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The Economic Journal, Vol. 64, n°254, June 1954

STABILISATION POLICY IN A CLOSED ECONOMY [1]

RECOMMENDATIONS for stabilising aggregate production and employment have usually been derived from the analysis of multiplier models, using the method of comparative statics. This type of analysis does not provide a very firm basis for policy recommendations, for two reasons. First, the time path of income, production and employment during the process of adjustment is not revealed. It is quite possible that certain types of policy may give rise to undesired fluctuations, or even cause a previously stable system to become unstable, although the final equilibrium position as shown by a static analysis appears to be quite satisfactory. Second, the effects of variations in prices and interest rates cannot be dealt with adequately with the simple multiplier models which usually form the basis of the analysis.

In Section I of this article the usual assumption of constant prices and interest rates is retained, and a process analysis is used to illustrate some general principles of stabilisation policies. In Section II these principles are used in developing and analysing a more general model, in which prices and interest rates are flexible.

SECTION I

Some General Principles of Stabilisation

1. *The Model* [2]

The model consists of only two relationships. On the supply side, it is assumed that the rate of flow of current production, measured in real units per year and identical with the flow of real income, is adjusted, after a time lag, to the rate of flow of aggregate demand, also measured in real units per year. On the demand side, it is assumed that aggregate demand varies with real income or production, without significant time lag.[3] The proportion by which any change in aggregate demand induced by a change in real income falls short of that change in income will be called the marginal leakage from the system. In the simplest

[1] This article is based on part of the material of a thesis submitted to the University of London for the degree of Ph.D. I am indebted to Mr. A. C. L. Day, Mr. A. D. Knox, Professor J. E. Meade, Mr. W. T. Newlyn, Professor Lionel Robbins and Dr. W. J. L. Ryan for helpful comments on an earlier draft.

[2] A mathematical treatment of models used and of the stabilisation policies applied to them is given in the Mathematical Appendix.

[3] A demand lag could be introduced in addition to the production lag, but has been omitted to avoid complicating the mathematical treatment.

case of a closed economy with government ignored and with constant investment it is equal to the marginal propensity to save. In all the illustrations given below the marginal leakage is assumed to be 0·25.

The response of production to changes in demand is assumed to be gradual and continuous. For aggregative models this is more realistic than the usual assumption that production changes in sudden jumps. Even if each producer were to have a rigid production plan which he altered only at intervals of several months, the planning periods of the thousands of individual

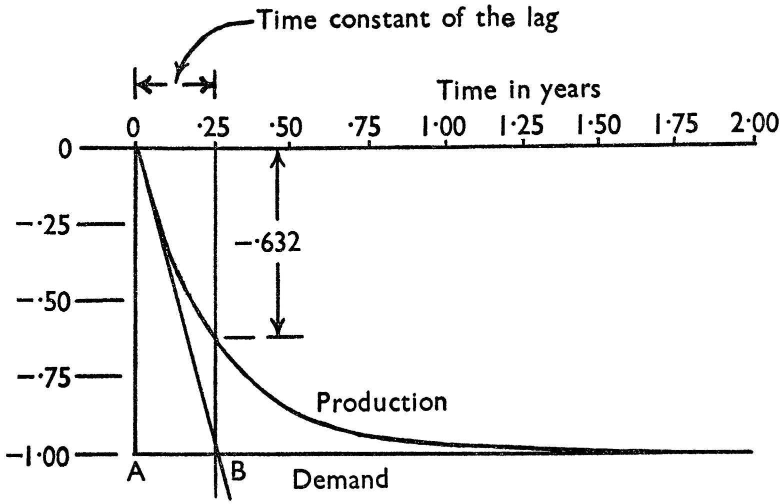

FIG. 1.—Single production lag.

producers would overlap, and the response of aggregate production to a sudden change in aggregate demand would consequently be more nearly approximated by a continuously changing variable than by one changing only at discrete intervals of time. To obtain a model in which this continuous change is represented, a distributed time lag is introduced by the hypothesis that whenever the production flow is different from the flow of demand, the production flow will be changing in a direction which tends to eliminate the difference and at a rate proportional to the difference.

The implications of this hypothesis are illustrated in Fig. 1, which shows the change that would occur in production if, from an initial equilibrium position, demand were to fall by one unit at time zero and to remain constant thereafter, on the assumption that the rate of change of production, measured in units per year per year, is four times the difference between demand and production, both measured in units per year. The factor of proportionality, 4 in this case, is a measure of the speed of response of production to changes in demand, and is indicated in Fig. 1 by the slope of the line OB, drawn tangential to the production-response curve at O. Its reciprocal is a measure of the slowness of response, or time taken to adjust production to changes in demand, and is called the time constant of the production lag. In this case it is equal to 3 months or 0·25 of a year, and is indicated in Fig. 1 by the length of the line AB. The time constant may also be defined as the time that would be taken, after a sudden change in demand, for production to change by an amount equal to 0·632 [1] of the full adjustment required for a new equilibrium, if demand were meanwhile to remain constant at its new value.

It is possible that a better representation of the real process of adjustment would be obtained by analysing the time lag into a number of separate components operating consecutively. For example, there may be a time lag in observing that an adjustment is necessary, another in making the decision to carry out the adjustment, and a third in actually making the adjustment. If two such lags are assumed, each with a time constant of $6\frac{1}{2}$ weeks so that the combined time constant is 3 months as in the previous example, the time path of the adjustment becomes that shown in Curve (*b*) of Fig. 2, while if three consecutive lags are assumed, each with a time constant of $4\frac{1}{3}$ weeks, the time path becomes that shown in Curve (*c*) of Fig. 2.[2] Although the slower adjustment obtained in the initial stages of the process with these multiple lags may be more realistic than that which results from the assumption of a single lag, the single lag is retained in the following analysis in order to simplify the mathematics.[3] In all the illustrations given below, the time constant of this single production lag is assumed to be 3 months.

In the complete model demand does not remain constant during the process of adjustment, but itself responds to changes in real income and production. It is therefore necessary to

[1] Or $1 - e^{-1}$, where e is the base of Napierian logarithms.

[2] If the number of consecutive lags is increased indefinitely, the time constants of the separate lags being simultaneously reduced so that the combined time constant remains fixed, the time path approaches the limit of a step function, jumping from 0 to -1 after a period of time equal to the combined time constant. I am indebted to Mr. J. Wise for providing me with a rigorous mathematical proof of this.

[3] Since aggregate production includes services, the provision of which responds instantaneously to changes in the demand for them, the more rapid initial response obtained by assuming a single lag may in fact represent quite a good approximation to the real process of adjustment.

distinguish between an initial or spontaneous change in demand, representing a disturbance or change in the relationships of the model, and the additional or induced changes in demand which result from the dependence of demand on production and in turn induce further changes in production by the familiar multiplier process. When these induced effects are taken into account the

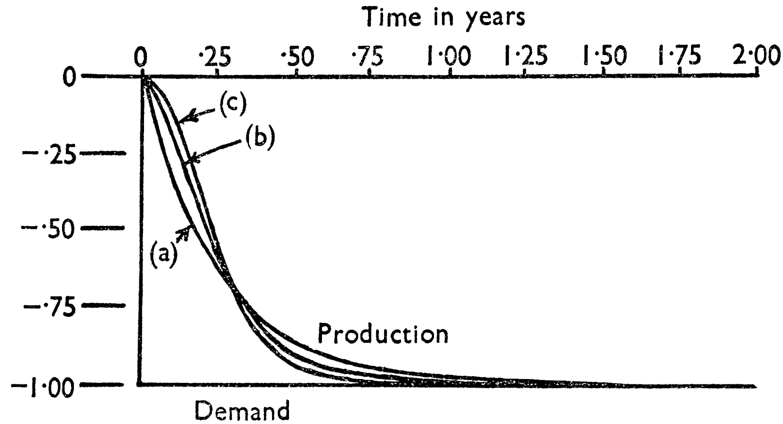

FIG. 2

Curve (*a*), single production lag.
Curve (*b*), double production lag.
Curve (*c*), triple production lag.

response of production, measured from an initial equilibrium value, to a spontaneous fall in demand of one unit, occurring at time zero and continuing thereafter, is shown by Curve (*a*) of Fig. 3. This is, of course, simply a continuous version of the ordinary multiplier process, the multiplier being the reciprocal of the marginal leakage, or 4.

2. *The Stabilisation Problem*

The adoption of a policy for stabilising production implies that there is some level of production which it is desired to maintain. The desired level may be that which, given the existing productive resources, would result in a certain level of employment, or it may be that which would result in a constant price index of consumers' goods, or the choice may be based on a number of other economic, political or social considerations. For the limited purpose of studying the principles of stabilisation in a closed economy the choice of desired production may be considered as given. The difference between the actual production and desired production at any time will be called the error in production.

Stabilisation policy consists in detecting any error and taking correcting action, by altering government expenditure, taxation, or monetary and credit conditions, in order to change demand in a direction which tends to eliminate the error. The amount by which aggregate demand would be changed as a direct result of the stabilisation policy (*i.e.*, excluding the further changes in demand which will be induced automatically through the operation of the multiplier process) if the policy were to operate without time lag will be called the potential policy demand, and the amount by which aggregate demand is in fact changed at any time as a direct result of the policy will be called the actual policy demand. Both may, of course, be either positive or negative.

The actual policy demand will usually be different from the potential policy demand, owing to the time required for observing changes in the error, adjusting the correcting action accordingly and for the changes in the correcting action to produce their full effects. A distributed time lag can again be introduced by the hypothesis that whenever such a difference exists the actual policy demand will be changing in a direction which tends to eliminate the difference and at a rate proportional to the difference. The time constant of this lag can then be defined in the same way as was done in the case of the production lag. The examples given below have been worked out for alternative correction lags with time constants of six months and six weeks respectively.

A number of different types of stabilisation policy will now be considered, corresponding to the different ways in which the correcting action taken may be related to the error in production.[1]

3. *Proportional Stabilisation Policy*

The simplest type of stabilisation policy is one in which the correcting action taken is such that the potential policy demand is made proportional in magnitude and opposite in sign [2] to the error in production. The ratio of the potential policy demand to the error, which is a measure of the strength of the stabilisation policy, will be called the proportional correction factor. As an example, a proportional correction factor of 0·5 would mean that

[1] The following treatment is an application of the general principles of automatic regulating systems and closed-loop control systems, in the analysis of which notable advances have been made in recent years. Cf. G. H. Farrington, *Fundamentals of Automatic Control,* Chapman and Hall, London, 1951; and Brown and Campbell, *Principles of Servomechanisms,* John Wiley and Sons, New York, 1948. On the use of closed-loop control theory in economics, cf. A. Tustin, *The Mechanism of Economic Systems,* Heinemann, London, 1953.

[2] That is, the principle of " negative feed-back " is used.

if production was 2% below the desired value the authorities concerned would attempt directly to stimulate demand by an amount equal to 1% of production (excluding the further increase which would be induced through the multiplier effects), and as the error was gradually reduced as a result of this action they would decrease the potential policy demand proportionately.

To show the effect of such a policy, it will be assumed that from an equilibrium position with production at the desired value there occurs at time zero and continues thereafter a spontaneous

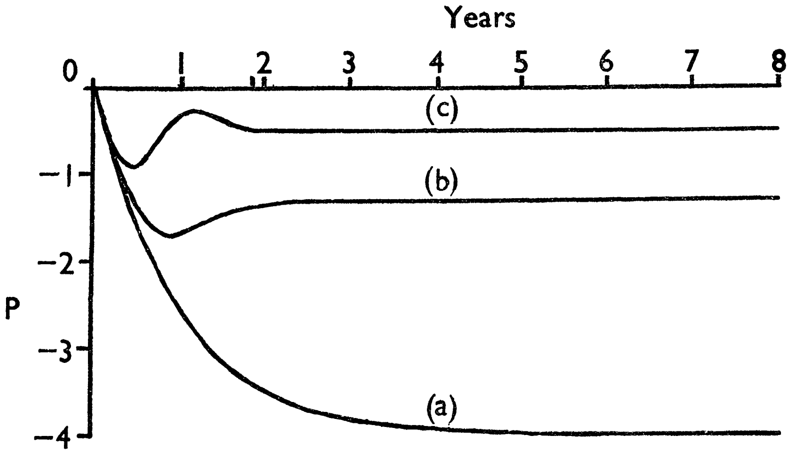

Fig. 3

Curve (*a*), no stabilisation policy.
Curve (*b*), $f_p = 0 \cdot 5$, $T = 6$ months.
Curve (*c*), $f_p = 2$, $T = 6$ months.

Note.—The symbols used in Figs. 3 to 9 inclusive have the following meanings :

P Change in production (measured from initial equilibrium).
f_p Proportional correction factor.
f_i Integral correction factor.
f_d Derivative correction factor.
T Time constant of the correction lag.

fall in demand of one unit. The resulting time path of production, if the proportional correction factor is 0·5 and the correction lag has a time constant of 6 months, is shown by Curve (*b*) of Fig. 3. The marginal leakage is assumed, as before, to be 0·25 and the production lag to have a time constant of 3 months, so the effect of the stabilisation policy can be seen by comparing Curve (*b*) with Curve (*a*). Curve (*c*) of Fig. 3 shows the effect of a stronger policy with a proportional correction factor of 2, the time constant of the correction lag again being 6 months. In the examples illustrated in Fig. 4 the time constant of the correction lag has been reduced to 6 weeks, the proportional correction factor again being zero for Curve (*a*), 0·5 for Curve (*b*) and 2 for Curve (*c*).

Two defects of a proportional stabilisation policy are immediately apparent. First, complete correction of an error is not obtained, since the correcting action continues only because the error exists. If the spontaneous change in demand is denoted by δ, the error in the final equilibrium level of production by ϵ, the proportional correction factor by f_p and the marginal leakage by l, in the final equilibrium the sum of the spontaneous and the policy changes in demand will be $\delta - f_p\epsilon$. The usual multiplier formula applies, so the total change in demand and production,

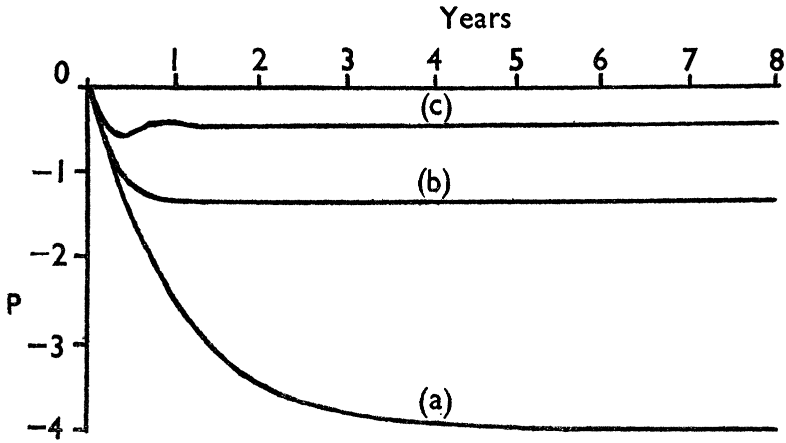

FIG. 4

Curve (a), no stabilisation policy.
Curve (b), $f_p = 0{\cdot}5$, $T = 6$ weeks.
Curve (c), $f_p = 2$, $T = 6$ weeks.

including the change induced by the multiplier process, will be $\frac{\delta - f_p\epsilon}{l}$. But the change in production is also the error, so that $\frac{\delta - f_p\epsilon}{l} = \epsilon$, from which $\epsilon = \frac{\delta}{l + f_p}$. When this type of policy is applied, therefore, the static multiplier becomes the reciprocal of the sum of the marginal leakage and the proportional correction factor, and a proportional correction factor of infinity would be required if the error were to be completely eliminated. The second defect of a proportional stabilisation policy is that it tends to cause a cyclical fluctuation in the time path of production, this fluctuation being the greater, the stronger the policy and the longer the time lag involved in applying it.

It may be noted that the proportional correction factor and the marginal propensity to save, or more generally any marginal leakage, have similar effects on the stability of the system. With a marginal propensity to save of zero, the simple multiplier system assumed so far would have no inherent regulation at all, *i.e.*, no stable equilibrium position would exist. With a positive marginal propensity to save, the change in demand resulting from a given change in production would differ from what it would have been if the marginal propensity to save had been zero by an amount proportional in magnitude and opposite in sign to the change in production. The marginal propensity to save therefore acts as a regulating mechanism of the proportional type inherent in the economy.

4. *Integral Stabilisation Policy*

An integral stabilisation policy is one in which the potential policy demand at any time is made proportional in magnitude and opposite in sign to the cumulated error up to that time, *i.e.*, to the time integral of the error instead of to the magnitude of the error. In terms of Figs. 3 and 4, with an integral stabilisation policy the potential policy demand at any time is made proportional to the area between the actual production curve and the desired production curve (or zero line) up to that time, whereas with a proportional stabilisation policy it is made proportional to the vertical distance between the two curves at that time. The ratio of the potential policy demand to the time integral of the error will be called the integral correction factor. If an error in production of 2% were to occur and to persist for a year, then with an integral correction factor of 0·5 the potential policy demand would be increased steadily from zero at the beginning to 1% of production at the end of the year. It is clear that with an integral stabilisation policy the final equilibrium position, if it exists, will be one in which the error is completely eliminated, since so long as even the smallest error persists the cumulated error or time integral of the error must be continuously increasing, and with it the magnitude of the correcting action, so that equilibrium is possible only when the error is zero.

It will be found, however, that in thus avoiding the first defect of a proportional correction policy, the second defect, the introduction of cyclical fluctuations, is greatly aggravated, and for this reason integral correction is rarely used alone in automatic control systems. There may, however, be a tendency for monetary authorities, when attempting to correct an " error " in production, continuously to strengthen their correcting action the longer the error persists, in which case they would be applying an integral correction policy.[1] Also, it will be argued in Section II of this article that flexible prices in an economy operate as an inherent regulating mechanism of the integral type. The integral relationship may therefore be of some importance in a number of economic adjustments.

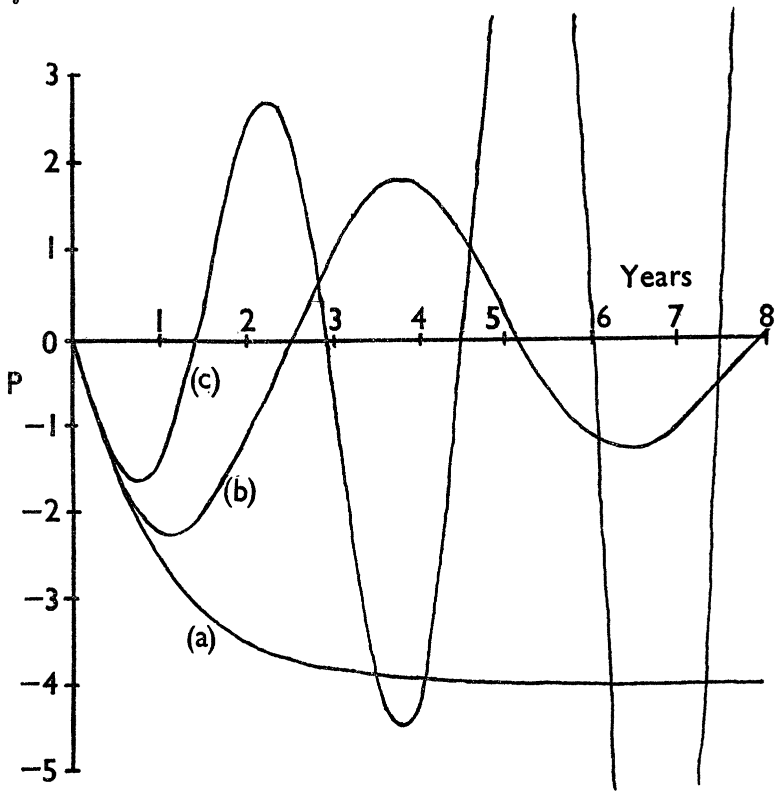

FIG. 5

Curve (*a*), no stabilisation policy.
Curve (*b*), $f_i = 0{\cdot}5$, $T = 6$ months.
Curve (*c*), $f_i = 2$, $T = 6$ months.

Figs. 5 and 6 show the effects of applying an integral stabilisation policy. The assumptions of the basic model and the type of disturbance are the same as in the previous examples. Curves

[1] International adjustments are not dealt with in this article; but it may be worth noting here that a country which attempts to regulate its current balance of payments, whether by means of internal credit policy or quantitative import control, and in doing so responds mainly to the size of its foreign reserves (*i.e.*, to the time integral of its current balance of payments) is applying an integral correction policy which is likely to cause cyclical fluctuations similar to those illustrated in Figs. 5 and 6. The short cycles which have occurred in the balances of payments of a number of countries since the war may be in part the result of such action.

(*a*) again show the response of production to unit spontaneous fall in demand when there is no stabilisation policy. In Fig. 5, Curve (*b*) shows the effect of adopting a stabilisation policy with an integral correction factor of 0·5, and Curve (*c*) the effect of a stronger policy with an integral correction factor of 2, the time constant of the correction lag being 6 months in each case. Curves (*b*) and (*c*) in Fig. 6 show how the response is modified when the time constant of the correction lag is reduced to 6 weeks, the integral correction factor again being 0·5 for Curve (*b*) and 2 for Curve (*c*).

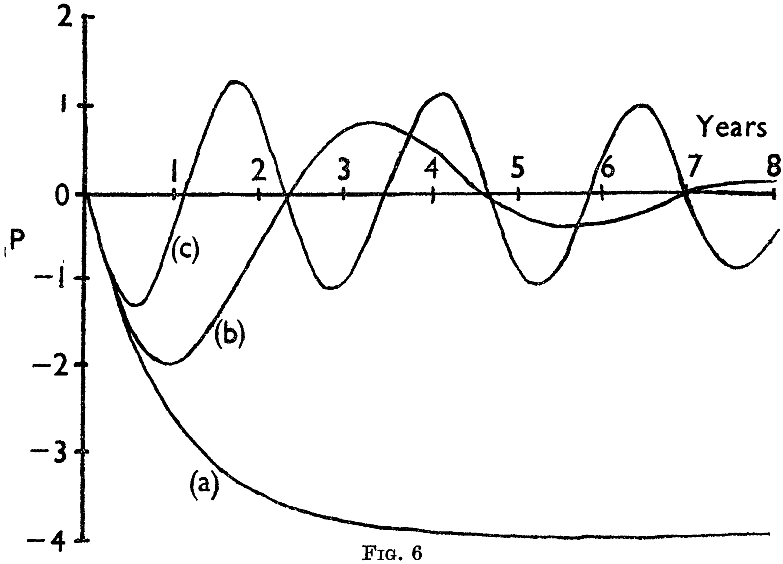

FIG. 6

Curve (*a*), no stabilisation policy.
Curve (*b*), $f_i = 0{\cdot}5$, $T = 6$ weeks.
Curve (*c*), $f_i = 2$, $T = 6$ weeks.

It will be seen that even with a low value of the integral correction factor, cyclical fluctuations of considerable magnitude are caused by this type of policy, and also that the approach to the desired value of production is very slow. Moreover, any attempt to speed up the process by adopting a stronger policy is likely to do more harm than good by increasing the violence of the cyclical fluctuations, particularly when the time lag of the correcting action is long. With an integral correction factor of 2 and a correction lag of 6 months, as illustrated in Curve (*c*) of Fig. 3, the system has become dynamically unstable. In such a case the oscillations would increase in amplitude until limited by non-linearities in the system and would then persist within those limits so long as the policy was continued.

5. *Proportional Plus Integral Stabilisation Policy*

A combination of proportional and integral stabilisation policies gives much better results than either policy alone. This can be seen from Figs. 7 and 8, in which Curves (*a*) again show the response of production to unit spontaneous fall in demand in the absence of stabilisation policy. Curve (*b*) of Fig. 7 shows how this response is modified if a stabilisation policy is adopted having a proportional correction factor of 0·5 plus an integral correction factor of 0·5, the time constant of the correction lag being 6 months. The proportional element in the policy helps to speed up correction and to limit the fluctuations caused by the

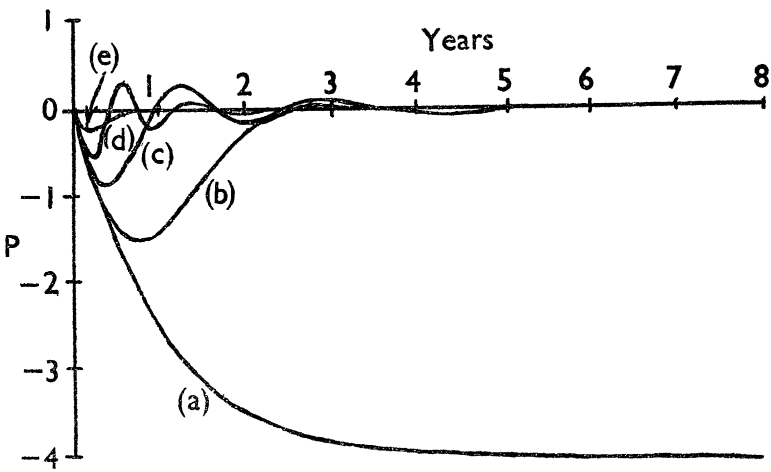

FIG. 7

Curve (*a*), no stabilisation policy.
Curve (*b*), $f_p = 0{\cdot}5, f_i = 0{\cdot}5, T = 6$ months.
Curve (*c*), $f_p = 2, f_i = 2, T = 6$ months.
Curve (*d*), $f_p = 8, f_i = 8, T = 6$ months.
Curve (*e*), $f_p = 8, f_i = 8, f_d = 1, T = 6$ months.

integral policy, while the integral element provides the complete correction unobtainable with the proportional policy alone. Curve (*c*) of Fig. 7 shows the response when a stronger policy is adopted, keeping the same proportion between the two elements, both correction factors being raised to 2, while in Curve (*d*) they are raised to 8, the correction lag remaining at 6 months in each case.

In the examples illustrated in Fig. 8 the time constant of the correction lag is reduced to 6 weeks. Curve (*b*) shows the response when both proportional and integral correction factors are 0·5. Comparing this with Curve (*b*) of Fig. 7, it may appear paradoxical that with a shorter correction lag a longer time elapses before something near full correction is obtained. The reason for this is that the more rapid operation of the policy results in a smaller error in the early stages of the adjustment, so that the cumulated error which forms the basis of the integral element is reduced, so reducing the speed of the later stages of the adjustment, which depend mainly on the integral element. The shorter the correction lag, therefore, the greater must be the integral correction factor if rapid correction is to be obtained. Conversely, of course, the longer the correction lag the smaller must be the integral correction factor if overshooting and fluctuations are to be avoided. A

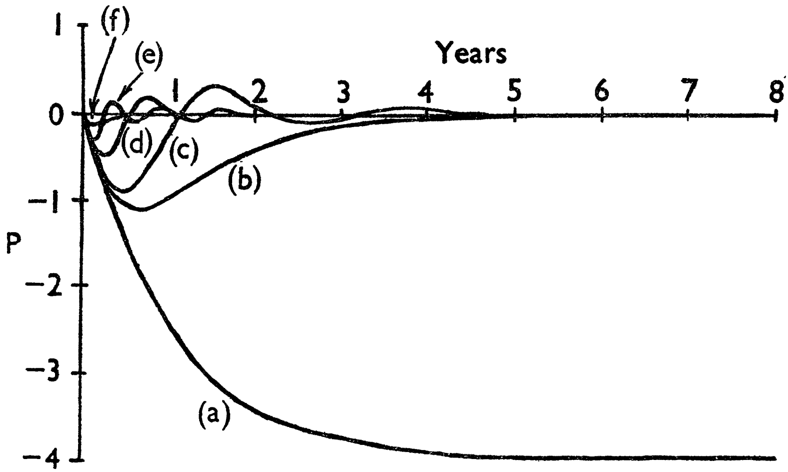

FIG. 8

Curve (*a*), no stabilisation policy.
Curve (*b*), $f_p = 0{\cdot}5$, $f_i = 0{\cdot}5$, $T = 6$ weeks.
Curve (*c*), $f_p = 0{\cdot}5$, $f_i = 2$, $T = 6$ weeks.
Curve (*d*), $f_p = 2$, $f_i = 8$, $T = 6$ weeks.
Curve (*e*), $f_p = 8$, $f_i = 32$, $T = 6$ weeks.
Curve (*f*), $f_p = 8$, $f_i = 32$, $f_d = 0{\cdot}25$, $T = 6$ weeks.

proportional correction factor of 0·5 plus an integral correction factor of 2 gives the response shown by Curve (*c*) of Fig. 8, while for Curve (*d*) the proportional and integral correction factors are 2 and 8, and for Curve (*e*) 8 and 32 respectively.

The fluctuations in the responses shown in Curves (*b*), (*c*) and (*d*) of Fig. 7 and in Curves (*c*), (*d*) and (*e*) of Fig. 8 could be eliminated by a sufficient reduction in the integral correction factor in each case; but only at the cost of increasing both the maximum size of the error and the time taken to correct it. A better method is available which not only eliminates the fluctuations, but also reduces both the maximum size of the error and the time taken to obtain complete correction.

6. *The Addition of Derivative Correction*

This method is to add to the potential policy demand, as determined by the proportional and integral relationships, a third element, proportional in magnitude and opposite in sign to the rate of change, or time derivative, of production. The effect of this is to make demand lower than it would otherwise have been whenever production is rising, and higher than it would otherwise have been whenever production is falling, so tending to check movements in either direction without affecting the final equilibrium position. The ratio of the magnitude of this element in the potential policy demand to the rate of change of production is called the derivative correction factor.

Curve (*e*) of Fig. 7 shows the response of production to a unit fall in demand when a proportional plus integral plus derivative stabilisation policy is applied, the correction factors being 8, 8 and 1 respectively, with a correction lag of 6 months. This may be compared with Curve (*d*) of Fig. 7, which shows the response when the derivative element of the policy is omitted, the other elements being unchanged. Similarly, the addition of a derivative element with a correction factor of 0·25 to a proportional plus integral policy with correction factors of 8 and 32 respectively and a correction lag of 6 weeks modifies the response from that shown by Curve (*e*) of Fig. 8 to that shown by Curve (*f*). It will be noticed that in order to maintain a suitable balance between the three elements in the policy when the length of the correction lag is reduced, it is necessary to reduce the derivative correction factor in about the same proportion as the time constant of the correction lag, whereas the integral correction factor had to be increased in about the same proportion.

The reader may have observed that the application of a derivative correction policy introduces into the system the same type of relationship as that postulated by the acceleration principle, but operating in the opposite direction, the additional policy demand being opposite in sign to the rate of change of production, whereas the additional investment demand resulting from the operation of the acceleration principle is of the same sign as the rate of change of production. This means that so far as its effect on system stability is concerned, the acceleration principle acts as a perverse or destabilising derivative correction element. The usefulness of the acceleration hypothesis has sometimes been questioned. But stated in the moderate form, that when production is rising entrepreneurs will want to invest at a greater rate, and after a time will in fact invest at a greater rate, than they would have done if production had not been rising, and conversely in the case of falling production, there can hardly be any doubt that the principle is a valid one. It seems desirable, therefore, to investigate the effects of stabilisation policies when the basic multiplier model is modified by the inclusion of an acceleration relationship.

7. *Stabilisation of a Multiplier–Accelerator Model*

We may define the term potential acceleration demand as the increase in investment demand that would occur as a direct result of rising production if the rise were to continue long enough for investment demand to become completely adjusted to it (and conversely in the case of falling production), and the term acceleration coefficient as the ratio between the potential acceleration demand and the rate of change of production which causes it.[1] Since investment demand will not respond instantaneously to alterations in the rate of change of production, a time lag may be introduced by the hypothesis that the actual acceleration demand tends continuously to approach the potential acceleration demand at a rate proportional to the difference between them. The time constant of this lag is defined in the same way as in the case of the production and correction lags.

Curve (*a*) of Fig. 9 (drawn with a different production scale from that used in Figs. 3–8 because of the greater fluctuations obtained) shows the response of production to unit spontaneous fall in demand when there is a marginal leakage of 0·25, a production lag with a time constant of 3 months, an acceleration coefficient of 0·6 and an acceleration lag with a time constant of 1 year, and when there is no stabilisation policy. With these values an explosive cycle is generated,[2] the fall in production in the first phase of the cycle being about 14 times as great as the spontaneous fall in demand. The cycle would eventually be limited by non-linearities in the system, and would then persist within those limits.

[1] Defined in this way, the acceleration coefficient is also the ratio of the change in the desired stock of capital to the change in the annual rate of production and real income, or what might be called the marginal desired capital–income ratio.

[2] The system gives damped oscillations when the acceleration coefficient lies between 0 and 0·5, explosive oscillations when it lies between 0·5 and 1, and is explosive without oscillations when the acceleration coefficient is greater than 1. All these values would probably be raised if the production and acceleration lags were divided, as they no doubt should be, into a number of separate shorter lags (observation lags, decision lags, process lags, etc.), and if similar lags were introduced in the response of demand to changes in income.

The application of a stabilisation policy having a proportional correction factor of 2 and a correction lag with a time constant of 6 months would change the response to that shown by Curve (*b*) of Fig. 9, and the addition to this policy of an integral element with a correction factor of 2 would change the response to that shown by Curve (*c*). As might be expected, the effect of the acceleration relationship has been to increase both the magnitude and the duration of the fluctuations resulting from these policies. (These responses may be compared with those shown by Curve (*c*) of Fig. 3 and Curve (*c*) of Fig. 7.)

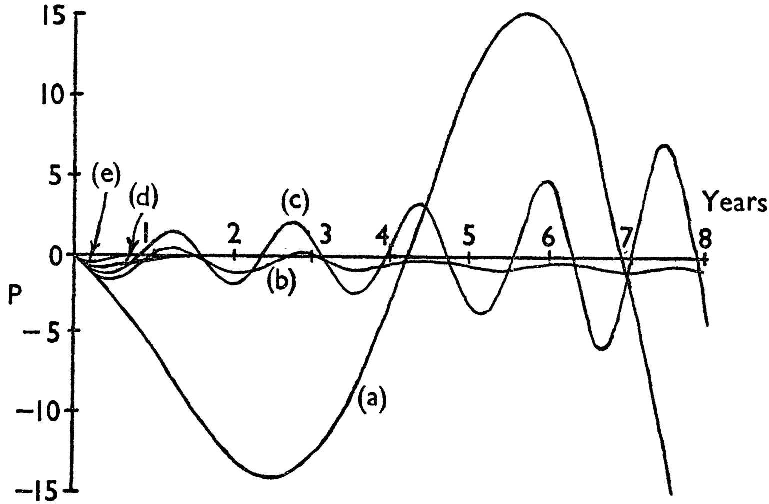

FIG. 9

Curve (*a*), no stabilisation policy.
Curve (*b*), $f_p = 2$, $T = 6$ months.
Curve (*c*), $f_p = 2$, $f_i = 2$, $T = 6$ months.
Curve (*d*), $f_p = 2$, $f_i = 2$, $f_d = 0{\cdot}55$, $T = 6$ months.
Curve (*e*), $f_p = 8$, $f_i = 8$, $f_d = 1{\cdot}3$, $T = 6$ months.

To eliminate these fluctuations it would be necessary to add a derivative element to stabilisation policy. A derivative correction factor of 0·3 would be needed to offset the acceleration coefficient of 0·6 (the derivative correction factor need be only half the size of the acceleration coefficient in this case, since the length of the correction lag is only half that of the acceleration lag), and an additional derivative correction factor of 0·25 would be needed to eliminate the fluctuations introduced by the proportional and integral elements of the stabilisation policy. Adding therefore a derivative element with a correction factor of 0·55 the response shown by Curve (*d*) of Fig. 9 is obtained. Finally, if the stabilisation policy was strengthened by multiplying each correction factor (excepting that part of the derivative correction factor which was needed to offset the acceleration coefficient) by four, the response shown by Curve (*e*) would be obtained.

These results appear to indicate that if any stabilisation policy [1] is to be successful it must be made up of a suitable combination of proportional, integral and derivative elements. A strong proportional element is needed as the main basis of the policy, sufficient integral correction should be added to obtain complete correction of an error within a reasonable time and an element of derivative correction is required to overcome the oscillatory tendencies which may be introduced by the other two elements of the policy. If the system itself has a considerable tendency to oscillate as a result of a perverse derivative relationship inherent in it in the form of the acceleration principle, the integral element in the policy should be made very weak or avoided entirely, unless it can be accompanied by sufficient derivative correction to offset the destabilising effects of the perverse derivative relationship.

8. *Diagrammatic Representation of the System*

The system of relationships used in this Section can be represented by a block diagram of the type frequently employed in the analysis of closed-loop systems. In Fig. 10 the lines represent variables and the squares represent relationships between variables, the particular relationship in each case being indicated by the symbol within the square. D indicates differentiation with respect to time, $\int$ indicates integration with respect to time and L indicates the operation of a distributed time lag in the adjustment of one variable to another. Brackets indicate multiplication by the parameter inside the brackets. The arrows show the direction of causation or the sequence of responses. Circles represent addition or subtraction according to the algebraic signs shown in each case.

The simple multiplier model is represented by the single closed loop at the bottom of the diagram. Production P is related to demand E through the operation of the production time lag L_p, and in turn influences demand through the marginal propensity to spend $(1 - l)$, l being the marginal leakage. The

[1] The general principles of stabilisation have been illustrated here with particular reference to aggregate production in a closed economy, but they are of quite general applicability. They could equally well be used, for example, in investigating the stability of adjustments in international trade, or the problems involved in commodity price stabilisation schemes.

loop immediately above this represents the acceleration principle, which adds another component to demand equal to the rate change of production $\frac{dP}{dt}$ multiplied by the acceleration coefficient k and subject to the acceleration time lag L_a.

The three loops at the top of the diagram represent the three types of stabilisation policy. The error in production ϵ is obtained by subtracting the desired production P_d from the actual production. (In the illustrations given in this article all variables are measured as deviations from their values in an initial equilibrium position with production at the desired level.

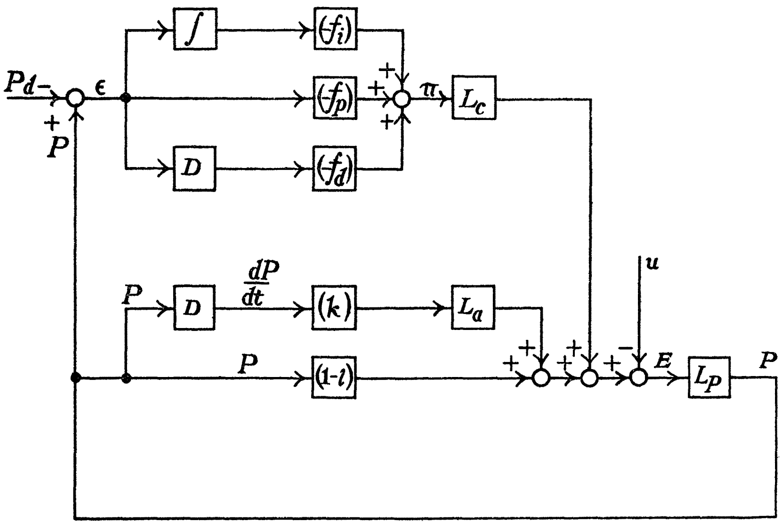

FIG. 10

P_d is therefore zero and ϵ equals P.) The error, the integral of the error and the derivative of the error are multiplied by $-f_p$, $-f_i$ and $-f_d$ respectively, f_p being the proportional correction factor, f_i the integral correction factor and f_d the derivative correction factor. The potential policy demand π is the sum of these products, and subject to the operation of the correction time lag L_c determines the actual policy demand which is added to the other components of aggregate demand.

The variable u represents a disturbance to the system, assumed throughout this article to take the form of a spontaneous fall in demand of one unit at time zero. Other forms of disturbance could, of course, be assumed and their effects investigated, and disturbances could be applied to other variables instead of, or in addition to, demand.

Section II

A Model with Flexible Prices

1. *The Relationship between Prices and Production*

If changes in the quantity and productivity of the factors of production are ignored, the change in the average level of product prices which results from a given change in the aggregate level of production will be the sum of two components. First, if the prices of the services of the factors of production (which will be referred to for brevity as factor prices) are absolutely rigid, product prices, tending to move with marginal costs, will vary directly with the level of production. This component of the change in product prices is probably not very large, and will be neglected in the following analysis.

Second, if factor prices have some degree of flexibility, there will be changes in product prices resulting from the changes which take place in factor prices. Even with flexible factor prices, there will be some level of production and employment which, given the bargaining powers of the different groups in the economy, will just result in the average level of factor prices remaining constant, this level of production and employment being lower, the stronger and more aggressive the organisation of the factors of production. If aggregate real demand is high enough to make a higher level of production than this profitable, entrepreneurs will be more anxious to obtain (and to retain) the services of labour and other factors of production and so less inclined to resist demands for higher wages and other factor rewards. Factor prices will therefore rise. The level of demand being high, the rising costs will be passed on in the form of higher product prices. Factor and product prices will continue to rise in this way so long as the high level of demand and production is maintained, the rate at which they rise being greater, the higher the level of demand and production.

Conversely, if aggregate real demand is so low that production at the level which would result in constant factor prices is unprofitable, entrepreneurs will be more anxious to force down factor prices, while at the lower level of employment factors will be less able to press for higher rewards and more inclined to accept lower rewards. Factor prices will therefore gradually move downwards, and the level of demand being low, the falling costs will be reflected in falling product prices. Prices will continue to fall in this way so long as demand and production remain low, the rate of fall being greater, the lower the level of demand and production.

We may therefore postulate a relationship between the level of production and the rate of change of factor prices, which is probably of the form shown in Fig. 11, the fairly sharp bend in the curve where it passes through zero rate of change of prices being the result of the greater rigidity of factor prices in the downward than in the upward direction. The relationship between the level of production and the rate of change of product prices will be of a similar shape if productivity is constant. In

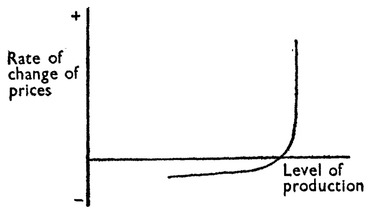

FIG. 11

spite of the marked curvature of the relationship, linearity may be assumed as an approximation for small changes in production. If the desired level of production is now taken to be that which would result in a constant level of product prices, we may say that the rate of change of product prices will be approximately proportional to the deviation of production from this level, *i.e.*, to the error in production, this approximation being better, the smaller the fluctuations which occur in production.

2. *Additional Relationships when Prices and Interest Rates are Flexible*

The model used in Section I can now be extended by dropping the assumption of constant prices and interest rates. The complete model is shown diagrammatically in Fig. 12, which is the same as Fig. 10 excepting for the addition of the group of relationships in the centre of the diagram. Four additional sequences can be distinguished.

(i) A deviation of production from the desired level will be accompanied by a change in the number of transactions and so a change in the amount of money needed for conducting them, even if prices are rigid. If the quantity of money is less than perfectly elastic, this will cause interest rates, i, to change in the same direction [1] as the error in production. As a result of this

[1] Provided that the liquidity preference schedule does not shift.

change in interest rates there will be a potential change[1] in investment demand and production, in the opposite direction to the error in production and (assuming linearity throughout) proportional to the error. This sequence of responses is represented in Fig. 12 by the closed loop of relationships (a), ($-b$),

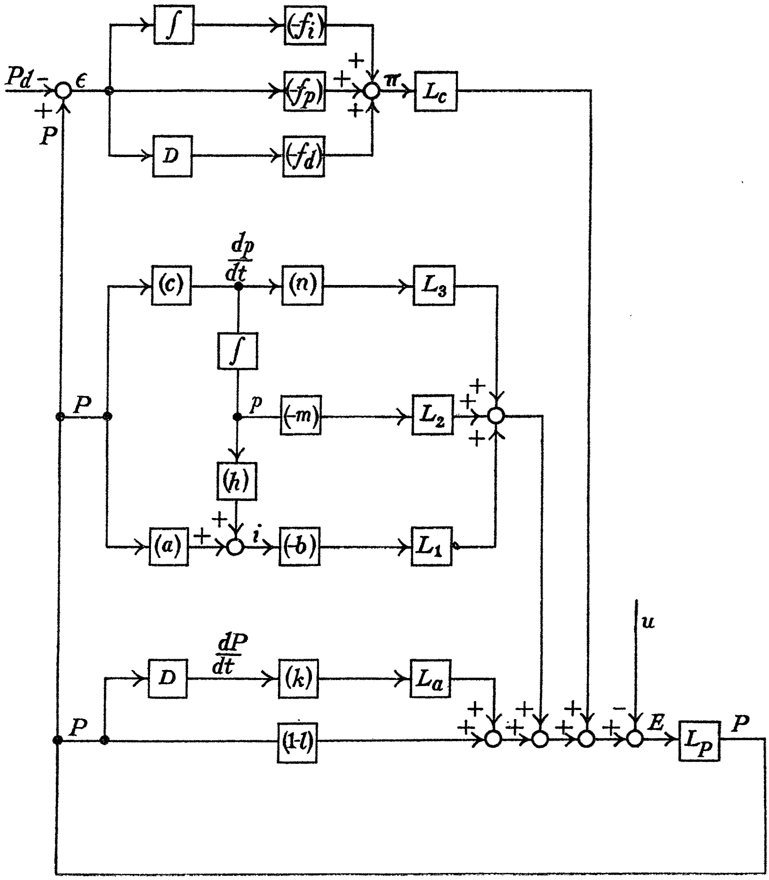

FIG. 12

L_1, L_P, giving a potential feed-back of $-abP$ and so acting as a regulating mechanism of the proportional type.

(ii) If prices, p, are flexible, the error in production will also cause prices to change at a rate proportional to the error. The amount by which prices have changed at any given time, being identical with the time integral of their rate of change up to that time, will be proportional to the time integral of the error. This

[1] It will be remembered that by a potential change in any variable we mean the change that would take place if no time lag was involved.

change in prices will cause a further change in the amount of money needed for conducting transactions, in addition to that caused by the change in the number of transactions conducted. There will therefore be an additional change in interest rates [1] in the same direction as the error, causing a further potential change in investment demand and production in the opposite direction to the error and proportional to the time integral of the error. This sequence of responses thus operates as a regulating mechanism of the integral type. The sequence is represented in Fig. 12 by the relationships (c), $\int$, (h), $(-b)$, L_1, L_P, which give a potential feed-back of $-chb\int P$.

(iii) Professor Pigou has pointed out [2] that even if the liquidity preference schedule was infinitely elastic at the prevailing level of interest rates, so that interest rates failed to move with a change in production and prices, a change in the level of prices would still influence demand by changing the real value of money balances and so the amount of saving at given incomes and interest rates. This potential change in demand would be in the same direction as the change in the real value of money balances, and therefore in the opposite direction to the change in prices. For small changes it would be approximately proportional to the change in prices and therefore proportional to the time integral of the error in production. The Pigou effect is therefore equivalent to another integral regulating mechanism inherent in the economy. It is represented in Fig. 12 by the closed loop of relationships (c), $\int$, $(-m)$, L_2, L_P, giving a potential feed-back of $-cm\int P$.

[1] The device of considering the quantity effects and price effects separately and then adding them can be justified as follows: Denoting interest rates, production and prices, measured from a zero base, by i_0, P_0 and p_0 respectively, and deviations from their initial equilibrium values by i, P and p, we have

$$i_0 = F(P_0, p_0)$$

Expanding this expression in a Taylor series and dropping all but the first two terms gives

$$\Delta i_0 = \frac{\partial i_0}{\partial P_0}\Delta P_0 + \frac{\partial i_0}{\partial p_0}\Delta p_0$$

or

$$i = \frac{\partial i}{\partial P}P + \frac{\partial i}{\partial p}p$$

as an approximation valid for small changes. For small changes $\frac{\partial i}{\partial P}$ and $\frac{\partial i}{\partial p}$ may also be considered constant, so we may write

$$i = aP + hp$$

which is the relationship shown in Fig. 12. Similar approximations are involved in considering the change in aggregate demand as the sum of a number of separate components.

[2] "The Classical Stationary State," ECONOMIC JOURNAL, December 1943, and "Economic Progress in a Stable Environment," *Economica*, August 1947.

(iv) Demand is also likely to be influenced by the rate at which prices are changing, or have been changing in the recent past, as distinct from the amount by which they have changed, this influence on demand being greater, the greater the rate of change of prices. Since the rate of change of prices in turn depends on the error in production, the potential change in demand and production resulting from these relationships will be approximately proportional to the error in production. This sequence of responses is represented in Fig. 12 by the relationships (c), (n), L_3, L_P, the potential feed-back therefore being cnP. The direction of this change in demand will depend on expectations about future price changes. If changing prices induce expectations of further changes in the same direction, as will probably be the case after fairly rapid and prolonged movements, demand will change in the same direction as the changing prices. That is, n will be positive, and there will be a positive feed-back tending to intensify the error, the response of demand to changing prices thus acting as a perverse or destabilising mechanism of the proportional type. If, on the other hand, there is confidence that any movement of prices away from the level ruling in the recent past will soon be reversed, demand is likely to change in the opposite direction to the changing prices. n will then be negative, and the response of demand to changing prices will act as a normal proportional regulating mechanism.

3. *Inherent Regulation of the System*

Some examples will be given below to illustrate the stability of this system under different conditions of price flexibility and with different expectations concerning future price changes. As before, it will be assumed that the marginal leakage is 0·25 and that the time constant of the production lag is 3 months. The time constants of the lags L_1, L_2 and L_3 will be taken as 6 months. The acceleration relationship will be omitted (by putting $k = 0$) so that the effects of price flexibility can be seen by comparing the response of the system with that of the multiplier model.

In deciding upon suitable values for the remaining parameters it will be convenient to think of the units being such that in the initial equilibrium position production is 100 units per year, the price index also being 100. Changes can be expressed in either absolute or percentage terms with negligible error so long as prices and production do not move too far from their initial equilibrium values. The product of the parameters a and b will be given the value 0·2, which is equivalent to assuming that if production were to fall by 1% or 1 unit, prices remaining constant, interest rates would fall sufficiently to stimulate investment demand by 0·2 of a unit. The product of the parameters h and b must be a little greater than this, since if the price level were to fall by 1%, production remaining constant, there would be the

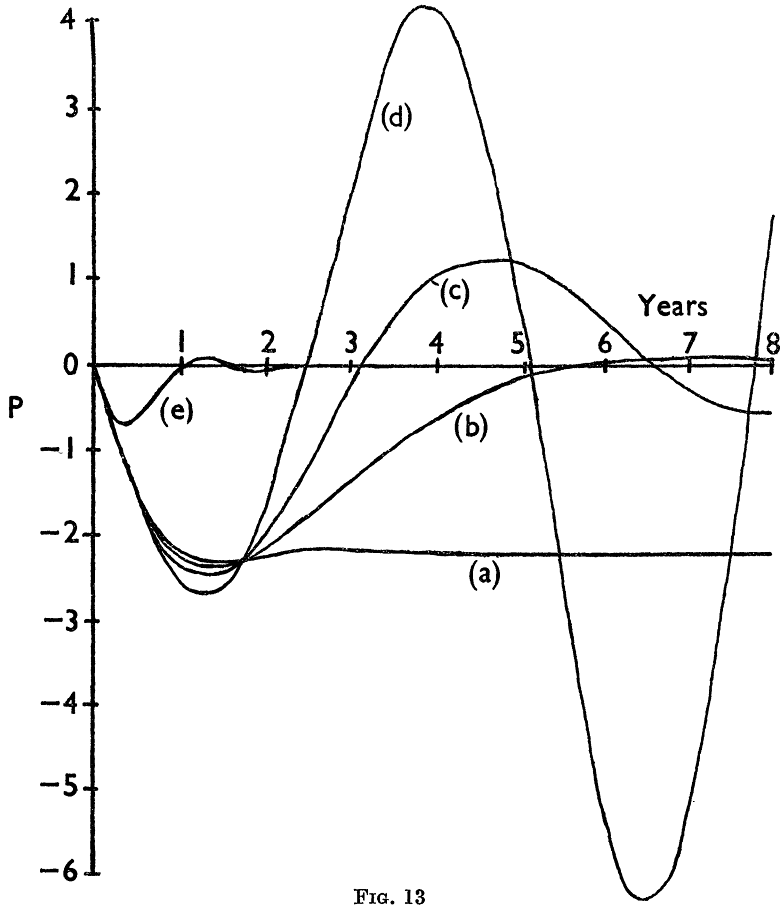

FIG. 13

Curve (a), zero price flexibility ($c = 0$).
Curve (b), $c = 0{\cdot}5$, $n = 0{\cdot}2$.
Curve (c), $c = 1$, $n = 0{\cdot}2$.
Curve (d), $c = 2$, $n = 0{\cdot}2$.
Curve (e), with stabilisation policy.

same reduction in the demand for money for transactions purposes; but the real value of money balances would be slightly increased so that the fall in interest rates would be rather greater. The product hb will therefore be given the value 0·25. m will be taken as 0·05, equivalent to assuming that a 1% fall in the price level would stimulate demand by 0·05% through its effect in increasing the real value of money balances, if there was no change in interest rates.

The responses shown in Fig. 13 have been worked out on the assumption that n has the value 0·2. This means, for example, that if prices are falling at the rate of 1% per year, demand will tend to be 0·2 of 1% lower that it would have been if prices had been constant. Given these assumptions, Curve (*a*) shows the response of production to unit fall in demand when there is zero price flexibility ($c = 0$). This response is similar to that of the multiplier model (Curves (*a*) of Fig. 3–8); but the final error is less owing to the stabilising effects of the flexible interest rates.

Curve (*b*) shows the response when $c = 0{\cdot}5$, *i.e.*, when for each 1% error in production prices change at the rate of 0·5% per year. The two integral regulating mechanisms now come into play, operating through interest rates and the Pigou effect respectively, so in the final equilibrium position the error is completely corrected. The total strength of the proportional regulating mechanisms is, however, reduced by the perverse effect of the response of demand to changing prices, with the result that the magnitude of the error in the early stages of the adjustment is slightly greater than in the case of zero price flexibility.

When $c = 1$ the response is that shown by Curve (*c*) and when $c = 2$ it becomes that shown by Curve (*d*). The strength of the integral regulating mechanisms increases with the increasing degree of price flexibility, while the total strength of the proportional regulating mechanisms decreases as demand responds perversely to the more rapid rate of change of prices, and both these effects tend to introduce fluctuations when price flexibility is increased beyond a certain point. When price expectations operate in this way, therefore, the system has fairly satisfactory self-regulating properties when prices are moderately flexible; but becomes unstable when there is a high degree of price flexibility.

If changing prices induce expectations of future changes in the reverse direction, n being given the value $-0{\cdot}2$, the response of the system when there are different degrees of price flexibility is as shown in Fig. 14. Curves (*a*), (*b*), (*c*) and (*d*) have been worked out for values of c of 0, 0·5, 1 and 2 respectively, and so can be compared with the equivalent curves in Fig. 13. In this case the response of demand to changing prices provides a normal proportional regulating mechanism which increases in strength with the increasing degree of price flexibility. This increasing proportional regulation limits the fluctuations which would otherwise be introduced as a result of integral mechanisms, the strength of which also increases with increasing price flexibility, and at the same time increases the speed with which the error is corrected in the early stages of the adjustment. We may conclude that the self-regulating properties of the system will be considerably

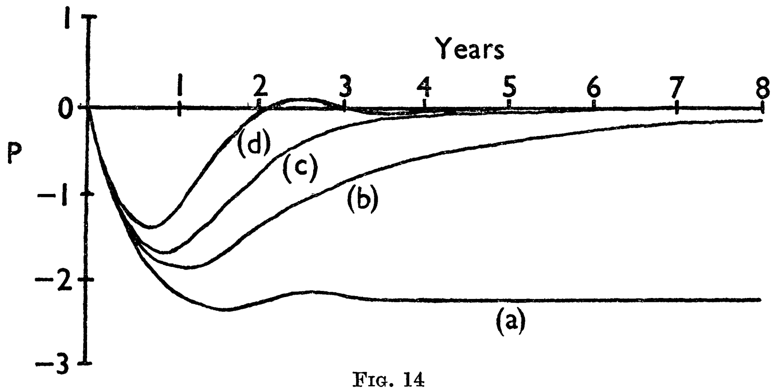

FIG. 14

Curve (*a*), zero price flexibility ($c = 0$).
Curve (*b*), $c = 0{\cdot}5$, $n = -0{\cdot}2$.
Curve (*c*), $c = 1$, $n = -0{\cdot}2$.
Curve (*d*), $c = 2$, $n = -0{\cdot}2$.

improved if there is confidence that any movement of prices away from the level ruling in the recent past will soon be reversed, and that if such confidence is sufficiently great the self-regulating properties will also be better, the higher the degree of price flexibility in the system.

4. *Stabilisation of the System*

If a stabilisation policy is applied to the system, the values of the correction factors required to produce any particular response will depend on the values of the parameters in the system. Examples of the values of correction factors which would result in the response shown by Curve (*e*) of Fig. 13 if the time constant of the correction lag was 6 months are given in the following table.

Proposals have sometimes been made for improving the stability of the economic system by "building in" additional regulating mechanisms. In the British Government's White Paper on Employment Policy [1] reference is made to a scheme for influencing consumption spending by varying the rates of social-insurance contributions above or below the standard rates by

[1] Cmd. 6527, 1944, paragraphs 68–71 and Appendix II.

Response of unstabilised system.	Stabilisation policy required to produce the response shown by Curve (*e*) of Fig. 13.		
	Proportional correction factor.	Integral correction factor.	Derivative correction factor.
Curve (*a*), Fig. 13	1·8	2·0	0·2
Curve (*b*), Fig. 13	1·9	1·85	0·2
Curve (*c*), Fig. 13	2·0	1·7	0·2
Curve (*d*), Fig. 13	2·2	1·4	0·2
Curve (*b*), Fig. 14	1·7	1·85	0·2
Curve (*c*), Fig. 14	1·6	1·7	0·2
Curve (*d*), Fig. 14	1·4	1·4	0·2
Curve (*a*), Figs. 3–8 . . .	2·0	2·0	0·2
Curve (*a*), Fig. 9	2·0	2·0	0·5

amounts depending on the percentage by which the actual employment is above or below the average percentage over a period of years. On the basis of the figures given in the White Paper to illustrate how the scheme might operate, this would introduce a proportional regulating mechanism with a correction factor of about 0·2.[1] If schemes of this sort could be administered successfully they would undoubtedly improve the stability of the system considerably, and would probably be more effective than attempts to forecast conditions a year or more ahead and adjust an annual budget accordingly.

Monetary policy would be a more convenient instrument for stabilising an economy. Some may doubt whether it is a sufficiently powerful instrument; but if the right type of stabilisation policy is being applied continuously, comparatively small correcting forces are sufficient to hold the system near the desired position once that position has been attained. It is quite likely, therefore, that a monetary policy based on the principles of automatic regulating systems would be adequate to deal with all but the most severe disturbances to the economic system.

[1] Professor Meade has suggested a scheme, operating through special monthly taxes and credits related to the level of employment, which would have similar effects. With the scales of taxes and credits suggested by him the proportional correction factor would be about 0·4. The cumulative expansion and contraction of the quantity of money resulting from the proposed method of financing the scheme would also introduce an integral correction element. Professor Meade must have been aware that there would be some disadvantage in having too much integral correction, since he recommended that this effect should be partially offset through appropriate action by the central bank. Cf. J. E. Meade, *Consumers' Credits and Unemployment*, Oxford University Press, London, 1938.

Mathematical Appendix

1. *Symbols Used*

- P aggregate production, in units per year, measured from the initial equilibrium value.
- E aggregate demand, in units per year, measured from the initial equilibrium value.
- α the speed of response of production to changes in demand. The time constant of the production lag is $\frac{1}{\alpha}$ years.
- D the differential operator $\frac{d}{dt}$.
- t time in years.
- l the marginal leakage.
- u unit step function, equal to zero when $t \leqslant 0$ and to unity when $t > 0$.
- β the speed of response of policy demand to changes in potential policy demand. The time constant of the correction lag is $\frac{1}{\beta}$ years.
- f_p proportional correction factor.
- f_i integral correction factor.
- f_d derivative correction factor.
- k acceleration coefficient.
- κ speed of response of demand to changes in the rate of change of production. The time constant of the acceleration lag is $\frac{1}{\kappa}$ years.

If the rate of interest is denoted by i, the price level by p, and potential demand by E', the parameters of the price relationships shown in Fig. 12 are defined as :

$$a = \frac{\partial i}{\partial P} \qquad b = -\frac{\partial E'}{\partial i} \qquad c = \frac{\partial\left(\frac{dp}{dt}\right)}{\partial P}$$

$$h = \frac{\partial i}{\partial p} \qquad m = -\frac{\partial E'}{\partial p} \qquad n = \frac{\partial E'}{\partial\left(\frac{dp}{dt}\right)}$$

2. *Distributed Time Lags*

A single distributed time lag in the response of production to changes in demand is represented by the equation

$$DP = -\alpha(P - E) \quad . \quad . \quad . \quad . \quad (1)$$

from which

$$P = \frac{\alpha}{D + \alpha} E \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Similarly, the value of any variable which responds with a distributed time lag to changes in another is obtained by multiplying the value which that variable would have if there were no time lag by an operator of the form $\frac{\alpha}{D + \alpha}$, where $\frac{1}{\alpha}$ is the time constant of the lag. This rule will be used whenever a time lag is introduced.

If, from equilibrium, demand were to fall by one unit at time $t = 0$, we may substitute $E = -1$ when $t > 0$, giving

$$(D + \alpha)P = -\alpha \text{ when } t > 0 \quad . \quad . \quad . \quad (3)$$

with the initial condition

$$P = 0 \text{ when } t = 0 \quad . \quad . \quad . \quad . \quad (4)$$

since immediately after the change production will still be at its initial value.

The solution [1] of equations (3) and (4) is

$$P = -(1 - e^{-\alpha t})$$

which is plotted in Fig. 1 and Curve (*a*) of Fig. 2 for $\alpha = 4$.

If there were two consecutive time lags in the response of production to demand, the time constant of each being $\frac{1}{8}$ year, equation (2) would become

$$P = \left(\frac{8}{D + 8}\right)^2 E \quad . \quad . \quad . \quad . \quad (2a)$$

After the unit fall in demand, this gives

$$(D + 8)^2 P = -8^2 \text{ when } t > 0 \quad . \quad . \quad . \quad (3a)$$

with initial conditions

$$P = 0, DP = 0 \text{ when } t = 0 \quad . \quad . \quad . \quad (4a)$$

The solution is

$$P = -[1 - (1 + 8t)e^{-8t}]$$

which is plotted as Curve (*b*) of Fig. 2.

With three consecutive time lags each having a time constant

[1] For methods of solving ordinary linear differential equations with constant coefficients see Piaggio, *Differential Equations*, G. Bell & Sons, Ltd., London, 1949, Chapter 3. For equations of order higher than the second a considerable saving of labour is obtained by using the Laplace Transformation instead of the classical methods of solution. See Carslaw and Jaeger, *Operational Methods in Applied Mathematics*, Oxford University Press, Oxford, 1948, and Gardner and Barnes, *Transients in Linear Systems*, John Wiley & Sons, New York, 1942.

of $\frac{1}{12}$ year the equations for the response to unit fall in demand would be

$$(D + 12)^3P = -12^3 \text{ when } t > 0 \quad . \quad . \quad (3b)$$

$$P = 0,\ DP = 0,\ D^2P = 0 \text{ when } t = 0 \quad . \quad . \quad (4b)$$

The solution is

$$P = -[1 - (1 + 12t + 72t^2)e^{-12t}]$$

which is plotted as Curve (c) of Fig. 2.

3. *The Multiplier Model*

The response of demand to changes in production or real income is represented by the equation

$$E = (1 - l)P - u \quad . \quad . \quad . \quad . \quad (5)$$

u being the spontaneous change in demand occurring at time $t = 0$. Substituting equation (5) in equation (1) and rearranging, we obtain

$$(D + \alpha l)P = -\alpha u \quad . \quad . \quad . \quad . \quad . \quad (6)$$

After the initial change in demand this becomes

$$(D + \alpha l)P = -\alpha \text{ when } t > 0 \quad . \quad . \quad . \quad (7)$$

with the initial condition

$$P = 0 \text{ when } t = 0 \quad . \quad . \quad . \quad . \quad . \quad (8)$$

since immediately after the change production will still be at its initial equilibrium value.

The solution of equations (7) and (8) for $\alpha = 4$ and $l = 0{\cdot}25$ is

$$P = -4(1 - e^{-t})$$

which is plotted as Curves (a) of Figs. 3–9 inclusive.

4. *Proportional Stabilisation Policy*

The potential policy demand is $-f_pP$, giving an actual policy demand of $-\dfrac{\beta f_p}{D + \beta}P$. Adding this to the demand shown in equation (5) we obtain

$$E = (1 - l)P - \frac{\beta f_p}{D + \beta}P - u \quad . \quad . \quad . \quad (5a)$$

Substituting this expression in equation (1) and rearranging gives

$$[D^2 + (\alpha l + \beta)D + \alpha\beta(l + f_p)]P = (-\alpha D - \alpha\beta)u \quad (6a)$$

After the initial change in demand u equals unity and the derivatives of u are zero, so that

$$[D^2 + (\alpha l + \beta)D + \alpha\beta(l + f_p)]P = -\alpha\beta \text{ when } t > 0 \quad (7a)$$

the initial conditions being

$$P = 0,\ DP = -\alpha \text{ when } t = 0 \quad . \quad . \quad (8a)$$

The solutions of equations (7a) and (8a) for different values of f_p and β, α and l having the same values as before, are given in the following table :

f_p.	β.	P.	Curve.
0·5	2	$-1{\cdot}33 - 1{\cdot}69e^{-1\cdot5t} \sin (111^\circ\, t - 52^\circ)$	Fig. 3 (b)
2	2	$-0{\cdot}44 - 0{\cdot}95e^{-1\cdot5t} \sin (227^\circ\, t - 28^\circ)$	Fig. 3 (c)
0·5	8	$-1{\cdot}33 - 1{\cdot}69e^{-4\cdot5t} \sin (111^\circ\, t - 128^\circ)$	Fig. 4 (b)
2	8	$-0{\cdot}44 - 0{\cdot}52e^{-4\cdot5t} \sin (412^\circ\, t - 58^\circ)$	Fig. 4 (c)

5. *Integral Stabilisation Policy*

With an integral stabilisation policy the actual policy demand is $-\dfrac{\beta f_i}{D + \beta}\int P dt$. Adding this to equation (5), differentiating with respect to time, then substituting in equation (1) and re-arranging the terms, we obtain

$$[D^3 + (\alpha l + \beta)D^2 + \alpha\beta l D + \alpha\beta f_i]P = (-\alpha D^2 - \alpha\beta D)u \quad (6b)$$

After the initial change in demand this becomes

$$[D^3 + (\alpha l + \beta)D^2 + \alpha\beta l D + \alpha\beta f_i]P = 0 \text{ when } t > 0 \quad . \quad (7b)$$

with initial conditions [1]

$$P = 0,\ DP = -\alpha,\ D^2P = \alpha^2 l \text{ when } t = 0 \quad . \quad (8b)$$

[1] The initial conditions are not intuitively obvious when the equations are of the third or higher order. The following rules, derived from A. Porter, *An Introduction to Servomechanisms*, Methuen & Co., Ltd., London, 1950, pp. 50–3, to which the reader is referred for their justification, lay down a simple procedure for finding the initial conditions of the system in these cases.

(i) Integrate the equation with respect to time between the limits $t = -0$ and $t = +0$. Terms containing integrals of P and u between these limits are equal to zero and may be omitted.

(ii) Repeat the integrations until only the term in P remains. This gives the value of P when $t = 0$.

(iii) Substitute this value of P in the preceding equation, setting u equal to 1 and any derivatives of u equal to zero. This gives the value of DP when $t = 0$.

(iv) Repeat the substitutions until all the initial conditions have been obtained.

The solutions of equations (7b) and (8b) for different values of f_i and β are given in the table below :

f_i.	β.	P.	Curve.
0·5	2	$0{\cdot}37e^{-2\cdot80t} - 2{\cdot}55e^{-0\cdot10t}\sin(68{\cdot}3°\,t + 8{\cdot}3°)$	Fig. 5 (b)
2	2	$0{\cdot}33e^{-3\cdot65t} - 1{\cdot}34e^{0\cdot33t}\sin(118°\,t + 14°)$	Fig. 5 (c)
0·5	8	$0{\cdot}02e^{-8\cdot27t} - 2{\cdot}88e^{-0\cdot37t}\sin(77{\cdot}0°\,t + 0{\cdot}34°)$	Fig. 6 (b)
2	8	$0{\cdot}04e^{-8\cdot91t} - 1{\cdot}35e^{-0\cdot05t}\sin(154°\,t + 1{\cdot}8°)$	Fig. 6 (c)

6. *Proportional plus Integral Stabilisation Policy*

The actual policy demand in this case is $-\dfrac{\beta}{D+\beta}(f_pP + f_i\int Pdt)$. Following the same procedure as in the previous example, we obtain the equations

$$[D^3 + (\alpha l + \beta)D^2 + \alpha\beta(l + f_p)D + \alpha\beta f_i]P = 0 \quad \text{when } t > 0 \quad (7c)$$

$$P = 0,\ DP = -\alpha,\ D^2P = \alpha^2 l \quad \text{when } t = 0 \quad (8c)$$

Solutions for different values of f_p, f_i and β are :

f_p.	f_i.	f_d.	P.	Curve.
0·5	0·5	2	$-1{\cdot}33e^{-t} - 2{\cdot}67e^{-t}\sin(99{\cdot}4°\,t - 30°)$	Fig. 7 (b)
2	2	2	$-0{\cdot}27e^{-t} - 1{\cdot}07e^{-t}\sin(221°\,t - 15°)$	Fig. 7 (c)
8	8	2	$-0{\cdot}06e^{-t} - 0{\cdot}51e^{-t}\sin(455°\,t - 7°)$	Fig. 7 (d)
0·5	0·5	8	$-3{\cdot}11e^{-t} + (3{\cdot}11 + 5{\cdot}33t)e^{-4t}$	Fig. 8 (b)
0·5	2	8	$-0{\cdot}11e^{-6\cdot86t} - 1{\cdot}62e^{-1\cdot07t}\sin(164°\,t - 4°)$	Fig. 8 (c)
2	8	8	$-0{\cdot}22e^{-4\cdot93t} - 0{\cdot}70e^{-2\cdot03t}\sin(396°\,t - 18°)$	Fig. 8 (d)
8	32	8	$-0{\cdot}06e^{-4\cdot2t} - 0{\cdot}27e^{-2\cdot4t}\sin(880°\,t - 13°)$	Fig. 8 (e)

7. *Proportional plus Integral plus Derivative Stabilisation Policy*

The actual policy demand becomes

$$-\frac{\beta}{D+\beta}(f_pP + f_i\int Pdt + f_dDP)$$

Applying these rules to equation (7b) we obtain :

1st integration $D^2P + (\alpha l + \beta)DP + \alpha\beta lP = -\alpha Du - \alpha\beta u$ . . . (a)

2nd integration $DP + (\alpha l + \beta)P = -\alpha u$ . . . . (b)

3rd integration $P = 0$ . . . . (c)

Substituting equation (c) in (b) and setting u equal to unity,

$$DP = -\alpha \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (d)$$

Substituting (c) and (d) in (a) and setting u equal to unity and Du equal to zero,

$$D^2P - \alpha(\alpha l + \beta) = -\alpha\beta$$

or

$$D^2P = \alpha^2 l \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (e)$$

Equations (c), (d) and (e) give the initial conditions of the system.

Following the same procedure as before we obtain the equations

$$[D^3 + (\alpha l + \beta + \alpha\beta f_d)D^2 + \alpha\beta(l + f_p)D + \alpha\beta f_i]P = 0 \quad \text{when } t > 0 \quad (7d)$$

$$P = 0,\ DP = -\alpha,\ D^2P = \alpha^2(l + \beta f_d) \quad \text{when } t = 0 \quad (8d)$$

Solutions for various values of f_p, f_i, f_d and β are :

f_p.	f_i.	f_d.	β.	P.	Curve.
8	8	1	2	$-0{\cdot}08e^{-1\cdot18t} - 0{\cdot}68e^{-4\cdot91t}\sin(315°\,t - 6°)$	Fig. 7 (*e*)
8	32	0·25	8	$-0{\cdot}07e^{-5\cdot03t} - 0{\cdot}31e^{-5\cdot99t}\sin(741°\,t - 13°)$	Fig. 8 (*f*)

8. *The Multiplier–Accelerator Model*

The acceleration demand is $\frac{\kappa k}{D + \kappa} DP$. Adding this to the demand shown in equation (5) and following the usual procedure, the equations of the system are found to be

$$[D^2 + (\alpha l + \kappa - \alpha\kappa k)D + \alpha\kappa l]P = -\alpha\kappa \quad \text{when } t > 0 \quad (7e)$$

$$P = 0,\ DP = -\alpha \quad \text{when } t = 0 \quad (8e)$$

With $\alpha = 4$, $l = 0{\cdot}25$, $\kappa = 1$ and $k = 0{\cdot}6$ the solution, plotted as Curve (*a*) of Fig. 9, is

$$P = -4{\cdot}00 - 6{\cdot}32\, e^{0\cdot2t}\sin(56°\,t - 39°)$$

When a proportional stabilisation policy is applied the equations become

$$[D^3 + (\alpha l + \beta + \kappa - \alpha\kappa k)D^2 + (\alpha\beta l + \alpha\kappa l + \beta\kappa + \alpha\beta f_p - \alpha\beta\kappa k)D + \alpha\beta\kappa(l + f_p)]P = -\alpha\beta\kappa \quad \text{when } t > 0 \quad (7f)$$

$$P = 0,\ DP = -\alpha,\ D^2P = \alpha^2(l - \kappa k) \quad \text{when } t = 0 \quad (8f)$$

With $f_p = 2$, $\beta = 2$ and the other parameters retaining their previous values the solution, plotted as Curve (*b*) of Fig. 9, is

$$P = -0{\cdot}44 - 0{\cdot}025\, e^{-1\cdot15t} - 1{\cdot}10\, e^{-0\cdot225t}\sin(227°\,t - 25°)$$

With a proportional plus integral stabilisation policy the equations of the system are

$$[D^4 + (\alpha l + \beta + \kappa - \alpha\kappa k)D^3 + (\alpha\beta l + \alpha\kappa l + \beta\kappa + \alpha\beta f_p - \alpha\beta\kappa k)D^2 + (\alpha\beta\kappa l + \alpha\beta\kappa f_p + \alpha\beta f_i)D + \alpha\beta\kappa f_i]P = 0 \quad \text{when } t > 0 \quad (7g)$$

$$\left.\begin{array}{l} P = 0,\ DP = -\alpha,\ D^2P = \alpha^2(l - \kappa k) \\ D^3P = \alpha^2(\beta f_p + \kappa^2 k) - \alpha^3(l - \kappa k)^2 \end{array}\right\} \text{when } t = 0 \quad (8g)$$

With $f_i = 2$, the other parameters retaining their previous values, the solution, plotted as Curve (c) of Fig. 9, is

$$P = -0{\cdot}07\, e^{-1{\cdot}43t} - 0{\cdot}13\, e^{-0{\cdot}69t} - 1{\cdot}08\, e^{0{\cdot}26t} \sin (231^\circ\, t - 11^\circ)$$

When a proportional plus integral plus derivative stabilisation policy is applied the equations of the system become

$$[D^4 + (\alpha l + \beta + \kappa + \alpha\beta f_d - \alpha\kappa k)D^3 + (\alpha\beta l + \alpha\kappa l + \beta\kappa + \alpha\beta f_p + \alpha\beta\kappa f_d - \alpha\beta\kappa k)D^2 + (\alpha\beta\kappa l + \alpha\beta\kappa f_p + \alpha\beta f_i)D + \alpha\beta\kappa f_i]P = 0 \text{ when } t > 0 \quad (7h)$$

$$\left.\begin{aligned} &P = 0,\ DP = -\alpha,\ D^2P = \alpha^2(l + \beta f_d - \kappa k) \\ &D^3P = \alpha^2(\beta f_p + \kappa^2 k - \beta^2 f_d) - \alpha^3(l^2 - 2\kappa kl \\ &\qquad + \kappa^2 k^2 + 2\beta f_d l + \beta^2 f_d^2 - 2\beta\kappa f_d k) \end{aligned}\right\} \text{when t} = 0 \quad (8h)$$

With $f_d = 0{\cdot}55$, the other parameters retaining their previous values, the solution, plotted as Curve (d) of Fig. 9, is

$$P = -0{\cdot}11\, e^{-0{\cdot}74t} + 0{\cdot}07\, e^{-2{\cdot}17t} - 1{\cdot}40\, e^{-1{\cdot}55t} \sin (158^\circ\, t - 2^\circ)$$

With $f_p = 8$, $f_i = 8$, $f_d = 1{\cdot}3$ and the other parameters as before, the solution, plotted as Curve (e) of Fig. 9, is

$$P = -0{\cdot}02\, e^{-0{\cdot}87t} - 0{\cdot}04\, e^{-1{\cdot}44t} - 0{\cdot}73\, e^{-0{\cdot}48t} \sin (300^\circ\, t - 5^\circ)$$

9. *The Model with Flexible Prices and Interest Rates*

If the time constants of the lags L_1, L_2 and L_3 in Fig. 12 are assumed for simplicity to be the same as the time constant of the correction lag, the total demand resulting from the system (excluding the acceleration relationship) is

$$E = (1 - l)P - \frac{\beta}{D + \beta}\,[(ab - cn + f_p)P + (cm + chb + f_i)\int Pdt + f_d DP] - u$$

Substituting this expression in equation (1) and carrying out the same operations as in previous examples, we obtain the following equations of the system :

$$[D^3 + (\alpha l + \beta + \alpha\beta f_d)D^2 + \alpha\beta(l + ab - cn + f_p)D + \alpha\beta(cm + chb + f_i)]P = 0 \text{ when } t > 0 \quad (7i)$$

$$P = 0,\ DP = -\alpha,\ D^2P = \alpha^2(l + \beta f_d) \text{ when } t = 0 \quad (8i)$$

With $\alpha = 4$, $l = 0{\cdot}25$, $\beta = 2$, $m = 0{\cdot}05$, $ab = 0{\cdot}2$, $hb = 0{\cdot}25$ and f_p, f_i and f_d equal to zero, the solutions for various values of c and n are shown in the table below :

c.	n.	P.	Curve.
0	—	$-2{\cdot}22 - 2{\cdot}30e^{-1{\cdot}5t} \sin (66{\cdot}5^\circ\, t - 75{\cdot}5^\circ)$	Figs. 13 and 14 (*a*)
0·5	0·2	$-0{\cdot}38e^{-1{\cdot}83t} - 8{\cdot}00e^{-0{\cdot}59t} \sin (32{\cdot}0^\circ\, t - 2{\cdot}72^\circ)$	Fig. 13 (*b*)
1	0·2	$0{\cdot}36e^{-2{\cdot}59t} - 3{\cdot}36e^{-0{\cdot}21t} \sin (53{\cdot}9^\circ\, t + 6{\cdot}1^\circ)$	Fig. 13 (*c*)
2	0·2	$0{\cdot}39e^{-3{\cdot}32t} - 2{\cdot}25e^{0{\cdot}16t} \sin (68{\cdot}3^\circ\, t + 10{\cdot}0^\circ)$	Fig. 13 (*d*)
0·5	− 0·2	$-2{\cdot}46e^{-0{\cdot}34t} - 2{\cdot}7e^{-1{\cdot}33t} \sin (75{\cdot}2^\circ\, t - 64{\cdot}0^\circ)$	Fig. 14 (*b*)
1	− 0·2	$-2{\cdot}11e^{-0{\cdot}66t} - 2{\cdot}86e^{-1{\cdot}17t} \sin (86{\cdot}6^\circ\, t - 47{\cdot}6^\circ)$	Fig. 14 (*c*)
2	− 0·2	$-1{\cdot}05e^{-t} - 2{\cdot}31e^{-t} \sin (112^\circ\, t - 27{\cdot}2^\circ)$	Fig. 14 (*d*)

If for each set of values of c and n in the above table, f_p, f_i and f_d are given the corresponding values shown in the table on page 315, the solution of the system becomes

$$P = -0{\cdot}26\, e^{-1{\cdot}14t} - 1{\cdot}10\, e^{-1{\cdot}73t} \sin (207^\circ\, t - 13{\cdot}6^\circ),$$

which is plotted as Curve (*e*) of Fig. 13.

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