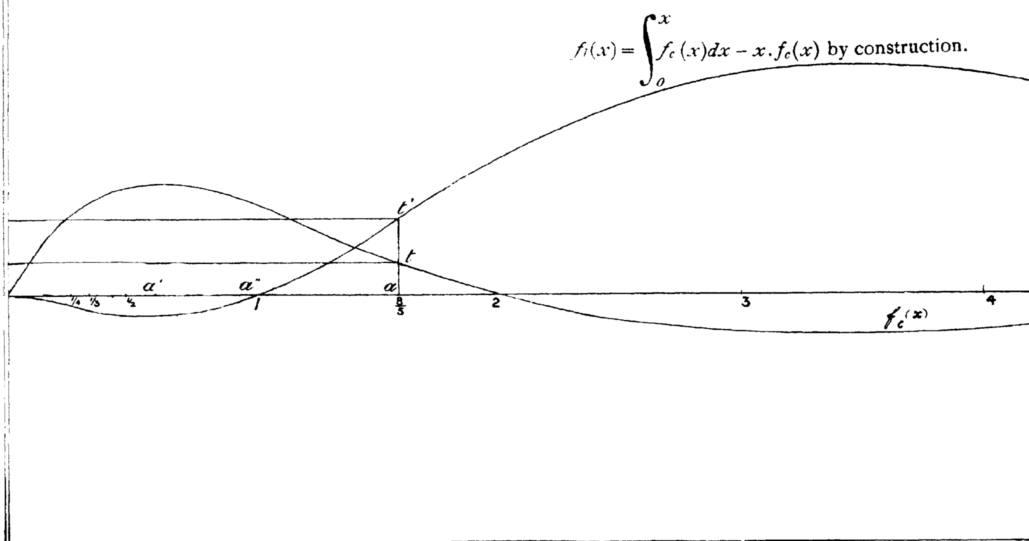
then we also have

$$m\Pi = \Psi\,(m\Lambda, mK).$$

We are to investigate the shares of P that fall to L and C; to shew that they are $\frac{dP}{dL}$. L and $\frac{dP}{dC}$. C respectively; and that the sum of them is P.[1]

[1] It may be convenient, on a first reading, to omit this demonstration and take up the argument at p. 31.

FIG. 3 $F(x)=p_l$



$f_l(x)=\int_0^x f_c(x)dx - x.f_c(x)$ by construction.
t'
t
O
α'
α''
α
1/4
1/3
1/2
1
$\frac{8}{5}$
2
3
4
$f_c(x)$

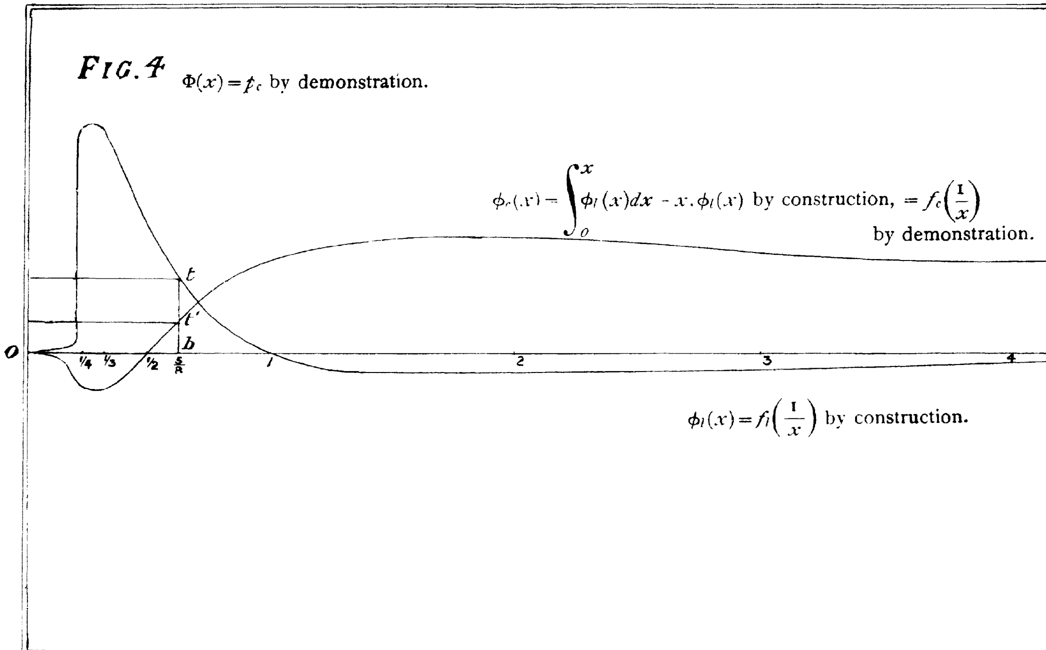
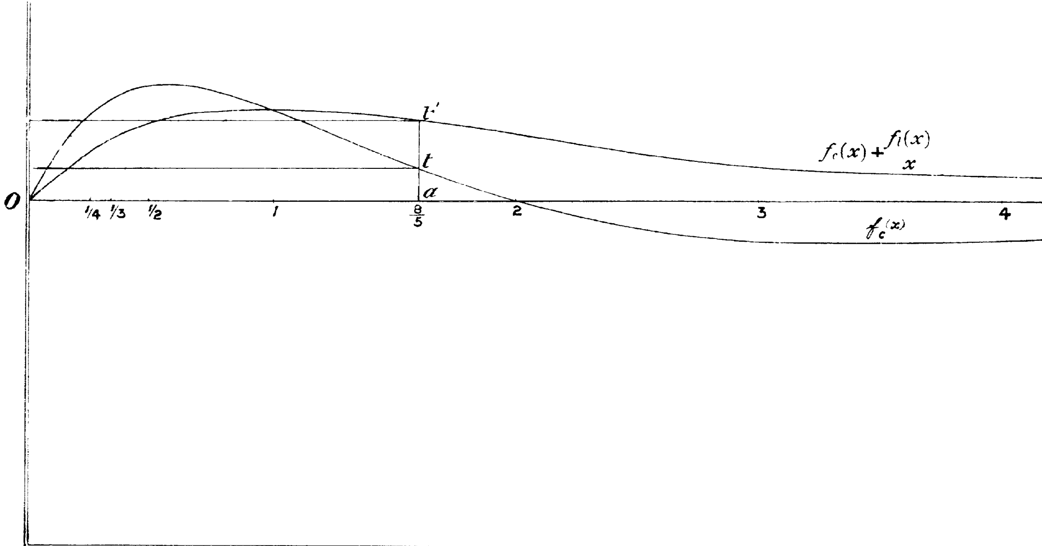


FIG. 4
$\Phi(x) = f_c$ by demonstration.
$\phi_c(x) = \int_0^x \phi_l(x)dx - x.\phi_l(x)$ by construction, $= f_c\left(\frac{1}{x}\right)$ by demonstration.
$\phi_l(x) = f_l\left(\frac{1}{x}\right)$ by construction.
t
t'
b
O
1/4
1/3
1/2
5/8
1
2
3
4

FIG. 5



Let us take

(i.) $p_l = F(c) =$ total product *per unit of land* on the supposition C : L $= c$.

Then $F'(c) =$ the rate at which increments of c, *i.e.*, increments in the ratio C : L, increase the yield per unit of land.

Take

$$\text{(ii.)}\quad f_c(c) = F'(c).$$

Then if we construct the curve

$$y = f_c(x). \quad \text{(Fig. 3.)}$$[1]

We shall have a curve corresponding to those used in the demonstrations of the "second form of the law of rent," and it gives

$$\text{(iii.)}\quad \int_0^x f_c(x)dx = F(x) = p_l.$$

Then rent per unit of land, or share of product-per-unit-of-land that falls to unit of land, is

$$\int_0^x f_c(x)dx - x \,.\, f_c(x).$$

Let us take

$$\text{(iv.)}\quad f_l(x) = \int_0^x f_c(x)dx - x \,.\, f_c(x).$$

[1] It will be convenient at this point to introduce a more suitable form of curve than is usually given in the text-books. The reasons for assigning certain definite properties to the curve (some of which are obvious on inspection) will be given in the supplementary section on the form of some economic functions, pp. 49 sqq. Meanwhile, I may appeal to Marshall's demonstration that a curve rising during the early part of its course is not to be regarded as inadmissible, and that a theoretical negative rent, accruing if we stop the increments of x near the origin, constitutes no objection. See *Principles*, vol. I., pp. 211, 214, 2nd ed.

Repeating (i.) in a general form for convenience we now have

(i.) $p_l = F(x) =$ Total product *per unit of land* for continuously increasing values of $C : L = c = x$.

(v.) $f_c(x) =$ Share in p_l that falls to C per unit of c, for $C : L = c = x$.

(vi.) $f_l(x) =$ Share in p_l that falls to the unit of L for $C : L = c = x$.

And by construction and definition we have

$$\text{(vii.)} \quad x \, . \, f_c(x) + f_l(x) = F(x) = p_l.$$

Now we have already

$$\text{(viii.)} \quad f_c(x) = \frac{dP}{dC}$$

$$\text{for} \quad f_c(x) = \frac{dp_l}{dc} = \frac{dp_l}{d \, . \, \frac{C}{L}}$$

$$= \frac{Ldp_l}{dC} = \frac{d(Lp_l)}{dC}$$

And since p_l is the product per unit of L, it follows that Lp_l is the total product of L, or

$$Lp_l = P$$

$$\therefore f_c(x) = \frac{d(Lp_l)}{dC} = \frac{dP}{dC}$$

What we have still to prove is

$$f_l(x) = \frac{dP}{dL}$$

In order to do this let us construct the curve (Fig. 3)

$$y = f_l(x)$$

We then have for any value Oa of C : L, or c, the share in the product per unit of L, which falls to the unit of C, expressed by the line at, and the share that falls to the unit of L expressed by the line at'.[1]

[1] The dimensions of the area F (x) are PL^{-1}, the dimensions of the line at are $Pc^{-1}L^{-1}$, or $PC^{-1}LL^{-1}$, or PC^{-1}, and the dimensions of the line at' are PL^{-1}.

The total share of L is $at'.\mathrm{L}$ and the total share of C is $at.c.\mathrm{L} = at.\frac{\mathrm{C}}{\mathrm{L}}.\mathrm{L} = at.\mathrm{C}$.

Now let us take

$$\phi_l(x) = f_l\left(\frac{1}{x}\right)$$

and construct (Fig. 4) the curve.[1]

$$y = \phi_l(x)$$

Let us consider the composition of this curve. Remembering that we have

$$\phi_l(x) = f_l\left(\frac{1}{x}\right)$$

we see that, at the origin, we shall be dealing with $c = \infty$ and shall be gradually diminishing it towards zero, as we move to the right; whereas we shall start with $l = \frac{\mathrm{L}}{\mathrm{C}} = \frac{1}{c}$ at zero, and shall be increasing it towards infinity.

Here then we have a curve that gives us the marginal returns per unit of land for increasing ratios of L to C, just as $f_l(x)$ gave them for decreasing ratios of L to C; and for any specific values K, Λ, of C, L we shall have corresponding values κ, λ of c, l such that

$$\lambda = \frac{1}{\kappa}$$

$$\phi_l(\lambda) = f_l(\kappa)$$

And for running values we shall have

$$\text{(ix.)} \quad \phi_l(l) = f_l(c)$$

Now let us take $\Phi(x)$ as the primitive of $\phi_l(x)$, so that we have

$$\text{(x.)} \quad \Phi(x) = \int_0^x \phi_l(x)dx$$

[1] This we can do; for we have established our right to consider the curve $f_l(x)$ as carried to infinity See p. 17.

and

$$\text{(xi)} \quad \Phi'(x) = \phi_l(x)$$

And taking

$$\text{(xii.)} \quad \phi_c(x) = \Phi(x) - x . \Phi' x$$

And constructing (Fig. 4) the curve

$$y = \phi_c(x)$$

We proceed to establish the relation

$$\phi_c(x) = f_c\left(\frac{1}{x}\right)$$

Which will link the four curves into a symmetrical system.

We have by construction

$$(1) \quad f_l(x) = F(x) - x . F'(x)$$

$$(2) \quad \phi_l(x) = f_l\left(\frac{1}{x}\right) = F\left(\frac{1}{x}\right) - \frac{1}{x} . F'\left(\frac{1}{x}\right)$$

Writing $F'(x)$ as $f_c(x)$ and changing sides we get from (1)

$$x . f_c(x) = F(x) - f_l(x)$$

Whence

$$\frac{1}{x} . f_c\left(\frac{1}{x}\right) = F\left(\frac{1}{x}\right) - f_l\left(\frac{1}{x}\right)$$

and

$$f_c\left(\frac{1}{x}\right) = x . F\left(\frac{1}{x}\right) - x . f_l\left(\frac{1}{x}\right)$$

Substituting

$$\phi_l(x) \text{ for } f_l\left(\frac{1}{x}\right)$$

We have

$$(3) \quad f_c\left(\frac{1}{x}\right) = x . F\left(\frac{1}{x}\right) - x . \phi_l(x)$$

But

$$x . F\left(\frac{1}{x}\right) \text{ is } \Phi(x)$$

For we can demonstrate that the two expressions disappear for $x = o$, and that their differential co-efficients are the same.

They disappear, for $x=o$. For $\Phi(o)$ being the amount produced by the unit of labour-plus-capital *without any land* is obviously zero. And $x \cdot F\left(\frac{1}{x}\right)$ becomes $o \times F(\infty)$, and this also is zero; for $F(\infty)$ is the limit of the product per unit of land when capital-plus-labour increases indefinitely and this is clearly finite.

Also the differential co-efficients of $\Phi(x)$ and of $x \cdot F\left(\frac{1}{x}\right)$ are equal, for when differentiated $x \cdot F\left(\frac{1}{x}\right)$ gives

$$F\left(\frac{1}{x}\right) - \frac{1}{x} F'\left(\frac{1}{x}\right)$$

Which by (2) is $\phi_l(x)$ or $\Phi'(x)$; therefore $x \cdot F\left(\frac{1}{x}\right)$ equals $\Phi(x)$.

If we now substitute $\Phi(x)$ for $x \cdot F\left(\frac{1}{x}\right)$ in (3) we have

$$f_c\left(\frac{1}{x}\right)=\Phi(x)-x \cdot \phi_l(x)$$

But by construction

$$\Phi(x) - x \cdot \phi_l(x) \text{ is } \phi_c(x)$$

Therefore

$$\text{(xiii.)} \quad \phi_c(x)=f_c\left(\frac{1}{x}\right)$$

We have now a perfectly symmetrical system of relations between the curves, which may be expressed thus:

(1) $f_l(x) + x \cdot f_c(x)=F(x)$

(2) $\phi_c(x)+x \cdot \phi_l(x)=\Phi(x)$

(3) $f_l(x)=\phi_l\left(\frac{1}{x}\right)$, or $f_l\left(\frac{1}{x}\right)=\phi_l(x)$

(4) $f_c(x)=\phi_c\left(\frac{1}{x}\right)$, or $f_c\left(\frac{1}{x}\right)=\phi_c(x)$

For specific values K, Λ, giving $x=\kappa$ for the F series, we shall have $x=\lambda=\frac{1}{\kappa}$ for the Φ series.

Hence $\quad F(\kappa)=f_l(\kappa)+\kappa \cdot f_c(\kappa)=\phi_l(\lambda)+\frac{1}{\lambda} \cdot \phi_c(\lambda)$

Whence $\lambda . F(\kappa)=\phi_c(\lambda)+\lambda . \phi_l(\lambda)=\Phi(\lambda)$ by (2)
For running values of $C : L=c$, we therefore have

$$\text{(xiv.)} \quad F(c)=c . \Phi\left(\frac{1}{c}\right)$$

But $F(c)$ is p_l, *i.e.*, the product of C and L, per unit of L, when $C : L=c$.

And since there are c units of C to every unit of L, the product per unit of C will clearly be $\frac{1}{c}$ of the product per unit of L, or the product per unit of L c times the product per unit of C.

Hence $$\text{(xv.)} \quad p_l=c . p_c$$

Substituting p_l for $F(c)$ in (xiv.), equating the right hand sides of (xiv.) and (xv.), and multiplying by $\left(\frac{1}{c}\right)$, we have $\Phi\left(\frac{1}{c}\right)= p_c$

But $\Phi\left(\frac{1}{c}\right)$, is $\Phi(l)$, hence

$$\text{(xvi.)} \quad \Phi(l)= p_c.$$

We are now rapidly approaching our goal. The purpose of this long and perhaps tedious demonstration has been to shew that the ordinary statement of the law of rent in its "second form" rigorously involves not only

$$\frac{dP}{dC}.C = \text{Share of capital-plus-labour.}$$

But also $$\frac{dP}{dL}.L = \text{Share of Land.}$$

This we are, at last, in a position to prove. For we have shewn Φ (l) to be p_c, or the product per unit of C, when the ratio of C to L is c;

But $$\phi_l(l) \text{ is } \frac{d\Phi(\lambda)}{dl} \text{ or } \frac{dp_c}{dl},$$

Whence it follows that (for any values of C, L and the corresponding values of c, l) we shall have the following equation :—

Share that falls to land, per unit, by hypothesis

$$= \mathrm{F}(c) - c.\mathrm{F}(c) = f_l(c) = \phi_l(l) = \frac{dp_c}{dl} = \frac{dp_c}{d\,.\,\frac{\mathrm{L}}{\mathrm{C}}} = \frac{d[\mathrm{C}.p_c]}{d\mathrm{L}} = \frac{d\mathrm{P}}{d\mathrm{L}}$$

and since this is the share that falls to land, per unit, the share of L units will be

$$\frac{d\mathrm{P}}{d\mathrm{L}}\,.\,\mathrm{L} \qquad \text{Q.E.D}$$

The other crucial proportion, viz.

$$\frac{d\mathrm{P}}{d\mathrm{L}}\,.\,\mathrm{L} + \frac{d\mathrm{P}}{d\mathrm{C}}\,.\,\mathrm{C} = \mathrm{P}$$

follows at once, from either figure. Taking the F series as our point of departure, we have

$$f_l(c) + c.f_c(c) = \mathrm{F}(c) = p_l$$

Or

$$\frac{d\mathrm{P}}{d\mathrm{L}} + \frac{\mathrm{C}}{\mathrm{L}}\,.\,\frac{d\mathrm{P}}{d\mathrm{C}} = p_l$$

Multiply by L

$$\frac{d\mathrm{P}}{d\mathrm{L}}\,.\,\mathrm{L} + \frac{d\mathrm{P}}{d\mathrm{C}}\,.\,\mathrm{C} = \mathrm{L}p_l = \mathrm{P}$$

I leave it to the reader to construct solid figures by adding an axis of Z on the F figure, on which to measure units of L, or one on the Φ figure, on which to measure units of C. These figures will enable him to read the actual product as a solid on either figure.

Taking the F figure we shall have

Dimensions of x .... c or CL^{-1}
,, ,, y .... $\mathrm{PL}^{-1}c^{-1}$ or $\mathrm{PL}^{-1}\mathrm{C}^{-1}\mathrm{L}$ or PC^{-1}
,, ,, z L
,, Plane of xy .... $\mathrm{CL}^{-1}\mathrm{PC}^{-1}$ or PL^{-1}
,, ,, ,, xz .... CL^{-1} L or C
,, ,, ,, yz (of no direct significance to us)... $\mathrm{PC}^{-1}\mathrm{L}$
,, Solid .... $\mathrm{CL}^{-1}\mathrm{PC}^{-1}\mathrm{L}$ or P

On such a figure we can read, either directly or by reciprocals, all the quantities that interest us. Total product, total of land engaged, total of capital-plus-labour engaged, ratio of capital-plus-labour to land or of land to capital-plus-labour (represented on this figure only by its reciprocal), ratio of product to land and ratio of product to capital-plus-labour.

6. We have now abundantly proved that the "second form of the law of rent," as all but universally accepted by economists, rigidly involves in its assumptions the following propositions:—The product being a function of land (L) and certain other factors that are not land (C) we have

$$\text{Share of the product that falls to } L = \frac{dP}{dL} \cdot L$$

$$\text{,, \quad ,, \quad ,, \quad } C = \frac{dP}{dC} \cdot C$$

and

$$\frac{dP}{dC} \cdot L + \frac{dP}{dC} \cdot C = P$$

But this does not in itself help us much, because the assumptions on which the "second form of the law of rent" is itself built are much too serious to be allowed to pass unchallenged. They are first that the function has the general property,

$$\text{if } \Pi = \Psi(\Lambda, K)$$

$$\text{then } m\Pi = \Psi(m\Lambda, mK)$$

and second that it is possible and legitimate to express a perfectly heterogeneous aggregate of factors in terms of a single unit.

It is time we examined these assumptions for ourselves.

We will begin with the first, and will throw it into a general form. P, the product, being a function, Ψ, of the factors A, B, C,...... is it true that if we have

$$\Pi = \Psi(A, B, C\ldots\ldots)$$

we shall also have $m\Pi = \Psi(mA, mB, mC\ldots\ldots)$?

When writers on rent assume this law in reference to land and capital, they are avowedly considering the mere material product, and it is significant that they always speak of rent as though it were paid in kind. Now it must of course be admitted that if the physical conditions under which a certain amount of wheat, or anything else, is produced were exactly repeated the result would be exactly repeated also, and a proportional increase of the one would yield a proportional increase of the other. The crude division of the factors of production into land, capital and labour must indeed be abandoned if we are clearly to understand the significance even of this proposition. We must regard every kind and quality of labour that can be distinguished from other kinds and qualities as a separate factor; and in the same way every kind of land will be taken as a separate factor. Still more important is it to insist that instead of speaking of so many £ worth of capital we shall speak of so many ploughs, so many tons of manure, and so many horses, or foot-pounds of "power." Each of these may be scheduled in its own unit, and when this has been done the enumeration of the factors of production may be regarded as complete. On this understanding it is of course obvious that a proportional increase of all the factors of production will secure a proportional increase of the product. But it is also obvious that this truism has no economic significance, for what we are interested in is not the amount of the material product but the amount of the industrial vantage that command of that product confers on its possessor, and it is clear that we have by no means exhausted the factors concerned in the production of this vantage when we have enumerated those concerned in the mere physical production. "Good-will" for example (measured perhaps in the number of families to whom the newsman supplies the Dailies and Weeklies, or the number of quarts of milk per diem that the milkman sells, or the extent of the doctor's or the schoolmaster's "connection") is a quite distinct factor in Production, if the "product" be regarded as "Industrial or Economic Vantage," and is often paid for on the basis of an estimate of its marginal efficiency per unit. Notoriety, in the same way (as measured in some such unit as the command of advertising stations of a given quality) is a factor of production, in the wider sense in which we are using the term. Of course managing ability, and indeed each distinguishable grade of managing ability is a distinct factor. Instructive instances will occur to the reader of the employment of some kind of superintendent or manager (such as a "Clerk of the Works") for half his time, under the belief that up to the last hour of his half time the marginal efficiency of the man's work is at least equal to his hour's salary at his market rate, but that, if further increased, his work would have a marginal significance less than that salary. "Travelling" is in many industries a very important factor of production.

The question we are examining, then, is this: If every one of the abilities, efforts, materials and advantages which contribute to production were severally increased in an identical ratio, would the product also be increased in that ratio?

If we are disposed to answer the question in the affirmative it may be objected that even when we make our formula perfectly general, and abstain from any attempt at an exhaustive enumeration of the factors, it is still unsafe to say that if all the factors were equally multiplied the product would be increased proportionately; because we have defined the product, not as a material output but as an industrial position or vantage; and in order that this may be doubled or trebled it is necessary not only that all the factors of production should be doubled or trebled, but also that the *area of operations* should be capable of corresponding enlargement on the same terms that have ruled hitherto; or in other words that there should be a fresh supply of people who want, or can be made to want, the commodity or service in question, on the same terms as those who now enjoy it. And to assume this is obviously unwarrantable.

And at this point comes dimly into view a problem of the utmost interest and importance, which will be touched upon, on one side, under section 7, paragraph (*f*) of this essay, and which suggests itself under another aspect here, but is far too vast to be dealt with thus incidentally. For the truth is that the real or "social" product is the total satisfaction accruing from the processes of industry to the whole community, including both the customer and the manufacturer;

and in this sense the body of customers and their desire for the product, themselves constitute factors of production. If *these* factors, like the others, receive a proportional increment then obviously the conditions are exactly repeated, and the product too will receive a proportional increment. But here we are checked again; for if the "physical product" is something narrower than the "commercial product" or vantage of the producer, the total or "social" product is something wider; and it is not the total vantage that we have to distribute, but the share therein of the producing firm, as against the rest of the world. And accordingly while some of the factors of commercial production are devoted to making the physical product, others are devoted to making, finding, or conquering from other producers, persons who want the physical product; and anything which would increase the share of the public in the social product, and decrease the share of the producer in exactly the same degree, would be regarded by the producer as pure loss; and it would indeed be a dead deduction from the commercial product. The producer regards the consumer not in his real significance as a factor of the social product, but merely in his commercial significance, as at once a necessity and a rival claimant whose share is to be minimised.

Thus the current assumption that a proportional increment of all the factors of production will secure a proportional increment of the product appears to be legitimate if we are speaking either of the physical or of the social product, but to be unwarranted exactly where we wish to make use of it, viz., where we are speaking of the commercial product.

We may go further and say that in case of an actual or virtual monopoly the assumption in question is manifestly false. For it is clear that the demand for any commodity is not indefinitely elastic; and that if each unit of physical product is backed by the same amount of pushing and other such factors of commercial production, the response will be slower as the amount increases. In a word if x be the amount of the product, so backed, turned out per annum, or per other unit of time, then the keenness of the want to which the last increment of x ministers will be a function of x, say $f(x)$, and the commercial vantage conferred by the command of x will be $f(x)$, and an increment of x, say h, will effect in the commercial product—not a proportional increment, which would be $x.f(x).h$,—but an increment of $h.\frac{d\{x.f(x)\}}{dx}$, or $h.f(x) + h.f'(x)..x$; where $f'(x)$ is, of course, negative. We see then that x is a function of A.B.C. ... .which meets the condition of proportional increments; but $f(x)$ and $x.f(x)$ are functions of A.B.C....... which do not meet that condition; and $x.f(x)$ is the $\Psi(x)$ we are investigating. Therefore $\Psi(x)$ does not meet the conditions of proportional increments.

It will be observed, however, that if we pass from the supposition of a virtual monopoly to the usual one of perfectly free competition, then the individual producer whose product is one qth of the whole amount produced, on increasing each one of the factors of his own production in the same proportion, will effect a proportional increment of x, say h, and this will correspond to an increment of his commercial product equal to $h.f(x) + h.f'(x).\frac{x}{q}$. Of these the value of $h.f(x)$ is not affected by the value of q. and is proportional to h; while $h.f'(x).\frac{x}{q}$ tends to diminish as q increases. Hence the statement that if all the factors of production are proportionally increased the commercial product will be proportionally increased also is never theoretically correct; but if $h.f'(x)$ is not sensible, then the proposition is sensibly true for small increments; and if $h.f'(x).\frac{x}{q}$ is small in relation to $h.f(x)$, the proposition is approximately true for small increments.

All this is very easily translated into the language of the market-place, and when so translated it will be found to approve itself to the practical sense.

The conclusion we have reached, therefore, is that where (1) the conditions are such that we can take h large enough to make $h.f(x)$ sensible, but small enough to make $h.f'(x)$ insensible; or where (2) Π is not the whole product, but the product of an individual producer, being one qth of the whole, and where it is possible to make $h.f(x)$ sensible, while $h.f'(x)\frac{x}{q}$ is not sensible, or is not significant in comparison with $h.f(x)$, there we are at liberty to assume that, for small values of n, if we have

$$\Pi = \Psi\ (A, B, C......)$$

we shall also have

$$(1+n)\Pi = \Psi\{(1+n)A,\ (1+n)B,\ (1+n)C.\ \ ....\}$$

It will be seen, on reflection, that the cases oomprised under (1) and (2) cover the cases usually regarded as normal in discussions of the abstract theory of production and distribution.

Let us now proceed to enquire whether this property involves the law of the co-ordination of the laws of distribution which we anticipated as probable. It is easy to show that it does so; for

$$P = \Psi\ (A, B, C......)$$

gives, for small values of n,

$$P + nP = \Psi\ (A + nA,\ B + nB,\ C + nC)^{1}$$

but since n is small the total increment nP is made up of the increments severally due to nA, nB, nC, etc., and (for small values of n) these are respectively

$$\frac{dP}{dA}.nA,\ \frac{dP}{dB}.nB,\ \frac{dP}{dC}.nC, \text{ etc.}$$

hence we have

$$\frac{dP}{dA}.nA + \frac{dP}{dB}.nB + \frac{dP}{dC}.nC + ... = nP$$

[1] Of course it does not follow that there is *no other* set of factors which will give $P + nP$ (or generally mP). Some factors may drop out altogether and others may be substituted for them. Or there may be various ways of increasing one factor and diminishing another so as to get an exact compensation.

or, multiplying by $\frac{1}{n}$

$$\frac{dP}{dA}.A+\frac{dP}{dB}.B+\frac{dP}{dC}.C+\ldots\ldots=P.$$

That is to say, under ordinary conditions of competitive industry, it is sensibly or approximately true that if every factor of production draws a remuneration determined by its marginal efficiency or significance, the whole product will be exactly distributed.

Q.E.D.

But it must be noted that we have not raised any commanding presumption that industries concentrated in a few hands come under this law.

The failure fully to confirm and generalise a property in the productive functions which would yield an admirably compact and complete co-ordination of the laws of distribution need not discourage us. Its suggestions as to the line of attack we must follow in dealing with monopolies, and with the true socialising of production, are so magnificent in their promise that we are more than consoled for the want of completeness in our immediate results.

One or two points may still be touched on in this connection. If, for example, the size of the business is itself an important consideration, then the fact will express itself in some such way as by the presence of some one or more factors (perhaps representing some special quality of managing capacity, or some kind of machine or building) which can only be added or subtracted in relatively large units, such as the whole working year of a high-class business man, or an enormous engine. This would introduce a serious discontinuity into an important factor and a serious indeterminateness into any empirical attempt to evaluate $\frac{dP}{dK}$ for this particular factor. But this is only a specially striking instance of the difficulty which is always present in economic investigations, and which has been faced perfectly frankly already. See p. 16. There always is a discontinuity in our functions, and such a formula as $\frac{dP}{dK}$ can under no circumstances have more than a relative appropriateness in economic investigations. This does not derogate from the extreme value of such expressions, as giving precise expression to the *theoretical limits of accuracy* in assigning the competitive shares in a product which the sundry factors of production would secure in an open market.

We may now turn to the second assumption that we have to examine. It concerns the legitimacy of treating all the factors except one selected factor as capable of being measured and expressed in one complex unit. Now the legitimacy of this may be deduced from the proposition we have just proved. For that proposition at once confirms and completes our conception of a quantitative relation between the services rendered by the several factors, each being measured by its effect on the product. And since this measurement may be supposed as precise as we please, it becomes a legitimate and intelligible hypothesis to assume that a *general command* of factors of production, expressed in terms of money, will be so specialised that no service will be secured at more than its worth, that is to say at a disproportionate sacrifice of any other service. Thus, given a market of productive services, there is a maximum productive efficiency in the expenditure of £1 therein, and it is perfectly legitimate to start with a unit of land, assume that the command of the other factors of production is so exercised as to secure the maximum productive result, and then treat the product as a function of land and pounds sterling. Two cautions however are necessary. In the first place, there is no kind of inherent propriety in singling out land as the special factor to be measured in its own unit, while all the other factors are expressed in pounds sterling. It would, for instance, often be more convenient, and would always be quite as legitimate, to express labour of some special kind in its own unit, and land, machinery, and perhaps even *other kinds* of labour itself, in terms of their market value in £ s. d. And, in the second place, we must be on our guard against calling the common-measure we adopt for all except the selected factor, "capital," and then talking of "capital" as a factor of production co-ordinate with land and labour. The fact is that in ordinary language we speak of the sum of resources with which a man starts in business (expressed in terms of gold) as his "capital," and we expect him to lay it out in land, labour, plant, etc., to the best advantage, in view of the market prices of all these things and their marginal efficiencies. In this sense "capital" is not *one* of the factors of production, but *all* of them. When we proceed to use "capital" in another sense, *viz.*, to signify all the factors except land and labour, its old associations hang about it, and we come insensibly to think that there is some intrinsic propriety in thinking and arguing about land as measured in its own unit of acres, labour as measured in its own unit of hours, and machines, buildings, tools, power of waiting for results, etc., etc., as measured in units of gold![1]

We see, then, that when properly safeguarded and generalised the two assumptions usually made in expositions of the law of rent may be legitimately accepted. A small proportional increment of each of the factors of production may be supposed to yield a proportional increment in the product; and we may, if we choose, select any one factor to measure in its proper unit while measuring all the rest in a common unit. And we see further that if these assumptions are made our co-ordinating law of distribution follows.

Hence we may claim for our theory of distribution the support alike of practical experience and of the best elaborated and least assailable portion of the current theory.

[1] The vicious habits of thought fostered by this arbitrary selection of some one or more factors to be always measured in their own units, while the rest are measured in units of gold, were pointed out to me long ago by Mr Graham Wallas, to whose suggestive criticisms of current theory I owe much. The connection of this confusion with the ambiguous use of the term "capital" was indicated to me by one of my own pupils.

Let us ask ourselves, then, exactly what it is we have proved. With the proviso that we are dealing only with cases in which the effect of a small increment of all the factors in lowering the market value of the product is imperceptible we have proved that if any periodic product P is shared amongst the factors that went to produce it in such way that any factor K gets $\frac{dP}{dK}\cdot K$, where $\frac{dP}{dK}$ is the rate at which under the actual circumstances an increment of K would affect the actual commercial product P, then the product will be just exhausted by the shares. If any factor gets more than the share thus determined some other factor will get less, and if any factor gets less some other factor will get more.

But when the representatives of the various factors are dealing with each other and making their bargains one with another, each has his market price based on an estimate, partly experimental and partly speculative, of the value of an increment of the factor in question to the industrial community at large, or $\frac{d\mathbf{P}}{dK}$. If the speculative element enters largely into these estimates it may happen that no arrangement is possible which will enable each factor to draw what it claims out of a common product; but if the estimates conform closely to actual industrial fact it will be possible for the factors to combine in such a proportion as to secure a product making in every case the actual $\frac{dP}{dK}$, with respect to the concrete product in question, correspond to the estimated $\frac{d\mathbf{P}}{dK}$, with respect to general industry. In practical cases there is usually a speculator who, besides himself contributing some direct factor (perhaps superintendence, or machinery, or both), buys out the other factors, speculatively, at their estimated values. If the sum of the estimated $\frac{d\mathbf{P}}{dK}\cdot$Ks turns out to be more than the sum of the actual $\frac{dP}{dK}$.Ks the speculator will have to "distribute" more than the whole product, and if less, less. But these gains and losses may be resolved into (1st) compensation for risk, and (2nd) the share that falls to this special speculating ability, regarded as a factor of production, and receiving its share of the product in accordance with the general formula $\frac{dP}{dK}$. K (which may in some cases be negative). So far as risk-taking is a necessary or useful condition for the conduct of industry it too must be regarded as a factor of production, and its share in the proceeds is regulated by precisely the same law as that which rules elsewhere.

Our law then may be regarded as perfectly general. In its strict form it merely asserts that the sum of the *actual* $\frac{dP}{dK}$. Ks covers the *actual* product. In this form it is not a law of distribution, but an analytical and synthetical law of composition and resolution of industrial factors and products, which would hold equally in Robinson Crusoe's island, in an American religious commune, in an Indian village ruled by custom, and in the competitive centres of the typical modern industries. The law so formulated holds, of course, even if there is no distribution amongst the factors at all; but if there is such distribution then we may assert, generally, that if any factor K obtains a share greater than $\frac{dP}{dK}$. K then one or more other factors must obtain less; but if none obtains more then all will obtain as much. In its practical form the law asserts that, in a freely competing community, no group of factors will willingly relinquish to any one of their number a larger remuneration than is fixed by the formula $\frac{dP}{dK}$, nor will any factor willingly accept for itself a smaller share than is fixed by the formula $\frac{d\mathbf{P}}{dK}$. There is equilibrium, therefore, when we have for any industry $\frac{dP}{dK}=\frac{d\mathbf{P}}{dK}$ for every K, and therefore

$$\frac{d\mathbf{P}}{dA}.A+\frac{d\mathbf{P}}{dB}.B+\frac{d\mathbf{P}}{dC}.C+\ldots\ldots=P$$

7. We have now completed our statement and examination of the law of distribution. It is far from my intention to develop its consequences in the present essay. But one or two miscellaneous notes and reflections may be added as an indication of the directions in which we may expect fruitful applications of the theorems now established.

(*a*) Before I was at all aware of the universality of the law of distribution we have been engaged in investigating I worked it out independently as the law of Industrial Interest. The results of this investigation were embodied in an article entitled On certain passages in Jevons's "Theory of Political Economy"' contributed to the American *Quarterly Journal of Economics* for April, 1889. I must refer the reader to that article for some of the applications of the law of distribution to the vexed problem of Interest. At the time the article was written Mr. W. E. Johnson, of King's College, Cambridge, to whom I was indebted for valuable suggestions, intimated his conviction that the principle involved was of universal application, and he has since worked out a theory of the laws of distribution and exchange based on the assumption of this universality. See his paper read before the Cambridge Economic Club, Easter, 1891. I did not, at the time, see the far-reaching significance of his remarks.

(*b*) The customary pictures of rent as a curve-bound surface, together with the style of reasoning based upon them, have fostered an inveterate delusion that there is somewhere a huge 'surplus' that may be cut into. Seeing that everything we ever investigate appears to give higher returns to the first than to the later increments, the imagination is vaguely haunted by great 'surplus' accumulations, away back at the origin, that are not touched by the "marginal" distribution. Now the first result of our investigation is to shew, with perfect clearness, that there is no such surplus at all. The marginal distribution accounts for the whole product, and though you can make any factor yield a higher rate per unit by diminishing the supply of it, you can only do so by making some of the rest yield a lower rate per unit. You can carry back each factor *successively* and make it seem to yield a large return per unit, but you cannot carry them all back *together* with the same result. It is the old fallacy of the argument from *any* to *all*. As this fallacy dies hard I have attempted one more diagramatic device that may be useful in combating it. Returning to the figure (Fig. 3) we used for the return to land and capital-plus-labour, land being constant, we will once more take

$$p_l = \int f_c(x) = \text{return per unit of L with the ratio C : L} = c.$$

Now at any point a we may erect att', as in Fig. 3, and we shall have $Oa \,.\, at + at' = F(Oa) =$ the value of p_l for $C : L = Oa$; and taking Λ, K, as the actual amounts of land and capital-plus-labour engaged we shall have

$$(1) \quad \Lambda(Oa \,.\, at + at') = \Lambda p_l = \Pi.$$

But the ratio $Oa : 1 :: C : L$ is the ratio $\frac{1}{Oa} : 1 :: L : C$, therefore the point b on Fig. 4 which corresponds to the point a on Fig. 3 will be given by the equation $Ob = \frac{1}{Oa}$. Erect $bt't$, as in Fig. 4, and we have $Ob.bt + bt' = \Phi(Ob) =$ the value of p_c, for $L : C = Ob$ or $C : L = Oa$, whence

$$(2) \quad K(Ob.bt + bt') = Kp_c = \Pi$$

But the relation $C : L = Oa = \frac{1}{Ob}$ gives

$$C = Oa.L$$
$$L = Ob.C$$

and enables us to write (1) and (2) respectively as

$$K \,.\, at + \Lambda \,.\, at' = \Pi$$

$$\Lambda \,.\, bt + K \,.\, bt' = \Pi$$

And since at' equals bt and bt' equals at, we may write either equation

$$K \,.\, at + \Lambda \,.\, bt = \Pi$$

And since at is $\frac{d\Pi}{dK}$ and bt is $\frac{d\Pi}{d\Lambda}$ the two figures, taken together, shew the whole product marginally distributed.

But it will be well to shew this on a single figure. In order to do this let us lay down the curve $y = f_l(x)$ in Fig. 5. Then instead of constructing the curve

$$y = f_l(x) = \int_0^x f_c(x)dx - x \,.\, f_c(x)$$

Let us calculate $\frac{f_l(x)}{x}$

And construct the curve

$$y = f_c(x) + \frac{f_l(x)}{x}$$

We may now take any ratio C : L $= c$, say Oa, and at the point a erect att', as in Fig. 5.

The whole rectangle Ot' now equals p_l, *i.e.*, the total product per unit of L; and the figure shows the whole of this product marginally distributed.

This is sufficiently obvious from inspection; but if a closer analysis is desired we may suppose units of L to be added on an axis of Z. Every additional unit of L will then correspond to c, or $\frac{C}{L}$, additional units of C. And l, or $\frac{L}{C}$, additional units of L will correspond to every additional unit of C. Hence the increments of product which will fall respectively to C and L, if the ratio between them is preserved, will be in the proportion $\frac{dP}{dC} : l.\frac{dP}{dL}$; but $\frac{dP}{dC}$ is at and $l.\frac{dP}{dL}$ or $\frac{f_l(x)}{x}$ is, by construction tt'. Therefore the marginal significance of increments that maintain the proportion between C and L will be in the proportion $at : tt'$; *i.e.*, p_l is marginally distributed between C and L in the figure.

As soon as we quite clearly understand that, under conditions usually regarded as normal, the marginal distribution exhausts the product, and that where every factor has taken a share regulated by its marginal efficiency, there is nothing left,—then, but not till then, shall we be in a position to attempt a scientific analysis of the ways in which the share of any one factor may be maximised.

(*c*) The idea that a "surplus" remains over and above the marginal distribution, has been fostered by a habit of confusing heterogeneous factors, alike in diagrams and in reasoning. Good-will, or notoriety, or even managing capacity, or monoply, have been confused with some special building or machinery with which they happen to be associated, and then their share of the product has been regarded as a return to "capital." And since it is evident that more plant brought into the concern at 'the margin' would not secure the same return, it has been concluded that in some cases "initial doses of capital" secure a higher return than marginal doses and provide a surplus over and above what is assigned to the several factors in the marginal distribution.

Or again a curve has been drawn of successive *Qualities* of land with their successively diminishing returns to "the same amount of capital and labour" (as if they ever got the same amount!) and since the picture looks like the one we have been dealing with (of product as a function of increasing ratio between other factors and land), and is also called a "curve of rent" the conclusions drawn from either of these pictures are then indifferently applied to both. A clear conception of the indefinite number of the factors of production, of the necessity of keeping each factor strictly homogeneous *as long as it is measured in its own proper unit*, and of the limits within which it is legitimate to express heterogeneous factors under the common measure of pounds sterling, would prevent such mistakes occurring.

(*d*) Such foolish questions as "does rent enter into the cost of production?" could never be asked, if the true law of distribution were kept in mind. Of course rent does not enter into the costs of production of the man who does not pay rent, and of course it does enter into the costs of production of the man who does pay it. But the whole question seems to be an inheritance from the old "cost of production" theory of exchange value. The argument seems to be: "The exchange value of wheat is determined "by its cost of production. But the man who pays rent "sells his wheat at the same price as the man who does not. "Therefore rent does not enter into the cost of production"! In reality of course, the product is a function of certain factors of production. The cost of production is the price paid to secure the co-operation of these factors. If two men produce the same thing but one of them avails himself of factors which the other does without, then different elements enter into the cost in each case; rent, for example, entering into the costs of one, but not of the other. It is really ludicrous to discuss gravely whether the absence of a certain item in one man's bill can be taken as removing it from the expenses of another man, on the ground that their total expenses are the same. It is extraordinary that the "cost of production" theory should have survived so rude a shock as it received from the investigations of the law of rent. "Rent is not the cause but the effect of the exchange value of the product" we read in our books. Precisely so, and since the law of rent is also the law of wages and the law of interest, it is equally true that "*wages* are not the cause but the effect of the exchange value of the product." And so too with interest. The economists have always seen that this fact was not inconsistent with the power of a combination of landlords, under given circumstances, to raise the exchange value of produce by standing out for higher rent; and so neither is it inconsistent with a similar power of combinations of men to raise the exchange value of the product by standing out for higher wages.

(*e*) From the social point of view it is impossible not to notice the significance of the fact that the return per unit to any factor is raised by the freedom with which the *other* factors are devoted to production, but lowered by the freedom with which it gives *itself*. As long as we think of labour in the mass and oppose material things, land and machinery, to it, we can only desire to see these latter as freely given and as scantily rewarded as possible; but if we remember that whenever we separate out one kind of labour then many other kinds of labour are included among the factors the increase of which brings about the rise of *its* remuneration and the fall of *their own*, it acquires a pathetic significance to reflect that to give self more freely is to give a larger share of the product to others, and retain a smaller share for self.

(*f*) Lastly I would just touch upon the fascinating subject of the analogies between the curves of Production and the curves of Satisfaction. In the ordinary individual or personal curve of consumption, the satisfaction is regarded as accruing to the individual, and the price paid is regarded as subtracted from the total amount. The early amounts of the commodity are regarded as yielding a surplus over their price because they yield a higher satisfaction than is represented by the marginal significance that regulates their price. Our investigations suggest another way of looking at the matter. Satisfaction may be regarded as a function of certain factors, one of them being a psyche or sensitive subject. If this be followed out it will be found that the "surplus" or "consumer's rent" is neither more nor less than the differential co-efficient or marginal significance of *psyche* as a factor in the production of satisfaction! Here we see once again that when the marginal distribution has been completed there is no surplus. Our ideal is for the whole satisfaction, without deduction, to fall in distribution to psyche, so that increments of psyche would be identical with increments of a satisfaction ideally maximised in its amount per unit of psyche.

8. We have now completed our examination of the Law of Distribution, and have indicated a very few of the suggestions and reflections that naturally rise from it. But since, in the progress of our investigation, we have been led to the close consideration of certain functions commonly dealt with by the economists, it will be of interest, in conclusion, to see whether we can ascertain anything definite as to their form.

The special function the economists have dealt with has generally been, product (in the narrow sense) per unit of land (of a uniform character) considered as a function of the ratio between all the other factors of production (measured in a common unit) and land ; or in our notation $p_l =$ the integral of $f_c(c)$. We wish to examine the form of f_c. Modifying our previous notation a little, we may put $f_{l,c}(c)$ for our old $f_c(c)$, and $f_{c,l}(l)$ for our old $\phi_l(l)$. We shall then have a notation which may easily be generalised. Now we have already established certain connections between $p_l = f_{l,c}(c)$ and $p_c = f_{c,l}(l)$; and in examining the form of $f_{l,c}$ we shall also examine that of $f_{c,l}$ and shall make the two enquiries assist each other. Further: singling out *any* factor K, and calling all the other factors, reduced to a common measure, S, we shall have

$$p_k = \text{the integral of } f_{k,s}(s) \text{ and } p_s = \text{the integral of } f_{s,k}(k)$$

Or, still more generally, putting all the factors into two complex groups S_1 and S_2 we shall have

$$p_{s_1} = f_{s_1,s_2,}(s_2) \text{ and } p_{s_2} = f_{s_2,s_1}(s_1)$$

And though we cannot make our conclusions strictly general, yet our investigation of the forms of $f_{l,c}$ and $f_{c,l}$ will lead us to conclusions, which, if used with due caution, will give us a clue to certain properties characteristic, as a rule, of the general form of f_{s_1,s_2}.

As we are about to give indefinitely high values to x it will be well to begin with $f_{c,l}(l)$ [the $\phi_l(l)$ of our Fig. 4], since the imagination will find more support, at first, in dealing with indefinite tracts of land than in dealing with indefinite amounts of capital-plus-labour. We are to conceive, then, a fixed amount of labour and appliances spread over increasing areas of land. We will make no inquiry at present concerning the first increments of land, but will take up the investigation at a point when the cultivator still would like more land but when each successive addition to the area over which he spreads his resources becomes less significant. In other words, we are at a stage of "diminishing returns" to land and increasing returns to labour, etc. This shows on the curve in descending values of y as x increases. But it is clear that this cannot continue indefinitely. As the labour is spread over ever wider areas of land the time must come when a further increment of land ceases to increase the product at all; that is to say the curve cuts the axis of x. If the labour is still further spread the increased allowance of land becomes absolutely hurtful and diminishes the output. And in the extreme case of an indefinite extension of the land cultivated by a definite allowance of labour, the product would obviously tend, without limit, to become nothing. It cannot sink below this, for we are dealing not with any "surplus" product, after replacing "capital," but with the gross yield; and that cannot be less than nothing. An analysis of these facts will show that they express themselves in the following properties of the curve we are considering. The total area included by the curve will equal zero; that is to say the positive area above the axis of x will exactly equal the negative area below it. This involves the curve becoming assymptotal to the axis of x, or coincident with it after a certain point. This property might indeed be independently deduced from considering that the successive applications of land, either lose all effect or tend to do so without limit as they become insignificant in relation to the total area under cultivation. It will appear further that the curve must have at least one negative maximum. Now translating all this back into the construction of the curve $f(c)_{l,c}$ [the $f_c(c)$ of our Fig. 3], in which land is constant, we see that it involves that curve passing through the origin (or coinciding with the axis of x from the origin to a certain small value of x). Because the integral p_l, *i.e.*, the total return per unit of land, for any value, κ, of x is always a definite fraction of the integral p_c, *i.e.*, the total return per unit of capital-plus-labour, for the value $\frac{1}{\kappa}$, or λ, of x. Therefore, if, as we have seen, both p_c and $f_{c,l}(l)$ are zero, for $l=\infty$, it follows that both p_l and $f_{l,c}(c)$ will be zero for $c=o$. Marshall has admitted curves, in this connection, that rise at first and then decline, and has thereby disarmed some very acute criticisms of Walras. But he inclines to think that this case is exceptional in England. *Principles*, vol. I., p. 212. I can hardly believe he will deliberately maintain this. When he speaks of the first dose being the most efficient he is clearly thinking of a definite dose such as a practical farmer might conceivably apply, but his figures being continuous and starting from the origin (his construction making it necessary that they should do so), he is clearly bound to deal with the smallest possible appreciable increments of the ratio of capital-plus-labour to land; and the consideration of such, in connection with the reciprocals on the curve of Fig. 4., will show that the curve must always pass through the origin (possibly coinciding with the axis of x near the origin). And this remains true if, with Marshall, we count only the *current* applications of labour-plus-capital. No amount of past applications will dispense with current labour, or make a finite amount of labour actually spread over an indefinite area of land, industrially significant. That is to say, in any case we shall have $f_{c,l}(l)=o$, for $l=\infty$; and therefore $f_{l,c}(c)=o$, for $c=o$.

Starting, then, with $f_{l,c}(c)=o$, for $c=o$, we will trace it through its values for increasing values of c. From our examination of $f_{c,l}$ we may deduce that $f_{l,c}$ will reach at least one positive maximum, and will then cut the axis of x. And we shall find, further, that the reasoning already applied to $f_{c,l}$ for the higher values of l holds, directly and independently, for $f_{l,c}$ for the higher values of c, though it is not quite so easy for the imagination to form concrete images of the further reaches in this latter case. We have then, a curve, starting at zero rising to a maximum (or several maxima) descending below the axis of x, sinking to a minimum (or several minima), becoming assymptotal to, or coincident with, the axis of x and yielding a total area, if carried to infinity, equal to zero.

Let us proceed to the more detailed economic interpretation.

Throughout the changes in the value of c, the share of the product falling to C will be at the rate $\frac{dP}{dC}$, or $f_{l,c}(c)$ per unit; and the share falling to L will be $\frac{dP}{dL} = f_{l,l}(c)$ per unit. At first this latter function will be negative. C will receive more than the whole product, and L's share will be negative. But each receives a share regulated by the effect on the product which a small increment in it would have. At this stage an increment of L, C remaining constant, would dimish the product.[1] The maximum negative rent would be reached if the cultivation were carried to the point of intensiveness represented by $c = Oa'$. (Fig. 3.) At a certain point of the down slope ($c = Oa''$ in our figure) the rent becomes o. Thence to the point at which the curve cuts

[1] It is not wholly impossible to imagine circumstances under which this might represent the industrial fact Marshall (*Principles* vol I. p. 215) declares that the function we call $f_{l,c}$ may have several maxima. Assume a case in which it has some such form as is shewn in Fig. 6. Now suppose that Oq per unit is the current remuneration of C. In the natural course of events we should have land of the quality represented in our curve either not cultivated at all or cultivated to the degree of intensiveness represented by Oa'. But suppose the land is only suited to a special kind of crop, and suppose that crop to be subsidised by some system of bounties or otherwise. It seems conceivable that the depression at t might check the further applications of C, and that the cultivator might receive the whole product Oat, minus payment of the negative rent, *i.e.*, plus the subsidy, Oqt. Perhaps such a case supposes a number of coincidences so great as to take it out of the range of legitimate hypothesis; but it may serve as a help to the imagination in a somewhat untried region of speculation, familiarity with which seems desirable.

the axis of x (when the whole product goes in rent) both rent and the return to capital-plus-labour will be positive. Were more capital-plus-labour still applied it would be doing a disservice. Increments of capital and labour would correspond to decrements of product; and the factor land would deal with capital-plus-labour as a discommodity, only consenting to receive its further co-operation for a consideration. By the law of indifference the share of C would now become negative throughout; L would receive, as usual, the whole (diminishing) product *minus* $\frac{dP}{dC}.C$, but $\frac{dP}{dC}$ being now a negative quantity this would mean more than the whole product for L's share. After a certain point further additions of capital and labour produce smaller proportionate decrements in the now vanishing product. At last the ratio of land to capital and labour is zero; small increments of land and of capital and labour, severally, are alike without significance, and the whole product is zero. Of course, all the later parts of this history correspond to imaginary circumstances which would never arise. If, however, it be objected that this fact makes it futile to consider them, we must answer that they correspond, point for point, to circumstances represented by the early values of x, which

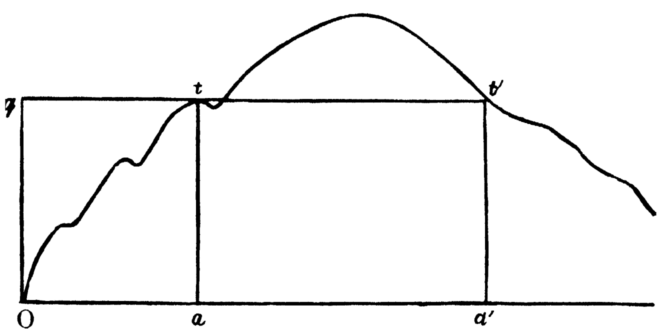


Fig. 6.

also would never arise in practice, but the theoretical requirements of which, should they arise, must inevitably be considered, since they help to determine the positive and finite area that represents rent.

We have now established certain properties as characteristic of the functions we are considering, *viz.*, $F(x)=o$ for $x=\infty$, or $x=o$, and $F'(x)=o$ for $x=\infty$ or $x=o$.

These properties then (if we revert to Figs. 3 and 4 and our former notation) ought to be common to our $F(x)$ and $\Phi(x)$. But we have also seen that $\Phi(x)$ stands in certain relations to $F(x)$ and may be derived from it by a definite process. It follows that if our reasoning is sound the process by which $\Phi(x)$ is derived from $F(x)$ must imply that all these properties, if characteristic of $F(x)$, will reappear in $\Phi(x)$. We proceed to demonstrate that this is actually the case; and so to gain a confirmation of the correctness of our analysis and our reasoning.[1]

Reverting to our Figs. 3 and 4, we have

$$f_c(o)=F'(o)=o$$
$$f_c(\infty)=F'(\infty)=o$$
$$F(\infty)=F(o)=o$$

We have to prove that—$f_l(x)$ being $F(x)-x\,.\,F'(x)$ and $\phi_l(x)$ being $f_l\left(\frac{1}{x}\right)$—we shall have

$$\text{(i.)}\quad \phi_l(o)=o$$
$$\text{(ii.)}\quad \phi_l(\infty)=o$$
$$\text{(iii.)}\quad \int_o^\infty \phi_l(x)dx=o$$

[1] I owe the working out of this proof to Mr. John Bridge, of Hampstead, to whom I here give my sincerest thanks for his invaluable assistance in the technical portions of the investigations throughout this essay.

(i.) and (ii.) result directly from the construction of the functions. We have to prove (iii.).

Now $$f_l(x) = \mathrm{F}(x) - x \, . \, \mathrm{F}'(x)$$

and $$\phi_l(x) = f_l\left(\frac{1}{x}\right) = \mathrm{F}\left(\frac{1}{x}\right) - \frac{1}{x} \, . \, \mathrm{F}'\left(\frac{1}{x}\right)$$

$$\therefore \int_o^\infty \phi_l(x)dx = \int_o^\infty \mathrm{F}\left(\frac{1}{x}\right)dx - \int_o^\infty \frac{1}{x} \, . \, \mathrm{F}'\left(\frac{1}{x}\right)dx$$

$$= \int_\infty^o \left[\mathrm{F}\left(\frac{1}{x}\right) - \frac{1}{x} \, . \, \mathrm{F}'\left(\frac{1}{x}\right)\right](-x^2)d \, . \, \frac{1}{x}$$

Putting v for $\frac{1}{x}$

$$= \int_\infty^o (\mathrm{F}v - v \, . \, \mathrm{F}'v)(-v^{-2})dv$$

$$= \int_\infty^o \frac{v \, . \, \mathrm{F}'v - \mathrm{F}v}{v^2} \, . \, dv$$

Of which the general integral is $$\frac{\mathrm{F}v}{v}$$

We have then to shew that

$$\frac{\mathrm{F}(o)}{o} - \frac{\mathrm{F}(\infty)}{\infty} \text{ is } o$$

Now $\mathrm{F}(o)$ is o $\therefore$ $\frac{\mathrm{F}(o)}{o}$ is indeterminate, but by the rule for $\frac{o}{o}$

the limit of $\frac{Fv}{v}$ will be the same as the limit of $\frac{F'v}{1}$, but this is o. And $\frac{F(\infty)}{\infty}$ is $\frac{o}{\infty}$ or o. Therefore

$$\int_{o}^{\infty} \phi_l(x)dx = o. \quad \text{Q.E.D.}$$

It has now been abundantly shewn that the implicit and explicit assumptions of the ordinary exposition of the law of rent rigorously involve an absolute identity of the law of rent and the law of return to "capital."