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STUDIES IN MACROECONOMIC THEORY

VOLUME 2
Redistribution and Growth

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REDISTRIBUTION AND GROWTH

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PREFACE

This volume collects most of my scholarly papers on the behavior and public control of distribution and growth in the market economy. Its principal subject is public finance; more broadly, the theory of economic policy. Stabilization questions, and hence the Keynesian side of fiscal policy, are the focus of my previous collection in this series.

It has been a rule in these volumes not to republish material from my previous books, so this volume does not contain my essay *Fiscal Neutrality toward Economic Growth* nor the papers in my *Golden Rules of Economic Growth*. By drawing on other expositions, though, including my first treatment of the Golden Rule, the collection manages to be substantially complete. The commentaries with which each group of papers is prefaced record my present position where it has moved away from my earlier opinions. The introductory essay previews and reworks some principal themes.

A basic view shared by these two collections is that every issue in public policy raises a problem of intergenerational social choice, yet such problems are not safely left as programming exercises in intergenerational utilitarianism; other criteria have to be sought and tested. This volume contains all the main chapters in a wide search for some better principle of social choice: the notion of fiscal neutrality toward national saving, the game-equilibrium approach to intergeneration inconsistency, and finally an arguably Rawlsian (though not Rawls's own) conception of intergeneration economic justice.

In beginning this project I worried that an audience incessantly excited by inflation and recession (and now energy shock) might find this second volume too tame compared to its monetary predecessor. When the neoclassical authors laid capital and value theory in a placid setting, ushering into economics a century of *wald und wiesen,* they were not writing for the modern sensibility. Nor has the economics of distribution and growth seen a paradigm shift like the rise of micro–macro theory in monetary economics.

In reviewing the papers in this volume, however, I found myself caught up again by their concerns. Below the still surface of fiscal and capital theory lie deep questions: the justice of balanced budgets, the efficiency of contemporary tax practice, the rationality of the market economy. And the recent rebellion against fiscal analysis lacking substitution effects—the miracle of the lump-sum tax—seems certain to be judged a major development, one which some papers here have helped along.

Much of this collection is the product of joint work. The collaborations more than a decade ago, with Manos Drandakis, Richard Nelson, and Karl Shell, arose accidentally out of associations with friends whose ideas I saw as rounding out my own additions to the positive theory of economic growth. Later, Janusz Ordover and John Riley played crucial roles in the difficult analysis of 'maximin' intergenerational justice, taking generous time away from their primary research interests to do it.

In studying economics these past three decades my debts of gratitude have piled too high to acknowledge more than a few of them, let alone to repay them. There is first my debt to James Nelson, dating from college days at Amherst. His demonstration that in economics one could be serious without being solemn encouraged me to pursue the subject.

I owe some of my early fascination with the theory of interest and growth to a graduate school where Irving Fisher was patron saint, and particularly to the brilliant instruction there by Tjalling Koopmans and James Tobin. A little later Paul Samuelson and Robert Solow were equally influential. To have found myself racing with them and the others for the next result in growth theory was the thrill of my youthful career.

Last there has been John Rawls, whose illuminating vision of a serious welfare economics drew me back to public finance and growth. This Rawlsian road is less traveled than the one I took earlier, but in the end that may have proved an advantage.

Let me acknowledge, in conclusion, the substantial material aid without which these collections would probably not have come into being. For several years I hardly put pencil to paper without assistance from the National Science Foundation, so most of my papers received its support, and its latest grants contributed to the preparation of these two volumes. A fellowship from the Guggenheim Foundation helped as well. There was a sabbatical from Columbia University during which I completed the first volume, and an invitation to the University of Mannheim where, in the peace of the nearby Odenwald, I finished the present volume.

INTRODUCTION: TAXATION, REDISTRIBUTION, AND GROWTH

It is only a quarter century ago that the theory of public finance was recaptured from Keynesian extremists. If ironing out business fluctuations is charged to the central bank instead of the Fisc, then fiscal policy—the choice of the tax structure and the algebraic budgetary surplus—can again be governed by neoclassical principles. Tax cuts that expand demands for current consumption and leisure divert non-idle resources from capital formation of sure use in the future.

The first halting steps in the rehabilitation of public finance theory were taken with the crutch of the fictitious lump-sum tax. In a series of papers, Paul Samuelson examined the economic behavior of a society taxing and transferring among its members in such a way as to maximize any social welfare function with the ordinal Bergson-Samuelson properties that every person's consumption of every good "counts" for social welfare but with diminishing marginal social-welfare weight. Such a well-ordered society, one not hung up in social conflict, would expunge Pareto-type inefficiencies, end unequal material endowments among like persons, provide collective goods without user charges, and possess social indifference curves—much as well-functioning families are said to do.

The optimum (social-welfare maximizing) distribution of taxes, though, and the consequent distribution of well-being somehow escaped attention.[1] The role of transfers and taxes in optimum redistribution was neglected as well in the rudimentary theory of optimal distortionary taxation begun by a few French and English economists in the 1950s. In those models the population of income earners is taken to be homogeneous, so no redistribution would contribute to social welfare; tax revenue is wanted for some other reason. In such a world, distortionary taxes really are "second best"—other than as devices to correct certain externalities (un-

[1] Years later James Mirrlees saw that if the social-welfare function being maximized is symmetrical, as when reflecting an even-handed ethical criterion plus identical tastes and capacities for enjoyment, then the strong will be taxed to a point at which they are worse off than the weak. (Such poetic-justice reversals cannot be engineered by non-lump-sum taxes on income, sales, etc.)

internalized by the market)—and the absence of the first best lump-sum tax is unexplained.

The engaging question for public finance theory in the era to which I am referring, roughly from Korea to Vietnam, was the total size of tax revenue to be collected in excess of the government's transfers, subsidies, and expenditure—the budgetary surplus (or deficit) to be chosen. The neoclassical or full-employment growth model built by Robert Solow and Trevor Swan was read as a message about budgetary policy: Stiffer taxation to reduce the fraction of national product spent for private consumption would, by allowing more investment, gradually raise the capital stock to a higher track and thus also the national income.

The discovery of the Golden Rule of Accumulation showed that each tax-induced constriction of the propensity to consume (expressed as a ratio to national income) will ultimately boost the level of national consumption, following its initial contraction, onto a higher track as well, if and only if the social rate of return to investment—the profit rate purified of imperfectly competitive elements—would otherwise remain indefinitely above the growth rate of the capital stock. On that golden condition, then, heavier taxation unaccompanied by greater public spending would eventually produce a higher standard of living for future generations. Of course, in that "long run" we are all dead. Yet the intergenerational utilitarianism of Frank Ramsey, which had become the prevailing philosophy of the time, mandated such a sacrifice on our part for the sake of our successors.

The fiscal requirements of an ascent to the Golden Rule state were made explicit in the growth model with overlapping generations of life-cycle worker–savers developed by Peter Diamond. Raising the capital-output ratio to its Golden Rule level, if we assume its steady-state tendency corresponding to the original fiscal policy was short of that level, would entail a sustained reduction of the public debt as a ratio to national output; this is so at least in the vicinity of the Golden Rule state where, in fact, debt "crowds out" capital dollar for dollar. The public debt was thus a "burden" if and insofar as its maintenance (as a ratio to output) by a tolerant fiscal policy left the capital-output ratio short of its Golden Rule level.[2]

It is impressive to see how much the theory of fiscal policy has

[2] If there was any contrary doctrine voiced in opposition to the prevailing wisdom it was the notion that taxation should seek merely to "neutralize" the false wealth represented by existing holdings of public debt plus the false wealth that would be added if the intertemporal plan for public expenditure were financed by further public debt creation. The claim was that if that fiscal stance of "neutrality" toward economic growth were adopted, the resulting volume of private saving and investment "determined in the market" would accurately and

changed complexion in one decade. The fiction of the lump-sum tax has been abandoned. While not an "impossible" levy, it is an inoptimal one because its imposition, by risking unfairness, would lower the expected value of social welfare; maybe the object of the levy will not be the breadwinner he seems to be. If people's respective wage rates in each activity, properly netted of equalizing differentials reflecting nonpecuniary differences among jobs, cannot be observed by the fiscal authorities, then collecting tax revenue requires taxing the observable and measurable things that people do—their earning, spending, saving, etc. What drove the lump-sum tax from economics, however, was the demonstration by James Mirrlees that modeling the heterogeneity of the workforce, without which the lump-sum tax would be unobjectionable, and optimizing the structure of necessarily "distortionary" taxes in such a model presented no insuperable analytical difficulties.

The recognition of distortionary taxes has implications of obvious importance. Even the structure of taxes that raises revenue "to capacity" does not indicate 100% tax rates since at such confiscatory rates the activities being taxed, and with them the tax base, would disappear. A stiffer tax rate intended to raise more revenue, to reduce the public debt, and to boost the capital stock might be counterproductive through a chilling effect on the activity being taxed or some other taxed activity.

The other change in public finance theory is more attitudinal than analytical. Among the general public the goal of material progress from generation to generation suffers dwindling support. Among economists, too, the long trudge to the Golden Rule state, where profit rate and growth rate have converged to a common constant that is presumably zero, no longer seems widely compelling—its utilitarian underpinning no longer axiomatic. The diminished enthusiasm for "growth" is another reason why interest has shifted from the welfare benefits of heavier taxation (for the sake of future people) to the welfare benefits that will accrue to all generations from selectively lighter taxation. It is nonetheless true, of course, that if we are to go beyond the mild objective of finding Pareto improvements we shall need some detailed conception of intergenera-

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properly express the intertemporal–intergenerational preferences of the present society as they actually exist.

Some valid arguments can be made on behalf of that position when applied purely to a Barro and Bailey world of dynastic families whose generations are consecutively interlinked by bequests. But those arguments all fall to the ground once we admit, in the spirit of the life-cycle model of saving, families that are constitutionally opposed to bequests or that would make negative bequests if legally enforceable. Then a policy of fiscal neutrality at best possesses strategic value in the intergeneration pension game.

tional as well as intragenerational justice in order to identify social-welfare gains from tax reforms.

This essay is a brief and informal introduction to the welfare economics of fiscal policy in its recently transformed state. It focuses on some classic questions that happen to have interested me and to which I have sometimes contributed. The first section addresses the effects of tax finance on the potential social welfare of future generations. The next section, on the just structure of taxation, takes up the taxation of capital and the question of progressivity. The last section discusses the determination of the shadow rate of interest and the shadow budget constraint in a setting of intergenerational justice.

I

Cicero said that man plants trees for future generations. Had he rather remarked that man levies taxes for future generations he would have founded fiscal theory (instead of capital theory). *Real politik* aside—after all, the next generation might repudiate the public debt if the present generation were to abuse its power to finance through debt issue—the present generation gains nothing from tax financing in whole or in part its government spending instead of borrowing. If there is a gain from tax financing it must accrue to the next generation.

The overlapping-generations model without bequests, extended to encompass heterogeneous workers able to vary continuously their supplies of effort and of private saving, leads to two propositions in this regard. If a tax measure taken by the government notionally belonging to the present generation of worker–savers over their two-period lifetime raises the potential social welfare of the next generation, all the spending activities of the government being held constant, the fiscal measure must also raise the present discounted value of the total tax revenues collected from the present generation over its two-period lifetime, thus reducing the final budgetary deficit and public debt, *when the discounting is done at the original social rate of return to investment* (as distinct from the *ex post* rate of return).[3] The converse is not true, however, and the second proposition differs from it: If a tax measure produces a rise of the final budgetary surplus and hence a decline of the public debt, when these changes are calculated on the basis of the original social rate of return, *and* if any resulting increase of the public debt or decrease of the capital stock with which the next generation is confronted in sufficiently small (small enough to save the next generation from the workings of diminishing returns),

[3] The implicit or notional final public debt here is equal at par to the *negative* of the following: the new capital stock plus its social rent, the latter figured at the original social rate of return, minus the new claim to retirement consumption.

then there results an unambiguous increase in the potential social welfare of the next generation. This point can be illustrated with the aid of Figure 1, where it can be seen that a rightward and parallel movement of the straight line passing through the initial point does not guarantee that the new point will lie on a better social-welfare contour for the next generation.

The key to this result is to see that the next generation is the recipient of a certain producer's surplus: In the illustrative one-product case, it is the excess of the final output, $F(K, \ldots)$, that the next generation optimally chooses to produce with the capital bestowed on it by the present generation, K, minus the consumption claim against that output which must (in justice) be paid to the present generation in its retirement, X. (Of course the next generation's own bestowal of capital to its successor is another minus, just as its own consumption claim in retirement is a plus; however we are free to treat these further terms as preoptimized constants

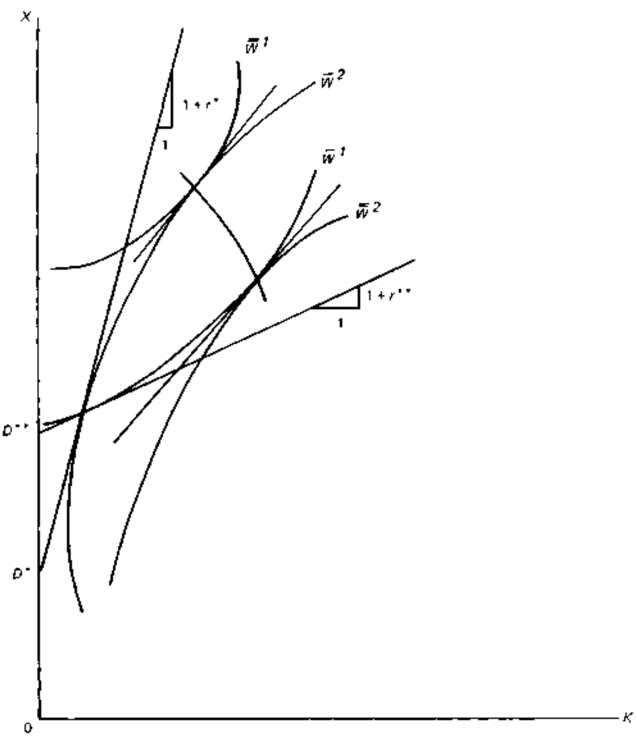

Figure 1. This diagram indicates the contours of constant potential social welfare and associated concepts.

for present purposes.) Any tax measure that increases K and decreases X, as does the reliable lump-sum tax, clearly increases this surplus as long as $F(\cdot)$ is increasing in K at the optimal levels of the labor inputs; and under competitive conditions at any rate, where the public debt left by the present generation is the excess of its retirement consumption over the principal plus competitive interest on the capital it leaves, $X - KF_K(K, \ldots)$, any tax-rate measure that decreases X or increases K must at the same time reduce the (aforementioned) notional public debt—the debt calculated from $X - KF_K(K^0, \ldots)$, where the original social rate of return corresponding to the *ex ante* capital stock, K^0, serves as the implicit interest rate. More generally, any tax measure (we may think here of a tax-rate increase) that while decreasing K causes X to decrease at a rate per unit of K that is large enough to satisfy the derivative condition $dX/dK > F_K(K, \ldots)$ serves to increase the next generation's producer's surplus and at the same time reduces the notional public debt—that is, makes negative the sum $dX - dKF_K(K, \ldots)$, which is the change of the debt figured at an interest rate equal to the original social rate of return. The same is true of a tax measure that while increasing X causes K to increase sufficiently more to satisfy the same inequality condition.

Suppose, for example, that the government associable with the present generation of worker–savers over their two-period lifetime, having already determined somehow upon certain benefits to be paid in the form of social-security type transfer payments and employment subsidies to low-paid workers (as well as any public expenditures on goods and services), introduces a tax rate on wage earnings—no other taxes being levied. It is obvious that there is a huge range over which the tax rate will raise a positive volume of tax revenue (at the end of which the tax rate is too high to support taxable economic activity). It is also clear that if the tax rate happens, in the balance between income and substitution effect, to leave the sum total of weighted labor supplies unchanged, each supply weighted according to its marginal productivity, then, provided both first-period and second-period consumptions are normal goods, households will decrease their consumption in both periods, hence increasing K and decreasing X. (The deducible rise of K implies that private saving falls by less than the rise of public saving.) If in fact the total weighted labor supply is increased by the tax rate and associated fall in living standards, the aforementioned results will be reinforced. If instead the labor supply is diminished, this reduction multiplied by the wedge between the marginal productivity of weighted labor and the after-tax wage rate of weighted labor gives the reduction of K resulting on this account if X should happen to be constant; there is in general a secondary effect on the producer's surplus that attenuates and could offset the constant-labor effect. But the aforementioned wedge and thus the whole secondary effect of the induced reduction of labor supply can be made arbitrarily small by the choice of a sufficiently small tax rate. For a "small" tax rate on wage income at least, therefore, there is an unambiguous increase in the next generation's potential social welfare through the net-resultant fall of retirement-age consumption claims and rise (or at least not offsetting fall) of capital.

Similarly, the introduction by the present generation of a small tax rate on the interest (or wealth) received in the retirement period, all other taxes absent, must surely generate tax revenue (as long as it does not quench the economic activity being taxed, namely private saving). If the tax rate happens to leave the supply of private saving unchanged, then households will after have less after-tax income to consume in retirement, hence will saddle the next generation with less X and no decrease of K. If instead the supply of saving is diminished, K will be decreased with attendant ambiguities for the next generation's producer's surplus. Yet the net gain for the next generation's potential social welfare is unambiguous if the tax rate introduced, while positive, is small enough.

By not taxing at all, therefore, despite its relentless government spending, the present generation would impose on the next generation a larger public debt for it to redeem, at the expense of the latter's consumption, and (by failing to tax wages at all) a smaller capital stock to work with, at the expense of the latter generation's production.

The main conclusion, however, is the more general one, which is worth restating a little more precisely. Corresponding to any provisional pair of terminal conditions (K, X) determined explicitly or implicitly by the present generation is some social rate of return to investment—denote it r^*—that is given by $1 + r^* = F_K(K, \ldots)$, where the labor inputs affecting the marginal productivity of capital depend of course upon both K and X. Whenever a tax measure can be taken, or a package of measures can be taken in concert, on a small enough scale to engineer arbitrarily small changes of K and X which satisfy the condition $(1 + r^*)dK - dX > 0$—which means that discounted tax revenue and the final budgetary surplus would be increased if they were figured at the original social rate of return—then taking such measures would increase the potential social welfare of the next generation.

If the minister of the Fisc instructs his technicians to pretend that private savers and the government can borrow or lend at the original social rate of return when calculating the budgetary implications of any new tax-rate proposal, it is not without reason! An estimated rise of the *actual* public debt as a result of the fiscal proposal, unlike a rise of the notional public debt calculated from the original social rate of return, would not be decisive evidence that the proposal would diminish the potential social welfare of the next generation. The point is easily grasped with the help of the following observation. Any tax measure or package of tax measures that increases the capital stock bestowed on the next generation while leaving the actual (algebraic) budgetary deficit no worse—the "actual" deficit, if the interest rate at which everyone can borrow or lend is continuously equal to the social rate of return to investment at any rate—must also increase the next generation's producer's surplus in precisely the same way that, say, any increase in the supply of labor, after being paid its consequently diminished marginal product, leaves a larger producer's surplus in the form of competitive rents to the owners of the fixed factor, such as land or capital. In other words, when the present generation permits itself the retirement award given by the fixed-deficit relation, $X = KF_K(K, \ldots) +$ constant, any fiscal act on its part causing K to increase runs into diminishing marginal productivity of capital; but the next generation is made the beneficiary of the diminishing returns because—by the factor-price relation—the marginal productivity of its labor is thereby increased. There is some room, therefore, for some increase of the budgetary deficit without thus depriving the next generation of some increase of its potential social welfare.

Hence "tax reform" in the shape of reduced taxation of certain economic activities may very well raise the potential social welfare of the next generation even if it widens a little the ultimate budgetary deficit. But if the reform increases the budgetary deficit calculated from the original social rate of return, so that the present generation overcompensates itself for diminishing returns, then the next generation must suffer a loss of potential social welfare. Such a calculation signals that the extra capital stock, even if remunerated at the undiminished marginal productivity of the initial stock, would be insufficient to pay for the extra retirement-period consumption claim induced by the tax reform; so if the next generation devoted the whole increment in the marginal productivity of its labor to supplement the retirement-age consumption of the old, thus offsetting the diminishing marginal productivity of capital encountered, the extra retirement-consumption claim would still not be entirely met.

II

The interests of the present generation in building capital for future production and in having wealth for its later enjoyment are quite opposite to those of its successor. Other things equal, including the old-age consumption claims it will be able to exercise against the output of the next generation, the present generation will suffer a loss of its potential social welfare with each increase of the capital stock, K, that it decides (implicitly or explicitly) to make available to the next generation of workers; transfer payments and subsidies will have to be lighter or taxes heavier (if possible) in order to contract first-period consumption or expand the supplies of labor or both (thus to increase the supply of public plus private saving), and there is no way to spare social welfare if it was being maximized before. The present generation will have a gain of potential welfare with each increase of its total wealth claims cum interest, X. Of course these relationships are confined to capital-wealth pairs, (K, X), that are technologically feasible for the next generation.

Consider now any chosen endpoint for capital and wealth, and denote it by (K^{**}, X^{**}). It may very well be not the product of explicit public choice so much as the destiny implicit in prevailing fiscal practice prior to some scientific appraisal, or if a target it may be only a provisional one, or merely a hypothetical endpoint for purposes of our analysis. The double star serves as a reminder that the endpoint under consideration may be only "second best" for the potential social welfare of the present generation, even when care is taken to protect the potential welfare levels of succeeding generations.

There is an iso-potential-welfare locus passing through (K^{**}, X^{**}), just as there is a potential-welfare contour passing through any other endpoint we might specify, and we may take such loci to be continuously differentiable without serious loss of generality. The slope of this locus at (K^{**}, X^{**}), a pure number giving the amount of extra X that would be needed to compensate for having to produce an extra unit of K, is sometimes called the social rate of discount; that number minus 1 may be called the *shadow rate of interest* and be denoted by r^{**}. There also corresponds to (K^{**}, X^{**}) a vector of shadow wage rates which simply give the associated marginal productivities of each sort of work at the welfare suboptimum.

Rather than think of the present generation's optimization as one subject to two quantitative constraints, $K \geq K^{**}$ and $X \leq X^{**}$, we are free to conceive of the optimization as if it had been subject instead only to a single linear restraint that happened (or was cleverly chosen) to yield the same solution for the optimum tax structure as that emerging from the true problem. In other words, there is associated with the target point (K^{**}, X^{**}) and the corresponding optimum tax structure a *supporting hyperplane* the equation of which is $X - (1 + r^{**})K = D^{**}$ where r^{**} and D^{**} are constants, the latter constant term being determined so as to make the hyperplane go through (K^{**}, X^{**}); and if the present generation in its optimization pretends it can choose any (K, X) on the hyperplane (or worse), it will be led as if by an invisible hand to choose (K^{**}, X^{**}) and hence choose the identical tax structure as it chooses when confined to the point (K^{**}, X^{**}). Because D^{**} is the difference between the old-age consumption permitted the present generation and the "principal plus shadow interest" on the capital accumulated, it might be called the shadow public debt (at par value). Of course, the public debt actually marketed, including the indebtedness represented by the social-security payments promised in retirement, is equal to $X - (1 + r)K$ where r is the market rate of interest.

With these shadow concepts in hand we can analyze the optimization of the tax structure of the present generation (for a social-welfare maximum) in its canonical, linearly constrained version—not worrying yet about how (K^{**}, X^{**}) is to be chosen in view of the conflicting interests of the two adjacent generations. No matter what constellation of shadow interest rate, vector of shadow wage rates, and shadow debt limit (or government shadow budget constraint) is finally settled on, the able technicians of the Fisc can work out the computer program that will at the push of a button calculate for any such constellation of shadow parameters the corresponding structure of after-tax rates of reward to working and saving that maximize the specified objective function—whether this be a traditional social welfare function of the Bergson-Samuelson type, $W(u^1(c_1^1, \ldots), \ldots, u^n(c_1^n, \ldots))$ in obvious notation, or something in the spirit of Rawls's 'maximin.' By a series of iterations it will be possible to ensure that the volume of saving calculated at the Fisc is consistent with the target, (K^{**}, X^{**}).[4]

In view of the globalness of the information needed to calculate the optimum, it is a relief to see how much can be said about the optimal structure of taxation while knowing little or nothing about people's demand functions, their well-being, and even the social welfare function other than that it is of the Paretian inefficiency-abhoring kind (as in Bentham, Nietzsche, and Rawls). One result of striking generality has to do with the after-tax rate of return to saving that it is optimal to offer on the private (nongovernmental) saving that members of the present generation may choose to do. Another result pertains to the optimal graduated taxation of wage income around the top income bracket.

The question of a return to saving may be examined on the preliminary assumption that all after-tax wages are to be in some proportion to the shadow wages, and likewise after-tax interest is to be some proportion of shadow interest—the two shadow-tax rates to be chosen optimally. A

[4] Perhaps it should be stipulated that such a calculation is not literally practicable in any real-life society, owing not only to the immense detail of the required information—every type of person's demand functions and, under some welfare functions, every type of person's well-being—but also to the need for this information at virtually every possible trial solution.

useful way to approach the question is to ask what there might be inoptimal about placing all the responsibility on the wage tax to meet the shadow budget constraint while setting social-security transfer payments and the rest at their optimum levels, thus letting savers receive the whole shadow rate of interest.

The analysis of that question goes beautifully in a simple case: If all persons saved the same proportion of their wages, no one would gain at the expense of the others from the introduction of a small shadow-interest tax in order to increase after-tax wages; there is either a general gain or a general loss even if tastes are different (provided that everyone saved equiproportionately). There is a general *loss* if and only if the proportionate increase of the after-tax wage rates per unit of decrease of the after-tax rate of return to saving that is required to meet exactly the shadow budget constraint (i.e., to keep total shadow-tax revenue constant) is *less* than the common ratio of personal saving to wages, since for each person the required increase of the after-tax wage rate needed to *compensate* him for the reduction of the after-tax rate of return on his saving is that person's ratio of saving to wage income. The former will indeed be less than the latter if and only if the latter *compensating* substitution of a little interest taxation for some wage taxation would have a *negative indirect effect* on shadow-tax revenue—particularly wage-tax revenue since that was the only revenue at stake—since the increase of the after-tax wage whose *direct effect* on revenue would exactly offset the direct effect on revenue of the decrease in the after-tax return to saving is precisely equal to the compensating increase. The net indirect effect is negative if and only if the reduction of the rate of return induces a fall of labor supply (which is the only activity that was generating revenue) that more than offsets the rise of the labor supply induced by the compensating increase of the after-tax wage rate.[5]

Some rather similar results emerge if, in the spirit of John Rawls's 'maximin' criterion, the social-welfare maximum entails maximizing total (shadow) tax revenue so that the government's economic aid to the least well-off persons or least well-rewarded workers can be as large as possible

[5] The advanced student of the subject will see that looking only at the indirect effect of wage-tax revenue of a substitution of some interest-tax revenue for some wage-tax revenue in the neighborhood of the *zero* tax rate on interest income is not necessarily decisive; such a local test may point us toward a relative maximum of the social welfare function that is in the wrong direction from the absolute maximum. A correct statement is that if in the neighborhood of the optimum the indirect effect on wage-tax revenue of a larger interest tax or lesser interest subsidy that is compensated by a lessening of the wage tax rate should be negative, then interest is optimally subsidized, and only the drain on revenue of further interest subsidization deters further subsidy.

(without exceeding the budget constraint). If a rise of the after-tax rate of return to saving would fail to have the Hicks-Lucas effect of stimulating labor supply, then interest-type income will not be optimally exempt from taxation. For optimality of a zero or negative tax on interest income the Hicks-Lucas effect must overcome the downward effect on labor supply of the after-tax wage rate decline whose direct effect on revenue (labor constant) would just offset the direct effect on revenue (saving constant) of the higher after-tax rate of return. So, again, it is the indirect effect on wage-tax revenue through induced labor-supply changes of a certain kind of tax substitution that is crucial for the tax treatment of interest. The indirect revenue effect of an interest-tax substitution will be positive, indicating the optimality of an after-tax return to saving below the shadow rate of interest, if and only if working-age consumption is more complementary to effort (more substitutable for working-age leisure) than retirement-age consumption is—as if the former were a needed fuel.[6]

The other great question about optimum tax structure is the progressivity of wage-income taxation. We were told when children that every earner of wage income ought to pay in tax an ever larger share of each successive dollar earned. However that shibboleth can only get in the way of maximizing social welfare functions of the Paretian (envy-free) kind. To tax at all the highest earner's last dollar of earnings is to impose a spiteful penalty on that earner at no gain, at even a loss, of tax revenue. It is only the earners below the top whose last dollars should be shared with the Fisc lest bigger earners receive a profligate tax windfall on some of their infra-marginal earnings. Of course, no lovely theorem such as that is without attacks against its premises.

III

To determine, or at any rate to confine and characterize in certain respects, the endpoint to be chosen, (K^{**}, X^{**}), with its associated shadow prices and shadow budget constraint, we need to specify something about intergenerational relations, particularly the prevailing conception of justice between generations or the lack of such. Here I shall suppose, for the sake of discussion at any rate, that conditions of likemindedness and trust exist between generations to such an extent that the present generation will seek every opportunity for a mutual gain of social welfare and will always apportion the gains between the two generations in a way regarded by them as just.

[6] Of course, a zero tax on shadow interest implies that if the market rate of interest is greater (less) than the shadow rate, a tax (subsidy) on market interest is required to get the after-tax rate of return to saving down (up) to the shadow rate of interest.

There is an opportunity for a mutual welfare gain at any (K, X) point where the terms on which the present generation would rationally be willing to supply more capital differ from the terms the next generation would rationally be willing to meet. Where the increase of X that the present generation must charge for supplying one extra unit of K in order to have an unchanged potential social welfare—this increase less 1 is the shadow rate of interest, r^{**}—is less than the increase of X that the next generation could pay without gain or loss to its potential social welfare—this increase is the social rate of return to investment, r^*—there is room for a mutual gain through increased K and increased X. As K is steadily increased, with X always increasing at a rate per unit of K greater than r^{**} and smaller than r^*, so that both generations are gaining, the social rate of return to capital must steadily be falling owing ultimately to the diminishing marginal productivity of capital and the shadow rate of interest must be rising owing (loosely put) to the diminishing marginal utility of retirement-age consumption. Where r^{**} and r^* finally meet, as they must, we have one (K, X) point on the Edgeworth-Pareto efficient-contract locus from which no departure of (K, X) can produce a mutual gain of potential social welfare.

A fundamental task in the application of public finance theory to economic policy, therefore, is to find ways to reach the Edgeworth-Pareto locus for a mutual welfare gain. From that unambiguously improved position it remains to make such additional changes in the budgetary surplus and tax rates as to satisfy, or to reinstate if it existed before and was lost in the move to the locus, the prevailing notion of justice between the generations. Given that the tax structure is chosen to reach the efficiency locus and to keep the economy on the locus in the face of redistributional moves, the role of the budgetary surplus is to lever the economy up or down the locus in order to provide the desired distribution of potential welfare between the generations.

What is perhaps most interesting about "tax reform" from a theoretical standpoint is the possibility that removing those inoptimal fiscal features of the existing system that inhibit the accumulation of capital and wealth may require that the present generation resist any instinct to offset any resulting loss of tax revenue—removing the over-taxation of interest-type income is a probable case in point—or take positive steps to offset any gain of tax revenue—here the elimination of over-progressivity serves as an example—lest it suffer a net loss of social welfare owing to the diminishing returns encountered by the increased investment induced by the fiscal reform. Revision of the tax structure without an adjustment of the budgetary deficit risks promoting the social welfare of one generation at the expense of the other.

It would be a limited kind of justice, moreover, that was content with saving either generation from harm. How much more natural for the generations to agree upon equality in the division of the benefit from their economic cooperation. Since the generation with the least social welfare can always pull itself up to the level of the next-to-least generation, the intergenerational equalization of social welfare is an obvious feature of maximum-optimal growth in the overlapping-generations model (with or without heterogeneity of the work force).

The dynamic programming of capital and wealth for maximin-optimal growth from generation to generation has been shown to have an interesting implication for the path of the public debt under any initial conditions making capital and wealth smaller than the respective stationary-state levels required for intergeneration efficiency (marked by intersection of the "stationariness locus" with the appropriate efficiency locus). The simple geometry of maximin-optimal growth makes this result transparent in the case where the supply of labor in each generation is institutionally fixed or perchance invariant to the perturbations in the case at hand: Namely, when the present generation acts to increase capital in the interest of maximin-optimal growth—as, for example, by lifting the excessive taxation of interest income—it must award itself a dose of deficit finance that serves to redistribute by an equalizing amount the loss to the present generation from the diminished marginal productivity of capital and the gain to the next generation (and its successors) of the enlarged marginal productivity of labor consequent upon the present generation's increased investment.

By way of conclusion, having said all of the aforementioned, I must admit that these findings, insofar as they have immediate applicability, are applicable only to the world as a whole and only when acting in a cooperative spirit. The national economics of public finance in one country amidst other nations goes quite a bit differently as long as international fiscal coordination is lacking. Until the tax loophole of international migration of capital and labor is effectively closed, a single country cannot unilaterally achieve intergenerational equality of social welfare within its own borders, taxing future generations when they have natural advantages for the benefit of the present generation, for much the same reasons that the poor cannot achieve equality with the rich within generations. International fiscal economics is a new field, only the surface of which has been scratched as yet. The difficulty is to model the world as one standing between perfect mobility, which would be a world incapable of justice in the absence of international cooperation, and perfect immobility. As the garrulous Cicero might have said, man leaves unsolved problems for future generations.

Acknowledgments

Much of the research reported here is work I carried out in collaboration with Janusz Ordover, John Riley and Karl Shell. Any misrepresentations or overextensions of that joint work as may be found here are solely my responsibility. The scant references to a few seminal papers in the text and bibliography are not meant to do justice to the many important contributors to the subject. Portions of this essay were drawn upon in a symposium on just taxation at the 1979 annual meeting of the Eastern Division of the American Philosophical Association held in New York City.

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INTRODUCTION

What makes economics exciting, if anything does, is the possibility that it will lead to action. (On that theory, art is relaxing because there is no risk it will prompt us to act.) When ground is broken in a new field, though, the action in mind is not often apparent; otherwise the digging would have started sooner. Each groundbreaker seems to work from some hidden agenda, as though trying not to attract competitors to the site.

The rise of neoclassical growth economics in the 1950s and 1960s is a case in point. As a Cezanne is not about actual fruit and furniture—it is about space—the early growth model is not about any actual evolution of capital and output; its real significance lies in policy space. While the model illuminated a little the postwar recovery of productivity, a model that omitted all sources of technical progress and all sources of market maladjustment to economic disturbances could hardly have been meant primarily to explain any country's history of economic growth or its prospect for growth in the future.

The particular subject of the early growth literature, however much hidden, is the scope of budgetary policy to alter the course of the nation's productive capacity and (not the same thing) its standard of living. A key parameter of the model, the saving-output ratio, is not meant as a behavioral constant. The famous ratio is meant rather as a control variable, adjustable through tax collections for public saving at any time in the present and future. The first major result of the theory, the existence of a Natural Rate of growth (so the steady-state growth rate is independent of the saving-output ratio) is not intended to suggest that every or any society's growth must settle to a steady rate; such a future is hardly likely. The suggestion is rather that new fiscal austerity would not add indefinitely to, nor new fiscal laxity subtract indefinitely from, the growth rate of productivity. Only the level of productivity, and generally the levels of the other variables, would be altered in the limit.

The proposition that I dubbed the Golden Rule of capital accumulation constitutes the second law of growth theory, after the logically prior law of the Natural Rate. Its message, too, was intended for fiscal policy making: There comes a point after which another notch of austere taxation, as measured by the consequent saving-output ratio, would not even begin to repay with an eventually greater rate of consumption; it would always be more jam tomorrow and never more jam today.

The golden proposition appeared in 1961 as a challenge to the near mania that had been building over several years for faster economic growth: The prodigous saving-output ratios observed in some countries, ratios as high as a third in Russia and Japan, were being held up as a standard to be emulated by nations, such as the United States, that had grown soft in their economic maturity. Thanks in part to the sobering effect of the Golden Rule, however, the ordinariness of American and European public thrift ultimately regained respectability. Calls for heavier taxation to curtail consumption and promote investment went out of vogue. High rates of saving, where they persisted, raised suspicions of a capital-retentive personality.

For me at least, the Golden Rule was significant primarily as a warning against fiscal miserliness, against unrequited saving, not as a prescriptive rule of saving. The gentle satire in the "Growth Fable" (as when the Solovians clamor for an impetuous dash to the Golden Rule state), my reply a year later to the comment by I. F. Pearce, and my "Second Essay" in 1965 all testify to that. It was my impression that the social rate of return to investment was already perilously near to the Natural Rate (i.e., to its Golden Rule level inasmuch as the observed profit rate was a return to "good will" as well as to tangible capital). I also had a vague sense of unease, of which I was not at first conscious, with the utilitarians' idea that generations must forever make sacrifices for their successors unless and until it does no good.[1] The quest for a less unsatisfactory theory of optimum national saving is the theme of the papers in part III.

The public-finance content of growth economics became explicit in the further development of the neoclassical growth model by Peter Diamond in order to study some prevailing questions about the burden of the public debt. The departure there was a utility–theoretic treatment of private saving in place of the behavioral postulate of a Keynesian consumption function by Solow and Swan. Diamond's assumption that all

[1] The relation of the Golden Rule state to intergenerational utilitarianism, especially the *per capita* version developed by T. C. Koopmans and C. C. vonWeizsäcker, is the subject of a chapter, "The Ramsey Problem and the Golden Rule," in my book *Golden Rules of Economic Growth* (New York: W. W. Norton and Co., 1966).

private saving is for "life-cycle" purposes has been a fateful one for nearly all succeeding papers in the same line of inquiry. The case for this extreme assumption rests on the hope that the life-cycle motive for saving pushes the real rate of interest low enough to wipe out the bequest motive and thus, if negative bequests are impossible, remove it from influence over the capital intensiveness to which the economy tends.

My paper with Karl Shell adopts a model in the same vein with which to reexamine the relation between capital and public debt from a different perspective—that of a national fiscal planner. Instead of asking what level of capital per worker would go with the public debt per worker that a permanently given deficit per worker would asymptotically produce, we ask what level of public debt per worker would have to be maintained through the appropriate deficit per worker in order to support the continuation of a given capital stock per worker. Does the fiscal engineering of greater capital intensiveness generally necessitate smaller public indebtedness? A higher tax rate? It is shown that in the neighborhood of the Golden Rule steady state, debt displaces capital dollar for dollar. Yet there is an unclassical range over which higher capital does not require lower debt at all. A temporary tax increase and debt reduction may trigger an unstable rise of capital requiring an ultimate increase of the debt in order to limit the capital deepening. (Of course, the Correspondence Principle of Samuelson, invoked by Diamond to exclude the unclassical possibility, is inapplicable when an "unstable" solution is being actively sustained by appropriate feedback public policies.) Perhaps it should be noted in passing, by way of a link to the next paper in this part and those in Part IV, that in economies with two or more distortionary taxes in use the steady-state relationship between capital and public debt would shift with each change in the mix of taxes producing a given deficit.

The last paper in this part, the first of two on interest-income taxation that I coauthored with Janusz Ordover, postulates separate proportional taxes on wage and interest income—the linear case. By suitable variations of the two tax rates, a larger welfare transfer can be paid to each member of the present generation of worker–savers without depriving the next generation of the capital-worker ratio to which the society has grown accustomed and without burdening the next generation with an increased quantity of public debt per worker: There exists some pair of tax rates on wage income and on future interest income that will satisfy the two constraints, *unique* if the transfer is large, for every level of the transfer—up to a point. The paper focuses upon the supply-elasticity conditions that must hold, and which characterize (to some degree) the associated tax rates, when the transfer is at its constrained maximum.

Ordover and I did not notice that as the transfer is increased, whether to its maximum or short of that, the necessary tax rate on interest does not rise—only the wage tax rate must rise, assuming it was set as low as possible to start with—*if* the fall of the after-tax wage contracts present-period consumption demand in the same ratio to future-period consumption demand that the increased transfer expands it; for in that case the adjustment of the wage tax that serves to protect capital formation (public plus private saving) also goes to insulate private saving and thus holds invariant the public debt as well. That result will not occur, and in particular the tax rate on interest must rise (fall) if the fall of the after-tax wage contracts present consumption in smaller (larger) ratio to future consumption than a reduced transfer would have done. It is a matter of cross-substitution effects, at the individual household level, and it thus brings to mind the stress by Corlett and Hague years ago upon the pivotal importance for taxation of the comparative net complementarities of consumer goods with household leisure.

We did not notice that property of the model, I suspect, because we had adopted (to begin with) a model in which capital and debt were each to be maintained at arbitrarily given levels, as by history, and in such a case the tax rate on interest might be non-zero (and one-signed) at all levels of the transfer payment—including the maximum level. The analysis carried out in the paper is enough to establish that, but it did not have the forceful clarity needed to prevent the profession from repeatedly slipping into error on the point. Ordover and I were finally driven to return to the matter in our 1979 paper. Bidding fond farewell to the world of steady-growth states, that paper provides the suitably dynamic setting that a proper tax analysis always needed.

THE GOLDEN RULE OF ACCUMULATION: A FABLE FOR GROWTHMEN

Once upon a time the Kingdom of Solovia was gripped by a great debate. "This is a growing economy but it can grow faster," many argued. "Sustainable growth is best," came the reply, "and that can come only from natural forces."

A few called the debate growthmanship. But most thought it would be healthy if it led to a better understanding of Solovian growth. So the King appointed a task force to learn the facts of Solovian economic life.

The committee reported that the labor force and population in Solovia grew exponentially at the rate γ. The number of working Solovians, N_t, at time t was therefore given by

$$(1) \qquad N_t = N_0 e^{\gamma t}, \qquad \gamma > 0.$$

The report expressed confidence that Solovia's supply of natural resources would remain adequate. It portrayed a competitive economy making full and efficient use of its only scarce factors, labor and capital, in the production of a single, all-satisfying commodity. Returns to scale were observed to be constant, and capital and labor were found to be so substitutable that fears of technological unemployment were dismissed.

The committee described the steady progress in Solovia's ways of production. It estimated that the efficiency of Solovian capital was increasing at the rate λ and that Solovian labor was improving at the rate μ. A continuation of these rates of technical advance was anticipated. Therefore production, P_t, at time t, was the following function of available capital, K_t, and the current labor force:

$$(2) \qquad P_t = F\,(e^{\lambda t} K_t\,, e^{\mu t} N_t), \qquad \lambda \geq 0,\ \mu \geq 0.$$

The report acknowledged further investigation of the production function might prove to be desirable.

Then the task force approached the growth issue. It doubted that technological advance could be accelerated and it took no positive stand on population increase. If γ, λ and μ were fixed parameters, then hope had to rest entirely on investment. While maintenance of the existing ratio of capital to labor would permit output per worker and per head to grow by virtue of technical progress, the report voiced the hope that higher incomes and perhaps a greater growth rate would be sought through a continuous increase in capital per worker, or what the task force called capital-deepening. It concluded by declaring the proper pace of capital-deepening to be a momentous question for Solovian political economy.

The King commended the task force for its informative and stimulating report. He invited all his subjects to join in search of an optimal investment policy. Solovian theorists considered dozens of fiscal devices for their effi-

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ciency, equity and effectiveness. Mathematicians, leading the quest for a growth strategy, grappled with extremals, functionals and Hamiltonians. Yet nothing practicable emerged.

Then a policy-maker was heard to say, "Forget grand optimality. Solovians are a simple people. We need a simple policy. Let us require that the fraction of output accumulated be fixed for all time, that is:

$$\frac{dK_t}{dt} = sP_t, \text{ for all } t, 0 \leq s \leq 1. \tag{3}$$

If we make investment a constant proportion of output, our search for the idea investment policy reduces to finding the best value of s, the fixed investment ratio."

"It's fair," Solovians all said. The King agreed. So he established a prize for the discovery of the optimum investment ratio. The prize was to be a year abroad to learn how advanced countries had solved the growth problem.

Soon a brilliant peasant, Oiko Nomos, claimed the prize. Solovians laid down their tools, picked up pencils and pads, and converged on their capital to hear the proposed solution.

Oiko spoke. "I begin with a definition. By a *golden age* I shall mean a dynamic equilibrium in which output and capital grow exponentially at the same rate so that the capital-output ratio is stationary over time. This is precisely the pattern of growth which might emerge asymptotically from the regime contemplated for Solovia where population growth and technical progress are expected to be exponential and the investment ratio is to be fixed for all time.

"Now I am obliged to make some assumptions which I hope later researches into the exact shape of our production function will support:

"First, I assume that Solovia is capable of golden-age growth. This simply means that, corresponding to every investment ratio Solovia might adopt, there exists at least one capital-output ratio which, if established, will be exactly maintained by the dynamic equilibrium which follows from equations (1)-(3).

"Second, I assume that Solovia's golden-age growth rate is independent of its investment ratio We may call this growth rate, g, the *natural* rate of growth, in that it depends not upon our investment decisions but only upon γ, λ, μ and possibly certain parameters affecting the shape of the production function. The existence of a natural growth rate implies capital and labor are substitutable in such a way that the capital-output ratio can adjust to any value of s so as to equate the rate of capital growth,

$$\frac{sP_t}{K_t}, \text{ to the natural rate of output growth, } g.$$

"We can express the output of an economy in a golden age and having a natural growth rate by the equation:

$$P_t = P_0 e^{gt}, \qquad g > 0 \tag{4}$$

where P_0 depends upon conditions at time zero.

"We come now to a crucial notion. Consider an economy which lacks a definite beginning and which has always enjoyed golden-age growth at the natural rate. It has traveled unswervingly up a single exponential path, a path stretching back indefinitely into the past. Along this path the output rate at any specified time (though not the rate of growth) depends, in general, upon the value of the equilibrium capital-output ratio. But this ratio depends upon the investment ratio that has reigned over the golden age; we noted earlier that under conditions of natural growth the capital-output ratio is simply:

$$\frac{K_t}{P_t} = \frac{s}{g}. \tag{5}$$

Therefore, the golden-age output rate at any time—the height of the growth path—is generally a function of the prevailing value of s. We can express this fact by replacing P_0 in (4) by the function $f(s)$. Thus:

$$P_t = f(s)e^{gt}. \tag{6}$$

"It has been observed that a large value of s corresponds to a small ratio of output to capital. Provided that the elasticity of output with respect to capital is uniformly smaller than one, a seeming condition for stability, the smaller the ratio of output to capital, the larger must be the absolute magnitudes of both output and capital. Hence $f'(s) > 0$.

"I shall call a golden age which lacks a definite beginning a *boundless golden age.* Such an age may be endless although that is not essential for the definition; but it must be endless looking backward.

"And now, if these concepts are clear and my assumptions granted, I wish to introduce the following lemma."

"A lemma, a lemma," the crowd shouted. It was plain that the Solovians were excited by the prospect.

Oiko resumed. "The lemma: *Each generation in a boundless golden age of natural growth will prefer the same investment ratio, which is to say the same natural growth path.*

"In deciding which growth path is best from its standpoint, a generation will look only at the amount of consumption which each path offers it. Given the constancy of s, every golden-age path is associated with a consumption path on which consumption grows exponentially at the same rate as output. Under conditions of natural growth, consumption along all these paths grows at the identical rate, g, so that these time paths of consumption cannot cross. Therefore, with resources limited, there must exist some uniformly highest, feasible consumption path. This dominant consumption path offers more consumption at every point in its history than any other natural-growth consumption path. All generations in such a history will naturally prefer this path, whence its corresponding investment ratio, to any lower consumption path. A rigorous demonstration is straightforward.

"Take the consumption rate of the 'generation' in a boundless and natural golden age at time t. By (3) and (6), this is:

$$C_t = (1\text{-}s)f(s)e^{gt}. \tag{7}$$

To find the value of s which maximizes C_t, we take the derivative with respect to s and equate it to zero. This yields:

$$(8) \qquad -f(s)e^{gt} + (1\text{-}s)\, f'(s)e^{gt} = 0.$$

"It is apparent that upon dividing (8) by e^{gt} all terms involving t vanish. The solution of equation (8) is therefore independent of the 'generation' whose consumption we choose to maximize. The s which is optimal for one generation in a natural boundless golden age is optimal for all. This proves the lemma."

Cries of "What a lemma!" resounded in the capital and Oiko was heartened by the reception. Anticipation ran high when he moved to speak again.

"And now I wish to announce a new and fundamental theorem. Theorem: *Along the optimal golden-age path, under conditions of natural growth, the rate of investment is equal to the competitive rate of profits.*

"Choosing the best value of s is simple enough in principle. A high value of s will be associated with a high golden-age output path. But too high a value of s will leave too little output available for consumption. Characterizing the exact optimum is a matter of calculus.

"Rewriting (8) in the form:

$$(8') \qquad \frac{s}{1\text{-}s} = \frac{f'(s)s}{f(s)}$$

we find that the optimal ratio of investment to consumption equals what we may call the elasticity of golden-age output at time zero with respect to the investment ratio. Looking at (6), it is obvious that, for every investment ratio, this elasticity must be the same at all points (dates) along the associated golden-age path. If this were not so, the golden-age growth rate would depend upon the investment ratio, contrary to our assumption of natural growth.

"The remaining task is to express this elasticity in explicit terms of the production function, and thus in terms of relative factor shares.[1] Now the production function indicates that $f(s) = F(K_0, N_0)$. Next we use the golden-age capital-output relation in (5) to write K_0 in the form $\dfrac{sP_0}{g}$. Upon making this substitution in the production function (2) we obtain an equation in golden-age output at time zero as function of itself, the investment ratio and the labor force:

$$(9) \qquad f(s) = F\left(\frac{sf(s)}{g}, N_0\right).$$

"Total differentiation of (9) with respect to s yields an equation in terms of $F_K(K_0,N_0)$, the marginal productivity of capital at time zero:

$$(10) \qquad f'(s) = F_K\,\frac{f(s)}{g} + F_K\,\frac{s}{g}\,f'(s).$$

[1] Oiko was seen at this point to wave gratefully to Richard Nelson for help with this proof.

Upon rearranging terms and using the capital-output relation (5) we find that

$$\frac{f'(s)s}{f(s)} = \frac{a}{1\text{-}a}, \text{ where } a = \frac{F_K(K_0, N_0)K_0}{P_0}. \tag{11}$$

"Looking at (8′) and (11) we see easily that

$$s = a \tag{12}$$

In competitive Solovia the variable a measures capital's relative share in total output at time zero. Now we have observed that the elasticity of golden-age output with respect to the investment ratio is everywhere equal on any particular golden-age path; it follows by (11) that a, the profit-income ratio, must also be constant along any particular golden-age path. Therefore, by (12), on the optimum natural growth path the investment ratio and the profit ratio are constant and equal. This proves the theorem.

"We may call relation (12) the *golden rule of accumulation*, and with good reason. In a golden age governed by the golden rule, each generation invests on behalf of future generations that share of income which, subject to (3), it would have had past generations invest on behalf of it. We have shown that, among golden-age paths of natural growth, that golden age is best which practices the golden rule."

The Solovians were deeply impressed by Oiko and his theorems. But they were a practical people and soon full of queries. How, Oiko, does your theorem apply to Solovia? What must we do if we are not already on the golden-age, golden-rule path? Should we abide by the golden rule even when out of golden-rule equilibrium?

"Perhaps," Oiko replied. "We might attempt to approach the golden-rule path asymptotically. However I urge that we, in our lifetime, take whatever steps are required to place Solovia securely on the golden-rule path. Associated with that path is a unique capital-output ratio. If our present capital-output ratio is smaller, then our consumption must be slowed until our ratio is no longer deficient. If our present ratio exceeds the golden-rule ratio, then we must consume faster until our capital-output ratio is no longer excessive.

"Once our capital-output ratio has attained its golden-rule value, we must make a solemn compact henceforth to invest by the golden rule. If the investment ratio remains ever equal to the profit ratio, no generation in all the future of Solovia will ever wish we had chosen a different, successfully enforced investment ratio. The foundations are thus laid for a quasi-optimal social investment policy."

The crowd dispersed, happy for their Kingdom's future. But there were skeptics who reminded the King of Oiko's assumptions. They questioned Solovia's immunity from technological unemployment. They wondered whether their production function admitted of a natural growth rate. So the King named a team of econometricians to investigate the shape of the Solovian production function.

The King's econometricians were eventually satisfied that production in Solovia took place according to the Cobb-Douglas function:

$$P_t = A(e^{\lambda t}K_t)^{\alpha}(e^{\mu t}N_t)^{1-\alpha} \qquad 0 < \alpha < 1 \tag{2'}$$

where α, a fixed parameter, was the elasticity of output with respect to the capital stock. They preferred to write it in the form:

$$(2'') \quad P_t = Ae^{\rho t} K_t^{\alpha} N_t^{1-\alpha}, \qquad \text{where} \qquad \rho = \alpha\lambda + (1-\alpha)\mu.$$

Solovians knew then they could have any capital-output ratio they desired, with full employment. The existence of a full-employment, golden-age equilibrium for every investment ratio was assured. Differentiating logarithmically, they quickly calculated from (1) and (2″) that in a golden age, capital and output would grow exponentially at the rate $\frac{\rho + (1-\alpha)\gamma}{1-\alpha}$, independently of the investment ratio. Thus did Solovia discover her natural rate of growth. What a triumph for Oiko. His assumptions were completely vindicated.

Joyously, the Solovians hurried to compute the golden-rule path. It did not take them long to realize that α was capital's share. On the golden-rule path, s would equal α. Next, using (5), they divided α by their natural growth rate to obtain the capital-output ratio on the golden-rule path. To their great relief, the resulting ratio exceeded their actual capital-ouput ratio by only a small factor. No wonder for they had invested most of their profits and consumed most of their wages anyway.

With Oiko's inspiring words still ringing in their ears, the Solovian people pressed the King for a program to attain the golden-rule path. So the King proclaimed golden-rule growth a national purpose and instituted special levies. Once the golden-rule path was reached, investment was continuously equated to profits and Solovians enjoyed, subject to (3), maximum social welfare ever after.

EDMUND PHELPS*

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PUBLIC DEBT, TAXATION, AND CAPITAL INTENSIVENESS

In his "Principles of Political Economy and Taxation" [*9*], Ricardo cautioned that the deficit-financing of public expenditures sets back the growth of capital. The reference was to economies with rapid adjustment to "full-employment" equilibrium. While the Keynesians denied the rapid-equilibration assumption, the postwar reevaluation of monetary policy led to a modern restatement of Ricardo's doctrine. In several papers, of which Samuelson's [*10*] is probably the best known, it was argued that, given the level of government expenditures, a tax reduction would increase consumption and thus restrict investment if monetary policy is used compensatorily to maintain aggregate income and employment at their targeted levels. This is a statical proposition, good for each instant in time, given the currently available capital stock. But the current capital intensiveness is dependent upon the past history of taxes, so that an intertemporal model is required if we are to deduce that a permanently increased capital intensiveness will be brought about by a permanent decrease in public indebtedness per man.

Analyses of this question have been few. In his parable of saving under population overlap, Samuelson [*11*] showed that the social "contrivance" of government-issued money (unbacked by government-owned capital or other interest-paying assets) would tend permanently to increase the rate of interest (thus tending to cure his economy from any inefficient permanent overinvestment); but whether ordinary public debt would do as well was left unexplored. Modigliani [*6*] in his life-cycle model of a stationary economy, argued that by permanently adding a dollar to the public debt, the government would ultimately and permanently displace exactly one dollar's worth of capital from private portfolios. Diamond [*2*] synthesized these two models and showed that, under certain stability and uniqueness conditions, a permanent addition to the debt per head would produce a permanent reduction of capital per head—though not generally in an equal amount.

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The present note shows that a government intent upon permanently increasing the economy's capital intensiveness through fiscal policy may find that as it succeeds the public debt per head has been increased and the necessary deficit per head has grown. Normally, however, there will be a "classical" range of capital-labor ratios in which an increase in long-run capital per head requires the government to pursue a policy that decreases the long-run public debt per head. These results depend only on the patent geometrical possibilities for the long-run consumption function, quite free of the model generating that function.

In a model in which the demand for consumption is a fixed fraction of disposable income, we show that there exists just one classical range and just one "anticlassical" range. Within the classical range, it is precisely at the Golden Rule capital–labor ratio—more generally, at the capital–labor ratio that maximizes sustainable private consumption—that a dollar of additional debt displaces exactly a dollar of capital. The reason is that it is just when consumption is maximal (across steady states) that one can, by the familiar envelope theorem, ignore the feedback effect of capital's displacement upon itself. As a corollary, it follows that, in this model, private wealth, defined as the sum of public debt and capital, is maximized at the Golden Rule capital–labor ratio. More generally, if the consumption function depends solely on private wealth and disposable income, then in the Golden Rule steady state, private wealth is maximized and a small increase of public debt would displace an equal dollar amount of capital. When in addition the consumption function depends upon the wage and interest rates, equal displacement and wealth maximization occur at the Schumpeterian zero-interest-rate steady state.

We then go on to discuss some welfare aspects of these behavioral relations. The point that an initial public debt is not a burden if it can be costlessly neutralized is reiterated. The absence of lump-sum taxes raises the possibility that the debt cannot be completely neutralized, because the available tax instruments may have substitution effects upon the allocation of time between work and leisure as well as between consumption and saving. The consequences of debt creation (or debt retirement) for future tax rates and corresponding future substitution effects must be considered, along with the current substitution effects of current tax rates, in the selection of the budgetary deficit program.

1. GOVERNMENT DEBT, SUSTAINABLE OUTPUT, AND CONSUMPTION

For ease of exposition, we begin with the simplest neoclassical model, showing later how the analysis can be extended to more general models.

Production follows the usual one-sector technology,

$$y = c + z = f(k), \tag{1.1}$$

where output per man, y, can be divided into consumption per man, $c \geq 0$, and investment per man, $z \geq 0$. At every instant, capital and labor are inelastically supplied, and the capital–labor ratio is denoted by k. If $n > 0$ is the constant rate of labor force growth, then the change in the capital–labor ratio is given by

$$\dot{k} = z - nk, \tag{1.2}$$

ignoring capital depreciation.

The household demand for consumption is a fixed fraction, $0 < (1-s) < 1$, of private disposable income, which is here comprised of rewards to privately owned factors and government transfers less taxes. Let us suppose that the government, through central bank action, is able to keep the economy along an equilibrium (full-employment) path with zero inflation. Then private demand for consumption goods per man is given by

$$(1-s)[f(k)+\phi],$$

where ϕ denotes net government transfers per head. If government expenditure is zero, then ϕ is equal to δ, the per capita deficit, so that in momentary equilibrium

$$c = (1-s)[f(k)+\delta]. \tag{1.3}$$

Government debt per head, denoted by Δ, therefore follows the simple law of motion

$$\dot{\Delta} = \delta - n\Delta, \tag{1.4}$$

so that in balanced growth equilibrium $\Delta = \delta/n$.

It should be noted that the model has two leading interpretations. First, as in Diamond [2], the public debt could be thought to consist of demand loans held by households (somewhat like postal savings deposits but fixed in consumption units) which are perfect substitutes for capital and therefore pay a dividend rate, r, which is always equal to the return on capital, $f'(k)$. Second, the model could be considered to be the "reduced form" of a more complete model like that of Foley, Shell, and Sidrauski [3] in which the public holds three assets, capital, money (noninterest-bearing government debt), and short-term bonds (like postal savings deposits)—none of which is a perfect substitute for any other.[1] In this interpretation of the model, it must be assumed that the

[1] For steady-state analysis only, one can admit bonds of any *finite* maturity, for then they will all be valued at par in steady states. Consols would raise valuation complications, though even these vanish as the steady state is approached.

central bank is able to hold the general price level constant by varying the debt–money ratio, through open-market purchases and sales, while the treasury controls the demand for consumption by its deficit policy.[2]

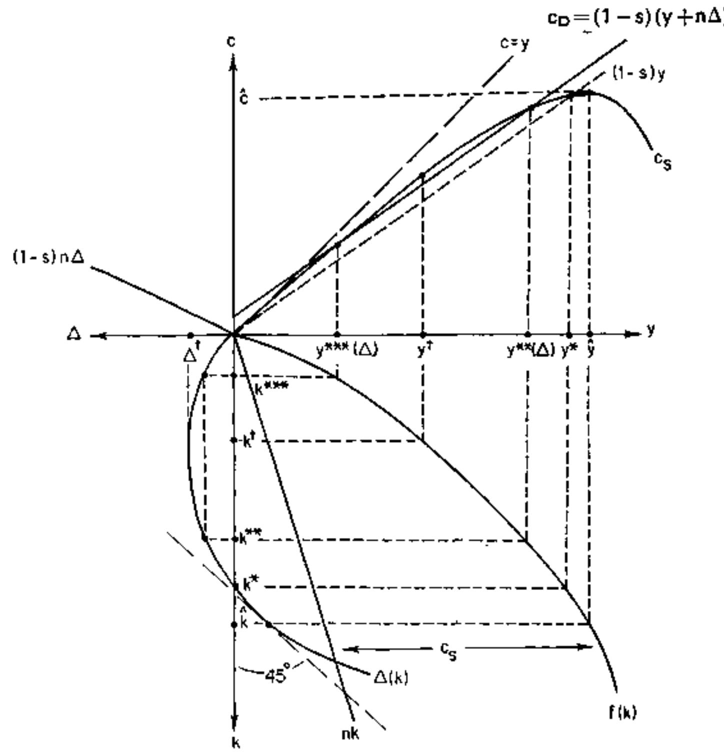

FIG. 1. Balanced-growth relations: sustainable per capita consumption supply, c_S, steady-state desired consumption, c_D, and steady-state per capita debt, $\Delta(k)$, as functions of the steady-state capital-labor ratio, k.

Balanced-growth ($\dot{k} = 0 = \dot{\Delta}$) solutions to the system (1.1)–(1.4) are described in Fig. 1. Output per head, $f(\cdot)$, and the nk-ray are plotted against the k-axis in the southeast quadrant. Sustainable per capita consumption supply, c_S, is equal to the difference between $f(k)$ and nk. Since $y = f(k)$, c_S can then be plotted against y in the northeast quadrant.

[2] It may be noted that should there exist a discrepancy between the central bank's liabilities and its assets, net public indebtedness will differ *pro tanto* from the Δ of (1.4). But no such discrepancy can exist in steady states, at least not for $n > 0$ and price-level stationarity.

Under the usual regularity conditions in production,[3] c_S achieves a maximum at the Golden Rule capital–labor ratio, $\hat{k}$, where the rate of interest equals the rate of growth, $f'(\hat{k}) = n$. Since steady-state consumption is less than steady-state output, the c_S locus lies below the $c = y$ ray. c_S is zero when y is zero, rises to a maximum at the Golden Rule per-capita output, $\hat{y}$, and falls to zero at the maximum sustainable per-capita output, where $f(k) = nk$. Since $f(\cdot)$ is strictly concave in k, c_S is a strictly concave function of steady-state per-capita output. Because of the Inada condition $f'(0) = \infty$, as y becomes small, the c_S schedule approaches tangency to the $c = y$ ray.

In balanced growth, $\dot{\Delta} = 0$, so that $\delta = n\Delta$ and desired per-capita consumption is equal to $(1-s)(y+n\Delta)$. Hence, in the present model, the steady-state desired per-capita consumption locus is a straight line in the northeast quadrant that intersects the vertical axis at $(1-s)n\Delta$. We are now ready to study existence and uniqueness of balanced growth states, along with important propositions in comparative dynamics.

If in the steady state debt per head is zero ($\Delta = 0$), then there exists the unique (Solow) steady-state output per head, y^*, for in this case c_D is a ray from the origin. For s sufficiently small, $y^* \leq \hat{y}$ and development is intertemporally efficient. If, however, s is large, then $y^* > \hat{y}$ and development is intertemporally inefficient.[4]

If the government is a long-run creditor, then, for given $\Delta < 0$, steady-state output per head is uniquely determined. Again, for sufficiently small s, steady-state output per head is less than or equal to $\hat{y}$ and development is intertemporally efficient, while for larger s, steady-state $y > \hat{y}$ and development is intertemporally inefficient.

The case in which the government is a long-run debtor is more complicated. When $\Delta > 0$, the steady state is unique if and only if the c_D line is tangent to the c_S curve. Since $c_D = (1-s)(y+n\Delta)$,

$$(\partial c_D/\partial k) = (1-s)(\partial y/\partial k).$$

But $c_S = y-nk$, so that $(\partial c_S/\partial k) = (\partial y/\partial k)-n$. Therefore at the point of tangency, $y^\dagger$, we have that $sr = n$, where $r = f'(k)$ denotes the marginal product of capital. The unique debt per head consistent with the $y^\dagger$ steady state is denoted by $\Delta^\dagger$.

For $\Delta > \Delta^\dagger > 0$, c_D must everywhere exceed c_S, so no steady state is possible. Hence, $\Delta^\dagger$ is the maximum sustainable debt per head. For given Δ such that $\Delta^\dagger > \Delta > 0$, there exist exactly two steady-state per-

[3] $f(k) > 0$, $f'(k) > 0$, $f''(k) < 0$ for $0 < k < \infty$, while $f(0) = 0$, $f(\infty) = \infty$, and $f'(0) = \infty, f'(\infty) = 0$.

[4] See Phelps [8].

capita outputs $y^{**}(\Delta)$ and $y^{***}(\Delta)$ with $y^* > y^{**}(\Delta) > y^\dagger > y^{***}(\Delta) > 0$.

Notice that government debt "matters." As Δ is increased, the c_D line is shifted upward. Therefore, in a steady-state equilibrium with positive debt per head, output per head is always less than in the Solow steady state ($\Delta = 0$). Similarly if $\Delta < 0$, output per head is always more than in the Solow ($\Delta = 0$) steady state, y^*.

If we restrict our attention to efficient steady states, where $y \leq \hat{y}$, then we know that the per-capita consumption is lower in the steady states with positive per-capita debt than in the steady state with zero per-capita debt. Similarly, steady state consumption is higher when $\Delta < 0$ than when $\Delta = 0$ as long as $y \leq \hat{y}$. It is easily seen from Fig. 1 that these propositions about steady-state per-capita consumption are reversed in the regimes for which $y > \hat{y}$.

Following previous authors, we ask the broader question: Is it in general true that a higher steady-state per-capita output must be accompanied by a lower steady-state per-capita government debt? We conclude from Fig. 1 that the answer is no. Notice that for each feasible y there exists exactly one steady-state Δ.[5] For $y > y^\dagger$, $d\Delta/dy < 0$, since $y^{**}(\Delta)$ falls as the c_D line is shifted upward. But for $y < y^\dagger$, $d\Delta/dy > 0$, since $y^{***}(\Delta)$ rises as the c_D line is shifted upward.[6] We summarize these results in the first proposition.

PROPOSITION 1. *Across steady states, there is a classical range where* $d\Delta/dk < 0$ *and an anticlassical range where* $d\Delta/dk > 0$, *with*

$$\text{sign}\,(d\Delta/dk) = \text{sign}\,(sr - n).$$

[The surprising decline of debt per head as capital per head falls, in the anticlassical (high-interest-rate) range, does not imply that debt per unit of output also falls in that range. Indeed, it is easy to show that the debt-output ratio is monotone-decreasing in the capital–labor ratio. Corresponding to every steady-state debt-output ratio is a deficit–output ratio and hence some fixed multiple between output per head and per capita disposable income. Hence, in this case, c_D, the long-run consump-

[5] Although the steady-state y is not a single-valued function of steady-state Δ, Δ is a single-valued function of y defined over the interval $[0, \bar{y}]$, where $\bar{y}$ is the maximum sustainable output per man.

[6] It might be argued that the anticlassical high-interest-rate regime has limited empirical relevance. If the growth rate is even as low as 3 percent and saving is equal to 10 percent of income, then $d\Delta/dk > 0$ only when the government is planning for a long-run return on investment in excess of 30 percent. We do not know, however, whether this anticlassical regime would seem more or less remote from observed and contemplatable capital intensities in more complex models.

tion function, starts at the origin. As the debt–output ratio is increased from its smallest (negative) sustainable value, the steady-state capital–labor ratio decreases, tending asymptotically to zero because of our assumption that $f'(0) = \infty$.]

Remember that if $c(y)$ is steady-state consumption per man, then $c(y)$ is at a maximum at $\hat{y}$ and sign (dc/dy) = sign $(\hat{y}-y)$. From Proposition 1, we can now deduce the next result.

PROPOSITION 2. *Across steady states, per capita consumption is positively associated with the debt both in the "anticlassical" range and in the inefficient portion of the classical range; that is,*

$$\frac{dc}{d\Delta}\begin{cases} > 0 & \text{if } r > n/s, \\ \textit{undefined} & \text{if } r = n/s, \\ < 0 & \text{if } n < r < n/s, \\ = 0 & \text{if } r = n, \\ > 0 & \text{if } 0 < r < n. \end{cases}$$

For purposes of construction, we draw the $(1-s)n\Delta$ ray in the northwest quadrant of Fig. 1. The intercept of c_D with the vertical axis is equal to $(1-s)n\Delta$. Therefore, by projecting these c_D intercepts through the $(1-s)n\Delta$ ray, we are able to derive the steady-state relation between capital per head and debt per head. This relation is described by the $\Delta(k)$ locus in the southwest quadrant. $\Delta(k)$ is zero when k is zero, rises to a maximum $\Delta^\dagger$ when $k = k^\dagger$, and falls to zero when $k = k^*$ (the Solow steady-state capital–labor ratio). For $k > k^*$, steady-state Δ is negative.

From (1.3), steady-state per capita consumption, c, is equal to $(1-s)[f(k)+n\Delta]$, since in balanced growth $\delta = n\Delta$. Therefore,

$$\left(\frac{1-s}{s}\right) n(k+\Delta) = f(k) - nk, \tag{1.5}$$

because $c = f(k) - nk$ in steady states. Differentiating (1.5) with respect to k yields

$$\left(\frac{1-s}{s}\right) n\left(1 + \frac{d\Delta}{dk}\right) = f'(k) - n. \tag{1.6}$$

Remember that at $\hat{k}$, the Golden Rule capital–labor ratio, the rate of interest, $f'(\hat{k})$, is equal to the growth rate, n. We conclude from (1.6), that $(d\Delta/dk) = -1$ if and only if $k = \hat{k}$. Therefore, in Fig. 1, $\Delta(k)$ is tangent to the 45° line at $\hat{k}$. We summarize this result in the next proposition:

PROPOSITION 3. *A dollar of government debt permanently displaces exactly one dollar of private capital (if it displaces capital at all) only in the Golden Rule steady-state, and not elsewhere.*

That the public debt permanently displaces an equal dollar amount of real capital from the portfolios of households was suggested by Modigliani [6]. It is of interest, therefore, that for our model, Modigliani's conclusion holds only in the neighborhood of the Golden Rule steady state. Proposition 3 is actually a simple instance of the familiar envelope theorem and consequently applies to a wider class of models. In more general models, steady state desired per capita consumption depends upon wealth per man, $w = k+\Delta$, disposable income per man, $h = y+n\Delta$, the wage rate, ω, and the interest rate, r. In balanced growth, therefore,

$$c_S(k)-c_D(w, h, \omega, r) = 0.$$

But in steady-state equilibrium,

$$h = f(k)+n\Delta = c_S(k)+nk+n\Delta = c_S(k)+nw.$$

Thus,

$$c_S(k)-\psi(w, y) = 0,$$

since y uniquely determines ω and r. Implicit differentiation in the above equation yields

$$\frac{d\Delta}{dk} = \frac{(dc_S/dk)-\psi_1(\partial w/\partial k)-\psi_2(dy/dk)}{\psi_1(\partial w/\partial \Delta)}.$$

But since $\partial w/\partial k = 1 = \partial w/\partial\Delta$ and $r = dy/dk$,

$$\frac{d\Delta}{dk} = \frac{(dc_S/dk)-\psi_1-r\psi_2}{\psi_1}.$$

If c_D depends only on income and wealth, i.e. if $\psi_2 \equiv 0$, then $d\Delta/dk = -1$ if and only if $k = \hat{k}$. That is, when c_S is maximal, the first-order change in Δ, *mutatis mutandis* (c_S allowed to vary) owing to a change in k is equal to the first-order change in Δ *ceteris paribus* (c_S constant) owing to a change in k. Most generally, equal displacement occurs if and only if

$$r-n-r\psi_2 = 0.$$

Thus in the Schumpeterian (zero-interest-rate) steady state, with no population growth, $d\Delta/dk = -1$, since $n = 0 = r$.

In Modigliani's model [6], the "desired" ratio of wealth to disposable income is constant, and disposable income equals national income in any stationary-state. But even in that special case, the feedback of capital displacement upon income is present for positive interest rates so that it is only in the Schumpeterian (Golden Rule) stationary state that exact displacement is ensured.

Differentiating (1.6) with respect to k yields

$$\left(\frac{1-s}{s}\right) n \left(\frac{d^2\Delta}{dk^2}\right) = f''(k) < 0. \tag{1.7}$$

Therefore, as shown in Fig. 1, $\Delta(k)$ is a concave function of k. Then, as a corollary to Proposition 3, we have for the simple model described by (1.1)–(1.4) that private wealth per man, $w = \Delta + k$, is maximized if and only if consumption is maximized, i.e., when $k = \hat{k}$.[7] Furthermore, this result holds for any consumption function which depends solely on private wealth and disposable income, since when $\psi_2 = 0$, $dw/dk = 0$ (i.e., $d\Delta/dk = -1$) if and only if $dc_S/dk = 0$. Even when c_D depends on ω and r, w is maximized in the Schumpeterian state, where $r = 0 = n$.

Some generalizations of the model described in Fig. 1 should be mentioned. If we introduce a fixed amount of government expenditure per man, both the c_S and c_D schedules are affected. The former curve is displaced downward by the amount of the public outlay per head. The variable ϕ now replaces δ, where $\phi = \delta - g$, g being the government expenditure per capita. Because $(1-s) < 1$, the balanced-budget theorem operates: The wedge of government expenditure reduces per-capita consumption supply, c_S, by more than demand, c_D. In the classical range, where the slope of c_S is less than the slope of c_D, the effect is to reduce steady-state investment per man, output per man, and capital intensiveness; the opposite results occur in the anticlassical range, where the slope of c_S is greater than the slope of c_D. Aside from the fact that a range of small capital–labor ratios are not sustainable with fixed $g > 0$, the analysis of the effects on k of a change in Δ are not essentially affected.

If per capita government expenditure is made a function of output per head, still different results occur. If $g = \gamma y$ with γ a fixed fraction, then the slope of the c_D function is $(1-s)(1-\gamma)$. The c_S curve will also have a smaller slope at every y since the government "takes its cut" of output left over after the capital formation necessary to maintain the capital–labor ratio. This means that the private-consumption-maximizing point, where one gives no weight to public expenditures in measuring total consumption, occurs at an interest rate greater than the growth rate, $r > n$ and $y < \hat{y}$. It is at this point, where the c_S curve has a zero slope, that a dollar of additional debt per head exactly displaces one dollar of capital per head. The critical interest rate is given by

$$r(1-\gamma) - n = 0, \tag{1.8}$$

and the point $y^\dagger$, separating the classical and anticlassical ranges, occurs

[7] See also Levhari and Patinkin [4], especially pp. 746–747.

where the interest rate is given by

$$r = \frac{n}{s(1-\gamma)+\gamma}. \tag{1.9}$$

In a life-cycle model, with or without bequests, the steady-state desired per capita consumption locus will not generally be linear. Imagine confronting each household with a configuration of a wage rate, an interest rate, and government transfers per head which are constant over time. There will correspond a steady-state, desired-wealth level, $w_D = (k+\Delta)_D$, and a steady-state per capita desired-consumption level, c_D. Since the interest rate and the wage rate are monotonic functions of output per man, y, such a model implies that

$$c_D = C(y, \Delta; n),$$

which need not, of course, be linear in y or in $\Delta(=\delta/n)$. Nonlinearity raises the possibility of multiple intersections of the c_D locus with the c_S curve. Then the functional relationship between Δ and k will exhibit a relative maximum or minimum corresponding to each tangency of c_S with c_D as the latter is shifted as a consequence of varying Δ. There will be a paradoxical anticlassical range corresponding to each relative maximum.

In the life-cycle model, c_D is ordinarily a monotonically increasing function of y. It is also plausible to suppose that c_D shifts upward as Δ is increased and that for positive Δ the c_D locus has a positive intercept with the vertical axis.[8] Since c_S is a concave function starting from the origin, if a steady-state with $\Delta > 0$ exists, $c_D = c_S$, then there must be at least one steady-state in which the slope of c_S is greater than or equal to the slope of c_D. If at the steady-state, the slope of c_S is strictly greater than the slope of c_D, then there must exist at least one other steady-state for which $(\partial c_D/\partial y) > (\partial c_S/\partial y)$. Thus, with $\Delta > 0$ the steady-state in the life-cycle model will only be unique in the singular case of a unique tangency of c_S with c_D.[9] If a non-tangency steady-state exists for given $\Delta > 0$, there must be an anticlassical range of capital–labor ratios for which $d\Delta/dk > 0$.

Another generalization is to allow for many capital goods. Then the c_S curve may have many local maxima and minima, though the Golden Rule point continues to be the global maximum. This too raises the

[8] An exception is the model of Bailey [*1*], where households (with bequest motives) presently value the discounted stream of taxes needed to service the public debt as exactly equal to the value of the outstanding interest-bearing public debt.

[9] This proposition depends upon the fact that c_S achieves a maximum. If we relax our assumptions about the regularity of the production function, $f(k)$, different results can occur. If c_S is monotonically increasing, then the steady-state may be unique for given $\Delta > 0$, and $d\Delta/dk$ would be positive for any $k > 0$.

possibility of multiple intersections with the c_D locus. Therefore, there may be multiple anticlassical ranges on this account as well. Further, even when c_D depends solely on wealth per head and disposable income per head, there may be many points of "equal displacement," each one yielding a relative maximum or minimum of total dollar wealth per head as a function of the dollar value of capital per head (since each one corresponds to a locally flat c_S locus).

2. DYNAMIC ANALYSIS

In the previous section, we studied steady-state behavior and derived certain propositions in comparative dynamics—in particular, that across steady-state sign $(d\Delta/dk) = \text{sign}(sr-n)$. In this section, we turn to the full dynamic analysis. This is important because, as we shall see, a certain stability analysis and a closely related assumption about uniqueness of the balanced growth state seems to be fundamental to Diamond's [2] claim that $d\Delta/dk < 0$ across steady-states. Further, such a dynamic analysis is crucial for understanding the relevance of the result.

Remember that in the one-sector, constant-saving-fraction model when long-run debt per man is chosen to be positive but less than the maximum value $\Delta^\dagger$, long-run output per man is not unique. At the lower output per man $y^{***}(\Delta)$, the slope of c_S exceeds the slope of c_D. Since, in Fig. 1, c_D shifts upward as Δ is increased, $dy^{***}/d\Delta$ and $dk^{***}/d\Delta$ are positive. The question immediately arises: If c_S has greater slope than c_D, is not the y^{***} equilibrium unstable? The answer is that this need only be true if we limit the government to the pursuit of policies with constant deficits per man. This point will require further analysis.

From Eqs. (1.1)–(1.3), we can derive the equation for capital accumulation,

$$\dot{k} = sf(k) - [(1-s)\delta + nk]. \tag{2.1}$$

In Eq. (2.1), the deficit per man, δ, can be set by the government at each instant to bring forth any desired investment consistent with existing endowments of capital and labor. On this postulate of full fiscal effectiveness, the government can achieve any technologically feasible time-path of consumption and capital accumulation.

For example, the government might follow some rule that makes planned per capita consumption, c^0, a function of the inherited capital–labor ratio.[10] The desired deficit per man, δ^0, can then be calculated

[10] In many contemporary planning models, a Ramsey-like optimal economic growth policy implies that planned consumption per head will be a uniquely determined increasing function of k. The implications of a Ramsey-optimal economic growth policy for

as a function of the capital–labor ratio, since from (1.3)

$$c^0(k) = (1-s)[f(k)+\delta^0(k)].$$

Given $c^0(k)$, we can calculate $\delta^0(k)$, and thus $z^0(k)$ and $\dot{k}^0(k)$.

The process of capital accumulation can be more fully analyzed in Fig. 2. *For the moment*, we assume, like Diamond [2], that the government

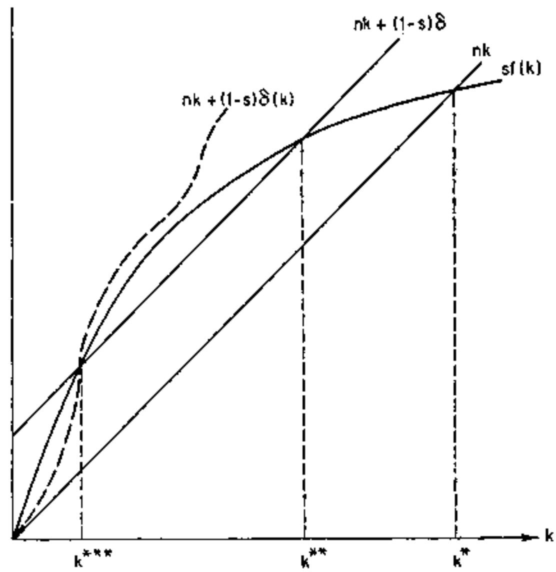

FIG. 2. Dynamic analysis.

holds the deficit per man, δ, constant throughout the adjustment path. For $\delta = 0$, $\dot{k}$ is the difference between the $sf(k)$ curve and the nk-ray. As Solow showed, k^* is unique and is globally stable, i.e.,

$$\text{sign}\,\dot{k} = \text{sign}\,(k^*-k) \quad \text{for } k > 0.$$

For fixed $\delta < 0$, the $[nk+(1-s)\delta]$ line lies below the nk-ray. Again, with δ held constant the unique long-run balanced growth equilibrium is globally stable.

For fixed $\delta > 0$, the story is more complicated. In the *neighborhood* of k^{**}, sign $\dot{k}$ = sign $(k^{**}-k)$, yielding that k^{**} is *locally* stable when the government holds δ constant. In the neighborhood of k^{***}, however, sign $\dot{k}$ = sign $(k-k^{***})$, so that k^{***} is unstable when δ is held constant.

fiscal and monetary policy are examined by Foley, Shell, and Sidrauski [3]. For a two-sector mixed economy with optimal fiscal and monetary policy, they show that in balanced growth, sign $(d\Delta/d\rho)$ = sign $(-d\Delta/dk)$ = sign $(n - sr)$, where $\rho > 0$ is the government's pure (subjective) rate of time discount for per capita consumption.

If the government maintains a constant deficit per head, then when the economy is disturbed from k^{***}, it will not return.

Diamond [2] excludes from his comparative-dynamics theorem steady states such as k^{***} which are dynamically unstable under fixed δ regimes. In the model described by Eqs. (1.1)–(1.4), $d\Delta/dk \geq 0$ if and only if the corresponding steady-state is unstable when δ is held constant. The k^{***} steady-state would, however, be globally stable if the government chose δ to be small when k is small and δ to be large when k is large. Such a rule, in which the deficit per man, $\delta(k)$, depends upon the capital–labor ratio, is described in Fig. 2. The dashed curve represents $[nk+(1-s)\delta(k)]$.For $k < k^{***}$,

$$sf(k) > [nk+(1-s)\delta(k)],$$

so that $\dot{k} > 0$. Similarly, for $k > k^{***}$,

$$sf(k) < [nk+(1-s)\delta(k)],$$

so that $\dot{k} < 0$. Hence, we have demonstrated how the government can choose a policy so that independent of initial endowments the economy ultimately tends to k^{***}.

For the simple constant-saving-fraction model, a steady-state equilibrium is locally stable for fixed δ *if and only if* $\partial c_D/\partial y > \partial c_S/\partial y$. In the life-cycle model, however, a steady-state with the slope of c_S greater than the slope of c_D (and thus $d\Delta/dy > 0$) is not necessarily unstable *even under constant δ regimes*. This is because the "momentary consumption functions' relating current per capita consumption to current income per head may be sufficiently steeper than the long-run c_D locus. Even though the long-run c_D locus is flatter than the long-run c_S locus at equilibrium, the "momentary consumption function" may be steep enough to ensure the stability of a constant-deficit-per-man policy.[11]

The reader will now readily see through our paradox of "capital deepening through fiscal ease"—in the anticlassical range. In the transition to a greater capital per head, the government must assuredly reduce at least temporarily the algebraic deficit, raising taxes and reducing consumption at first. But as the capital–labor ratio rises, the increase in sustainable consumption is so great, in the anticlassical range, relative

[11] Such cases do not figure in the general analysis presented by Diamond [2]. He excluded them from the text, apparently on the ground that he wished to deal primarily with the case in which the curves describing short-run interest-wage determination intersect in such a way that a Walrasian *tatonnement* process would be stable (see [2], p. 1132). In an appendix, however, he showed that the opposite assumption, allowing for a stable Marshallian adjustment process, in no way interferes with the convergence of the economy to its golden age equilibrium. He noted that in this case $dk/d\Delta > 0$ across golden ages.

to the increase in consumption demand which would occur if the original deficit per head were restored that even the *original* deficit is too small to establish equilibrium at the higher capital–labor ratio: A higher deficit is required. This is faintly reminiscent of the doctrine of secular stagnation with this important difference: In the present model, monetary policy can maintain full employment despite the lower yield on capital, but deficit spending is needed to keep the superfluity of private thrift from leading to still further capital deepening.

Welfare Aspects

Much of the debate on the "burden of the debt" is beclouded by semantic difficulties. Many writers have termed the public debt "burdensome" if long-run consumption per man is decreased when government debt per man is increased. Diamond [2] showed that in this sense the debt is not a burden for economies pursuing intertemporally inefficient development programs. For such cases, long-run $d\Delta/dk < 0$ but long-run $d\Delta/dc > 0$. We have shown that in addition to the inefficient, low-interest-rate range, there is also a high-interest-rate range within the efficient range for which steady-state $d\Delta/dc > 0$ and $d\Delta/dk > 0$.

Even in that part of the "classical" range that is short of the Golden Rule point, namely, $n < r < n/s$, so that $d\Delta/dc < 0$, the term "burden" is unfortunate for prejudicing fiscal policies which increase the debt. At any moment, a government fully aware of the consequences of its actions might choose an easy fiscal policy coupled with a tight monetary policy, i.e., elect to finance a given government expenditure partly through a deficit rather than by taxation. Such policies are not necessarily irrational merely because they promote current consumption at the expense of future consumption: The present benefit may be thought to outweigh the future loss from that policy.

But the central objection to the term "burden" is that the inherited stock of debt, as distinct from increases in it, cannot be a burden if, as in the context of the model described by Eqs. (1.1)–(1.4), the government can neutralize the allocative and distributive influences of that debt by means of suitable taxation. In choosing an optimal fiscal policy program for the (possibly infinite) planning period, the government must, at least implicitly, rank attainable growth paths. Suppose, for example, the government's criterion functional depends only on the time-path of per capita consumption. Then maximal attainable welfare at time t, W, depends only upon inherited endowments at that time:

$$W[k(t), \Delta(t)].$$

In our model, future consumption possibilities are enhanced when the inherited capital–labor ratio is higher, so that $\partial W/\partial k > 0$. If, as assumed in the previous sections, fiscal policy can achieve any technologically feasible consumption path, no matter what the size of the inherited debt, then $\partial W/\partial \Delta = 0$ for all Δ.

The proposition just enunciated requires no assumption about the availability of lump-sum taxes. If only the per capita consumption sequence figures in the social utility function, then the possibility that the fiscal instrument employed to control that variable may have side effects on other allocations is of no welfare significance.

If, in contrast, labor supply is not invariant and if the time-path of per capita leisure, as well as per capita consumption figures in the objective functional then it will not *generally* be possible, with just one fiscal instrument, to guide these two variables along the best technologically feasible path. Singular cases exist, the most obvious being that in which, for every per-capita consumption rate the government engineers, each household is automatically led by the market to choose the social-utility maximizing amount of leisure because, in a certain sense, social preferences between goods and leisure correspond to individual private preferences and because the kind of tax in use to control consumption, like the essentially fictitious "lump-sum" tax, does not "distort" the labor-leisure decision. When social and private preferences are alike (as between goods and leisure) but the kinds of tax in use for controlling consumption are "distorting," like the income tax, the technologically feasible optimum is not generally attainable. These taxes will distort, through the substitution effect, the leisure-goods allocation, as well as private thrift.

When the technologically feasible optimum is not generally attainable by virtue of an insufficiency or imperfectness in the fiscal instruments, it is also the case that the initial public debt cannot generally be exactly neutralized. The usual argument is that the additional taxes necessary to offset the wealth and income effects upon consumption demand will have substitution effects upon the effort-leisure choice.[12] But one cannot exclude the possibility that, by chance, the "neutralizing" increment in tax rates will serve to reduce consumption and leisure demand in just the right proportions so as to make reattainable the *status quo ante debitum.*[13]

Despite the analytical complexities of the matter, the size of the "tax rate" remains of some interest as a cause of resource misalignment. If we restrict our attention to that interpretation of our model in which short-term government bonds are perfect substitutes for capital and there is no money among the government liabilities, then there is a relationship

[12] See Meade [5].

[13] See Phelps [7].

between τ, the tax rate expressed in terms of taxable income, the latter defined to include the interest on the debt, and debt per head,

$$\tau = \frac{\gamma y + (r-n)\Delta}{y + r\Delta},$$

where $r = f'(k)$. On the one hand, the interest on the debt increases the tax rate necessary to yield the deficit that corresponds to the debt per head, Δ. This effect is attenuated by the accompanying enlargement of the tax base. On the other hand, interest apart, the larger the debt per head the larger the deficit must be, and hence the smaller the required tax rate, in order that the debt keep pace with the growing population.

We can study the derivative, $d\tau/d\Delta$, always remembering k is not a single-valued function of Δ, so that the derivative must be thought of as $(d\tau/dk)/(d\Delta/dk)$. The relation is

$$(y+r\Delta)^2 \frac{d\tau}{d\Delta} = [(1-\gamma)r - n](y - \Delta y') + \Delta r'[(1-\gamma)y + n\Delta],$$

where primes denote differentiation with respect to Δ. In the classical range, where the derivatives y' and r' are negative and positive, respectively, we see that at least for nonnegative Δ, the tax rate is increasing with the debt up to the modified Golden Rule point where $(1-\gamma)r = n$; at sufficiently larger capital-labor ratios, there appears to be some ambiguity, since reaching these low-interest rates may require negative debt. In the high-interest anticlassical range, $y' > 0$ and $r' < 0$. For large enough, positive Δ, therefore, the sign of the derivative is again in question.

We have uncovered and sought to explain some of the surprising relationships that exist among capital per head, public indebtedness per head, and the income tax rate in simple mixed-economy models in which the government may use fiscal instruments to influence household consumption demand and thus to control the growth-path of the economy. In such models, where long-run, steady-state behavioral loci may be misleading for stability analysis, and where stability analysis is itself beside the point when the government is in effect altering the private response to data changes in order to secure some desired result, there will generally exist some surprising anticlassical relations among these three variables—debt, taxes, and capital. A fiscal policy designed to reduce capital intensiveness for a near-term gain of consumption may end up permanently reducing per capita public indebtedness, and, whether or not it decreases the debt, it may also, though it need not, reduce the income tax rate.

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LINEAR TAXATION OF WEALTH AND WAGES FOR INTRAGENERATIONAL LIFETIME JUSTICE: SOME STEADY-STATE CASES

Every paper on the intrageneration redistribution of utilities rightfully makes reference to James Mirrlees. His celebrated paper is the first to study a problem in optimum redistribution without the magic wand of the lump-sum tax. What has proved seminal in that paper is its attractive representation of generational heterogeneity: all worker-consumers can be placed along a one-dimensional continuum with respect to their different efficiencies at producing. Otherwise they are all alike, are equally efficient at consuming, and have the same egoistic utility function that each individual maximizes.

The setting for the analysis is a stark one-period competitive-market economy of bread and leisure. Total output of bread is a constant-marginal-returns function of the aggregate efficiency-weighted man-hours worked—there is no capital. The wage paid to an individual is equal to his product, the efficiency-weighted hours he elects to work. The tax he pays is some function (alike for all individuals) of his wage income. The aggregate tax revenue thus collected is (after subtractions) disbursed to all individuals in the form of a demogrant or lump-sum credit equal for all individuals. Mirrlees's redistributional problem, then, is to find a tax-net-of-demogrant schedule against wage income that maximizes the additive Bentham-Bergson-Fleming social-welfare functional, $\int u(n)^{-\theta} dn$.

It has been the source of some astonishment that no problem in optimum redistribution by non-lump-sum means was even stated, let alone analyzed, until 1971. It is also surprising that the problem that came eventually to be formulated deals with an ornery "detail"—the shape or progressivity of a particular tax schedule. It might have been expected that the economics of redistribution would begin with the analysis of a problem both easier and perhaps more important, to wit: the optimum *mix* of *proportionate* taxes with which to finance redistributive transfers. The marginal utility of departures from linearity in the tax functions could always be left for later attention. Of course, to study such a problem one needs to endow the model economy with more than one factor of production (or more than one consumer good).

Our paper investigates the optimal mix of taxes on two factors of production, capital and (efficiency-weighted) labor. More precisely, we analyze the optimum proportionate taxation of wages and interest, or wages and wealth, with account taken of the public debt. The vehicle for analysis is a blend of Mirrlees's theory of efficiency-weighted labor supply and the life-cycle theory of the supply of wealth

* New York University and Columbia University, respectively. The present paper is an abridged version of a much more extensive working paper cited in the bibliography. Ordover gratefully acknowledges partial support from the National Science Foundation grant to New York University.

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found, for example, in the growth model constructed by Peter Diamond.[1]

The particular optimization problem studied here possesses two special features. The social welfare function we employ expresses the conception of economic justice championed by John Rawls: As a chain is no stronger than its weakest link, social welfare is only as great as the utility of the least well-off, the person or persons having minimum utility. Thus the redistributional optimum is "maximin."

The other feature is a restriction on the dynamics of capital and public debt. The basic notion is also found in a related study of interest taxation by Ordover, in which society is constrained to maintain at some predesignated level the ratio of capital to (geometrically progressing) population. We may not know to what quantity that ratio ought "in justice" to be equal. But whatever that quantity and whenever it is to be attained, society will want then to have that mix of taxes (and resulting demogrant) which is economically just *intragenerationally*, subject to the mandate that fiscal policy preserve this capital-population ratio for the use of future generations. As for the ratio of public debt to population, this too is to be maintained constant from generation to generation, but at a preoptimized rather than arbitrary level.

The present paper is more extensive. We first formulate the "general" (steady-state) problem in which both the public debt and the capital stock per worker are arbitrary constants. We then report our analysis of the optimal tax structure in the aforementioned problem and in the diagonally opposite problem—where the debt per worker is fixed arbitrarily and *capital* per worker is maintained intact at its optimal steady level. Lastly we cite some results obtained for the joint problem of the Golden Rule steady state in which both capital and debt per worker are maintained at their best steady levels.

The paper here also differs with regard to the channels of tax "distortion." As in the original Mirrlees paper, the disincentive effect of taxation falls on the quantity of hours worked (as well as the volume of private saving) rather than the quantity of investment in education, as in the "Polish view" adopted by Ordover and earlier Eytan Sheshinski. As a consequence the supply of efficiency-labor depends not only upon the after-tax wage rate paid to a unit of efficiency-work but also upon the after-tax rate of return to saving—the latter a factor emphasized by several general-equilibrium theorists from Hicks to Lucas.

Our emphasis is on the theory of optimal taxation under market capitalism, with its obvious agenda of questions: Should profits be taxed? Should they be taxed more heavily than wages? Might it be optimal to levy a proportionate wealth tax at a rate exceeding the average rate of profit on capital?[2]

The model has relevance to the theory of market socialism as well as to the welfare economics of capitalist taxation. Karl Marx, Oskar Lange, and some other socialist theorists explained that the returns to capital garnered by the state after payment of wages would be distributed to the people as an equal poll-subsidy/social-divided after subtraction of the state's expenditures on final goods plus any budgetary surplus appropriate to its plans

[1] Koichi Hamada has studied a different problem in the same setting.

[2] The answers are that "it depends"—depends upon the quantities of capital and public debt per worker and upon the elasticities of factor supplies and demands. The novelty here is that the answers are "technical," being reasoned from an explicit ethical postulate, namely, the maximin criterion. (Use of this analytical approach does not imply indifference or obliviousness to institutions, considered as ends or as means.) In some other approaches, it almost seems as though there is intended to be some kind of justice between people and machines, an animism in which machines usually are held morally inferior to humans.

for capital formation. They did not tell us whether a household might be allowed to accumulate wealth claims to future goods; nor did they say what rate of return—positive, zero, or negative—such abstract property, if allowed, ought to be paid for maximum social welfare. The present paper sheds some light on those unanswered questions.

I. The Model

We first discuss production possibilities, next markets and accounting, then consumer behavior.

A. *Resources and the Production Set*

Each generation lives for two periods. Its members work, if at all, over the first period; they consume, if at all, at the end of each of these two periods. The size of the working age population, N, grows exogenously at the geometric rate $n \geq 0$:

$$(1) \qquad N_{-1} = (1+n)^{-1}N$$

Each per worker variable used below is a certain aggregate divided by current N. We shall be constraining the economy to maintain a steady state in which by definition each such per worker variable is equal from generation to generation.[3]

The per worker quantity of effective labor supplied by the current workforce, denoted by l, is measured in standardized units of "efficiency man-hours." It is an endogenous variable, a function of the after-tax prices and transfers. The efficiency of each person's hours worked is measured by a parameter m that ranges from zero to some $M>0$. The history of any individual of type m is denoted by

$$(2) \qquad x^m = (x_0^m, x_1^m, x_2^m, x_3^m), \qquad m \in [0, M]$$
$$x_0^m + x_3^m = 1, \quad x_j^m \geq 0 \quad \text{all } j$$

where x_0^m is first period leisure, x_i^m is consumption at the end of age i ($i=1, 2$), and x_3^m is "effort" or "worktime" in the first period of life. Worktime is measured in *natural units*, namely as a fraction of the duration of the period.

The basic postulate here is that efficiency differences among individuals are purely labor augmenting; that is, the marginal rate of substitution in production between a man-hour of type m_1 and a man-hour of type m_2 is "constant," given simply by m_1/m_2, independent of the quantities of capital and labor types working alongside. It follows that there exists a labor aggregate which adds up the various man-hours worked when converted to efficiency units. To obtain from the quantity of effort x_3^m, the amount of labor service in efficiency units that it supplies, say l^m, we need only multiply the former by m.[4] Thus we are standardizing on the productivity of the $m=1$ types. Now let the proportion of the population of any and every generation having an efficiency *m or less* be given by a nondecreasing cumulative-distribution function, Φ, with the properties

$$(3) \qquad 0 < \Phi(0) < 1, \ \Phi(M) = 1,$$
$$\Phi(a) \leq \Phi(b) \qquad \text{if } 0 \leq a < b \leq M$$

Then per worker effective labor supplied is[5]

$$(4) \qquad l \equiv \Sigma l^m = \Sigma m x_3^m$$

and the other per worker quantities are

$$(5) \qquad x_i \equiv \Sigma x_i^m, \qquad i = 0, 1, 2, 3$$

We posit a standard one-sector neoclassical net aggregate output function, F. In any steady state with constant k, a certain portion nk of per worker net output,

[3] Note that N is the potential number of workers in the market sector, an upper bound on the number who choose to work.

[4] There is no loss of generality in multiplying by m instead of a monotone function of m because the frequency function of m can be adjusted as needed.

[5] For convenience of notation we replace the integral sign by the summation sign.

$F(k, l)$, must be allocated to net investment to keep k constant. Per worker government consumption, $\gamma \geq 0$, is likewise constant. The remaining fraction of net output is available for per worker consumption c.[6] The latter is distributed in a steady state between the per worker consumption of current workers and retired workers thus:[7]

$$c = x_1 + x_2(1 + n)^{-1} \tag{6}$$

Hence steady-state growth with constant k entails the *capital constraint*

$$F(k, l) - x_1 - x_2(1 + n)^{-1} = \gamma + nk = \text{constant} \geq 0 \tag{7}$$

B. *Market Organization, Prices, and Social Accounting*

We turn now to the demand side of the model and to the market aspects.[8] There are three goods and we may think of two perfect markets: a labor market where labor (homogeneous after conversion to efficiency units) is exchanged for present consumables, and a capital market where savings (unexercised claims to present goods) are traded for claims to future goods.

So far as proportionate taxation of households' market demands and supplies are concerned, there is no loss of generality in our assuming that expenditures on the commodities (x_1 and x_2) are untaxed. Hence let us consider a proportionate tax rate on wage income, denoted τ_w, and a proportionate tax rate on profits (or interest) from saving, τ_r. (We can also entertain the possibility of a proportionate tax on private wealth.) In the "natural" interpretation, then, our two markets *and* taxation determine *four* relative prices of the two factors: The before-tax real wage rate w, the after-tax real wage rate ω, the rate of return before tax on real private savings r, and the after-tax rate of return ρ. The term "real" denotes deflation by the "money" price of presently consumable commodities, and the term "wage rate" refers to the rate for a standard individual, the real wage per efficiency unit of labor.

We postulate the universal lifetime income guarantee, or demogrant, which comes in two installments, β_1 and β_2. For the moment we think of β_1 and β_2 as each nonnegative. Accordingly there is an obvious lifetime budget constraint for households of *any* type m which, when aggregated, yields the relation

$$(1 + \rho)^{-1}(x_2 - \beta_2) = \omega l - (x_1 - \beta_1) \tag{8}$$

Note that the right-hand side of (8) equals the per worker purchases of wealth by the present working-age generation. But aggregate wealth in a steady state must grow geometrically at rate n. Therefore, if d denotes the per worker stock of public debt existing at the beginning of the period, and so $k+d$ is the per worker quantity of wealth, then the workers must purchase $(1+n)(k+d)$ of wealth in order that per worker wealth be unchanged next period. Hence the *private wealth constraint:*

$$(1 + \rho)^{-1}(x_2 - \beta_2) = (1 + n)(k + d) = \text{constant} > 0 \tag{9}$$

Equations (8) and (9) imply the *social budget constraint:*

$$x_1 + \frac{x_2}{1 + n} = \beta_1 + \frac{\beta_2}{1 + n} + \omega l + \rho(k + d) - n(k + d) \tag{10}$$

[6] Obviously the requirements of growth and public expenditure may be too great to admit a solution with nonnegative consumption and efficiency-weighted leisure.

[7] In a state of steady growth at rate n, the (future) second-period consumption x_2 of the present generation of first-period people is $(1+n)$ times the second-period consumption of people now in their second period.

[8] For an extensive discussion of these matters, the interested reader may consult the authors' working paper.

This states that, with regard to aggregates, consumption equals disposable income *less* private saving. Let us also observe that, expenditure taxes nil, national output equals national income:

$$(11)\qquad rk + wl = F(k, l)$$

Hence, from (7)

$$(12)\qquad x_1 + \frac{x_2}{1+n} = rk + wl - \gamma - nk$$

Subtracting (10) from (12) we obtain the government's income statement

$$(13)\qquad (r-\rho)k + (w-\omega)l - \rho d = \beta_1 + \frac{\beta_2}{1+n} + \gamma - nd$$

The left-hand side is net tax revenue—net, that is, of after-tax interest on the public debt; this must cover public benefits and expenditures *less* the budgetary deficit. In (13), the deficit is constrained to maintain the given debt-worker ratio d_0, however determined—whether predetermined by actual history or hypothetically optimized. Hence (13) expresses a *public debt constraint* that, using (11), we may state more explicitly and analogously to (7) in the form

$$(14)\qquad F(k,l) - \rho(k+d) - \omega l - \beta_1 - \frac{\beta_2}{1+n} = \gamma - nd_0 = \text{constant}$$

The capital constraint in (7) prescribes a certain amount of national saving while the debt constraint prescribes how much of this shall be public saving, $-nd_0$, in lieu of private saving.[9]

C. *Consumer Behavior*

Regarding consumers, we suppose that households of any type m choose $x^m = (x_1^m, x_2^m, x_3^m)$ to maximize utility $u(x^m)$, which is independent of m, subject to their budget constraint—like (8), expressed here in terms of present value or discounted prices, $q^m = (q_1, q_2, q_3^m)$. The latter are defined thus:

$$q_1 \equiv 1, \quad q_2 \equiv (1+\rho)^{-1}, \quad q_3^m \equiv m\omega, \quad q_3 \equiv \omega$$

In these terms the maximization problem is

$$(15)\qquad V^m(q^m; \beta_1, \beta_2) = \max_{\{x^m\}} \left\{ u(x^m) - \alpha^m \left[\sum_1^2 q_i^m (x_i^m - \beta_i) - q_3^m x_3^m \right] \right\}$$

V^m is the indirect utility function, homogeneous of degree zero in the prices, with derivatives

$$(16)\qquad \begin{aligned} \partial V^m/\partial q_3^m &= \alpha^m x_3^m \\ \partial V^m/\partial q_2^m &= -\alpha^m x_2^m \\ \partial V^m/\partial \beta_i &= \alpha^m q_i \qquad i = 1, 2 \end{aligned}$$

where the Lagrange multiplier $\alpha^m > 0$ is the marginal utility of (present) money.

From the households' utility maximizations we can obtain the individual demand functions for x_1^m, x_2^m, and the individual supply functions for saving $\sigma^m \equiv q_2(x_2^m - \beta_2)$, and effective labor l^m. Aggregating these demands and "factor" supplies, we obtain the per worker market demands for x_1 and x_2, and the market supplies of labor l and wealth $\sigma \equiv q_2(x_2 - \beta_2)$. These market demands and supplies are each a function, given β_2, of the triplet $(1, q_2, q_3, \beta_1)$. These market functions must obey the lifetime budget constraint in (8), which we write here

$$(8')\qquad q_2(x_2 - \beta_2) = q_3 l - x_1 + \beta_1 \equiv \sigma$$

Let σ_i denote $\partial\sigma/\partial q_i$, and similarly for l_i, x_{1i}, and x_{2i}. Then differentiation of this budget equation yields

[9] The constraint on public saving in (14) could be replaced by the constraint on *current* private *all-generation* saving in (10) or, as well, by the constraint on after-interest private wealth in (9). But the overlife constraint on the present generation only is no substitute for these social constraints.

$$(17)\quad x_2 - \beta_2 + q_2x_{22} = 0 + q_3l_2 - x_{12} \equiv \sigma_2$$
$$q_2x_{23} = l + q_3l_3 - x_{13} \equiv \sigma_3$$
$$q_2x_{2\beta} = 1 + q_3l_\beta - x_{1\beta} \equiv \sigma_\beta$$

To narrow the analysis that follows, we wish to place some natural restrictions on the functions x_1, x_2, l, and σ. Thus, we shall suppose that all goods are strictly noninferior, subject to some inessential qualifications at boundaries, so that

$$(18)\quad x_{i\beta}>0,\ 0<q_ix_{i\beta}<1,\quad i=0,1,2;\ x_{3\beta}<0$$
$$l_\beta<0,\ \sigma_\beta>0\qquad (\text{for } q_2<\infty,\ q_3>0)$$

Concerning the wealth supply function we shall assume, *first*, that $\sigma(1,\infty,q_3,\beta_1)=0$: that is, no one will save at the confiscatory $\rho=-1$. (Since there may be public saving, the latter does not guarantee that $\rho=-1$ is inoptimal.) *Second*, a rise of the wage rate increases saving everywhere. Then, for all $q_2<\infty$ and $q_3>0$,[10]

$$(19)\qquad \sigma_3(1,q_2,q_3,\beta_1) = q_2x_{23} > 0$$

Third, because the "marginal propensities" $q_ix^m_{i\beta}$ are between zero and one, by noninferiority, we can also argue that

$$(20)\quad \sigma_2(1,q_2,q_3,\beta_1) = \begin{cases} <0 & \text{at } q_2=\infty \\ >0 & \text{at } q_2=0 \end{cases}$$

The argument is that σ increases with ρ in the neighborhood of $\rho=-1$, because there is no income effect to counter the positive substitution effect of larger ρ. Hence, σ "bends backward" at sufficiently large ρ (small q_2) in the (σ,ρ) plane and likewise in the (σ,q_2) plane.

Concerning the labor supply function we shall assume, *first*, that $l(1,q_2,0,\beta_1)=0$: no one will work at $\omega=0$. *Second*, we specify that

$$(21)\quad l_2(1,q_2,q_3,\beta_1) = \begin{cases} \lesseqgtr 0 & \text{at } q_2=\infty \\ >0 & \text{at } q_2=0 \end{cases}$$

[10] An interpretation of this inequality, though not a necessary implication, is that leisure and second-period consumption are everywhere net substitutes.

As $q_2\to\infty$ the income effect of higher q_2 becomes negligible, so l_2 has the sign of the cross-substitution effect, positive or negative. If negative then there must hold a *Hicks-Lucas* effect $l_2<0$ at sufficiently large q_2—that is, labor supply must be increasing in ρ around $\rho=-1$. As $q_2\to 0$, $x_2\to\infty$, so the income effect of a rise in q_2 is presumed to be decisive, decreasing leisure and increasing labor supply, whatever the sign of the cross-substitution effect. We shall refer to $l_2>0$ as the *anti-Hicks-Lucas effect.*

Finally, we have

$$(22)\quad l_3(1,q_2,q_3,\beta_1) = \begin{cases} >0 & \text{at } q_3=0 \\ =0 & \text{at } q_3=\infty \end{cases}$$

The labor supply function is increasing at $q_3=0$, where $l(\cdot)=0$, and it may bend backward at sufficiently large q_3.

II. The Analysis

Our problem is the selection of tax rates on wages and interest (or wealth) so as to maximize over feasible steady states the minimum level of utility, $\min_m V^m(\cdot)$. This bottom utility must equal $V^0(\cdot)$, for no one of any type m could do worse than the $m=0$ types. As is widely done in the optimal taxation literature, we actually maximize $V^0(\cdot)$ with respect to the after-tax factor rewards (ρ,ω) or equivalently (q_2,q_3), as if the optimization were for a socialist state. This is merely a calculating device. To each pair (ρ,ω) there corresponds a unique tax-rate pair (τ_r,τ_w) and the latter is often readily solved for.

This maximization is subject in the "general" case to any *two* of the three constraints—(7) on capital, (10) on private wealth supply, (14) on public debt. In terms of the first two of these constraints, the general problem can be expressed for any given β_2 as

$$(23)\qquad \underset{(q_2,q_3,\beta_1)}{\text{maximize}}\ V^0(q_2,q_3,\beta_1;\beta_2)$$

subject to

$$(23') \quad nk = F(k, l) - (q_3 l + \beta_1 - \sigma) - (q_2^{-1}\sigma + \beta_2)(1+n)^{-1} - \gamma \equiv J(q_2, q_3, \beta_1; k, \beta_2)$$

$$(23'') \quad (1+n)(k+d) = q_2(x_2 - \beta_2) \equiv \sigma(q_2, q_3, \beta_1; \beta_2)$$

In (23′) we have obviously substituted for x_1 from the lifetime budget constraint (8′) in terms of l and σ. Similarly, we represent x_2 by the same function of $(q_2, q_3, \beta_1, \beta_2)$, namely $q_2^{-1}\sigma+\beta_2$, as gives the *ex ante* x_2 of the current workers on the ground that the prices established must be the same over generations in any steady state.

The first-order conditions for a maximum in (23) involve the first derivatives V_2^0, V_3^0, and V_β^0 of the indirect utility function V^0. We have

$$(24) \quad V_2^0 = -(x_2^0 - \beta_2)V_\beta^0, \quad V_3^0 = l^0 V_\beta^0 = 0, \quad V_\beta^0 > 0$$

Here, V_β^0 is the marginal utility of β_1 to the poor; V_3^0 is zero because the $m=0$ type cannot earn wages.

The sign of V_2^0 is indeterminate until β_2 is specified. We shall confine the analysis reported here to the following special case: The parameter β_2 has been fixed such that, *at the maximum in* (23),

$$(25) \quad \beta_2 = x_2^0 \geq 0$$

To motivate (25) it might be argued that the government would want β_2 large enough to lift from the poor the burden of providing for their old age out of their β_1 yet not so large that the poor are induced to borrow against future β_2 for additional first-period consumption.

Consequently $V_2^0 = V_3^0 = 0$ in the neighborhood of the maximum. A change of q_2 or q_3 in this neighborhood will therefore increase $V(\cdot)$ only by increasing β_1 via one of the constraints. Hence the pair (q_2, q_3) which maximizes $V(\cdot)$ in (23) must be simultaneously maximizing β_1.

The maximization of β_1 may be formulated as a problem in what might be termed "stationary programming." Noting that $J_\beta < 0$, let us assign β_1 to meet the capital constraint in (23′). Then choose (q_2, q_3) to maximize β_1 so determined, subject to the wealth constraint in (23″). (Because $\sigma_3 > 0$ it is also natural to think of q_3 as tied to the wealth constraint, thus making q_2 the variable free for decision.) The *maximum* value of β_1 depends only on the exogenous k and d (given β_2 and the other parameters) and is denoted $b(k, d)$:

$$(26) \quad b(k, d) = \max_{\{q_2, q_3\}} \beta_1$$

subject to (23′) and (23″).

From (26), (23′), and (23″) we then obtain the functional equation

$$(27) \quad 0 = \max_{\{q_2, q_3\}} \{J(q_2, q_3, b(k, d); k) - nk - \lambda[\sigma(q_2, q_3, b(k, d)) - (1+n)(k+d)]\}$$

In this maximization, $b(k, d)$ and the Lagrange multiplier $\lambda(k, d)$ are "constants," independent of (q_2, q_3), being functions only of the predetermined state variables (k, d) and the other parameters.

The interpretation of (27) is clear. To maximize β_1, (q_2, q_3) are chosen, subject to $\sigma = (1+n)(k+d)$, to make $J(\cdot)$ as large as possible. For since $J_\beta < 0$, the larger is $J(\cdot)$ at any β_1, the higher is the β_1 that keeps $J(\cdot) = nk$.

At the maximum β_1 the following first-order conditions hold:

$$(28a) \quad J_2 - \lambda\sigma_2 = 0, \qquad J_3 - \lambda\sigma_3 = 0$$

$$(28b) \quad J_\beta - \lambda\sigma_\beta \leq 0$$

The argument for (28b) is that if it did not hold there would be room for some free self-sustaining rise of β_1 which would increase $\lambda\sigma(\cdot)$ by so much as to permit (via q_2 and q_3) a net increase of $J(\cdot)$, hence still larger β_1; but such is impossible at the maximum β_1.

By equating to zero the total derivative with respect to d of the right-hand side of (27) we also obtain the "marginal worth" of an increase in the debt per worker:

$$(29) \qquad b_d(k, d) = \frac{\lambda(1+n)}{-J_\beta + \lambda\sigma_\beta}$$

It follows from (28b) that λ and $b_d(k, d)$ are like-signed—positive, negative, or zero as the debt is too small, too large, or just right. It may be noted that λ and hence $b_d(k, d)$ have the sign of J_3 by virtue of (28a) and the specification that $\sigma_3 > 0$. If optimal $J_3 < 0$, for example, a larger debt, by entailing larger $\sigma(\cdot)$ and therefore larger q_3 at each q_2, would result in smaller $J(\cdot)$ at any given β_1 and thus imply a reduction of maximum β_1, that is, $b_d(k, d) < 0$.

A. *Optimal Taxes with Arbitrary Debt and Capital*

Space permits only fleeting attention here to the tax implications of the general problem as studied in our working paper. The succeeding sections on the more restrictive problems will serve to indicate the kinds of propositions developed and to convey the analytics employed.

For use in Sections IIA, B, and C, respectively, we define these *specific* (as distinct from *ad valorem*) tax rates:

$$(30a) \qquad t_\lambda = (1-\lambda)(1+\rho)^{-1} - (1+n)^{-1}$$
$$= [n - \lambda(1+n) - \rho]q_2(1+n)^{-1}$$

$$(30b) \qquad t_n = (1+\rho)^{-1} - (1+n)^{-1}$$
$$= (n-\rho)q_2(1+n)^{-1}$$

$$(30c) \qquad t_r = (1+\rho)^{-1} - (1+r)^{-1}$$
$$= (r-\rho)q_2(1+r)^{-1}, \qquad r \equiv F_k$$

$$(30d) \qquad t_w = w - \omega, \quad \tau_w = t_w\omega^{-1} \quad w \equiv F_l$$

Evidently t_n is a kind of "shadow" tax rate. It measures the wedge between ρ, the after-tax rate of return to saving, and n, the "natural" rate of interest—essentially, the marginal rate of transformation between x_1 and x_2 in the equation $J - nk = 0$. For $\lambda \neq 0$, however, t_λ is the analogous shadow tax rate. It appears if we write our maximand in (27) as

$$(27') \qquad \underset{\{q_2, q_3\}}{\text{maximize}}\ F(k, l) - q_3 l + t_\lambda x_2 + \text{constant}$$

where again the functions $l(q_2, q_3, \beta_1)$ and $x_2(q_2, q_3, \beta_1)$ are to be evaluated at $\beta_1 = b(k, d)$. The derivative of the maximand with respect to efficiency-labor is $t_w = F_l - q_3$ and the derivative with respect to x_2 is t_λ.

The first-order conditions for a maximum in (28) can thus be expressed as:

$$(31) \qquad l_2 t_w + x_{22} t_\lambda = -(1-\lambda)(x_2 - \beta_2)$$
$$l_3 t_w + x_{23} t_\lambda = l$$
$$l_\beta t_w + x_{2\beta} t_\lambda \leq 1$$

While a few results, largely in the nature of logical possibilities, are derivable from (31), we proceed now to the more restrictive problem with preoptimized debt in Section B and with preoptimized capital in Section C.[11]

B. *Optimal Taxes with Optimized Debt*

In the problem here, public debt per worker is a free variable. The portion of investment comprised of *private* saving is unconstrained, subject only to *nonnegativity* of private wealth supply. In the special case defined in (25) therefore, where $V_2^0 = 0$, the optimal tax problem may be

[11] An amusing footnote to the general problem of exogenous k and d is its Cantabridgian theory of the functional distribution of *disposable* national income between interest, $\rho(k+d)$, and wages, ωl:

> *Cambridge Lemma*: Because all interest (after tax) is consumed while some wages (after tax) are saved, the functional distribution of income (after tax), $\{\rho(k+d), \omega l\}$ must favor wages just enough to insure that private saving equal $n(k+d)$.

Of course, it is the factor unit rewards, ω and ρ, which are of policy significance.

expressed as

$$(32) \qquad nk = \max_{\{q_2,q_3\}} J(q_2, q_3, b(k); k)$$

subject only to

$$(32') \quad 0 \leq q_2(x_2 - \beta_2) \equiv \sigma(q_2, q_3, b(k))$$

where $b(k)$ is the maximized β_1.

At an interior solution, where the nonnegativity constraint is not binding, the first-order conditions are

$$(33) \qquad J_2 = 0, \quad J_3 = 0; \quad J_\beta < 0$$

A subsequent proposition will address the possibility of a corner solution at $\sigma(\cdot)=0$.

The following properties of the solution can now be shown.

PROPOSITION B1: *Optimal $\omega>0$ whether or not optimal $\sigma(\cdot)>0$.*

If $\omega=0$, $l=0$ whence $J(\ )<0<nk$ which is inadmissible.

PROPOSITION B2: *The corner solution at $\sigma(\cdot)=0$ occurs if and only if optimal $\rho=-1$.*

First note that $\sigma(\rho, \omega, \beta)=0$ for all ω and β if $\rho=-1$, that is, $q_2=\infty$. Conversely, the existence of a solution at $\sigma(\cdot)=0$ implies $\rho=-1$. For $\omega>0$ by Proposition B1, $\beta_1>0$ for the solution to permit $0<x_1^0(=\beta_1)$, while $\sigma(\rho,\ \omega>0,\ \beta_1>0)>0$ for all $\rho>-1(q_2<\infty)$. So the optimal allocation will display property owning if optimal $\rho>-1$.

PROPOSITION B3: *Optimal $l_3>0$ whether the solution is at the corner or is interior.*

At an interior optimum, $J_3=0$. At the corner solution where $\sigma(\cdot)=0$, $\rho=-1$ so that $\sigma_3=0$ for all ω. Hence $J_3=0$ at a corner solution as well. Recall now that

$$J(\cdot) = F(k, l) - x_1 - x_2(1+n)^{-1} - \gamma$$

Since $F_l>0$, $x_{13}>0$, and $x_{23}\geq 0$, $l_3\leq 0$ would imply $J_3<0$, a contradiction. The economics here is that at an allocation on some backward-bending segment of the labor-supply curve, where $l_3<0$, it would be possible to drop ω to a rising segment for a gain in $J(\cdot)$ at the initial β_1 and thus obtain some increase in β_1 without violating $J(\cdot)=nk$.

PROPOSITION B4: *Optimal $\rho=-1$ implies $l_2\geq 0$ (no Hicks-Lucas effect) in that neighborhood. Hence the presence of a Hicks-Lucas effect of larger ρ, at least around $\rho=-1$, is sufficient to rule out a corner solution.*

Using the budget relation $x_1=\omega l+\beta_1-(1+\rho)^{-1}(x_2-\beta_2)$ we write

$$(34) \qquad J(\cdot) = F(k, l) - \omega l - \beta_1 + (x_2-\beta_2)(1+\rho)^{-1} - x_2(1+n)^{-1} - \gamma$$

At the corner where $\rho=-1$ and thus $x_2=\beta_2$ the first-order conditions for a constrained maximum are

$$(35a) \quad \left.\frac{\partial J}{\partial \rho}\right|_{\rho=-1} = (F_l - \omega)\frac{\partial l}{\partial \rho} + t_n \frac{\partial x_2}{\partial \rho} \leq 0$$

$$(35b) \quad \left.\frac{\partial J}{\partial \omega}\right|_{\rho=-1} = (F_l - \omega)\frac{\partial l}{\partial \omega} - l = 0$$

recalling that $x_{23}=0$ at $\rho=-1$. Now $t_n\equiv(1+\rho)^{-1}-(1+n)^{-1}>0$ at $\rho=-1$. And $t_w=F_l-\omega>0$ since $l>0$ for a solution and $l_3>0$ by Proposition B3. Also $\partial x_2/\partial\rho\geq 0(x_{22}\leq 0)$ around $\rho=-1$. Hence (35a) implies $\partial l/\partial\rho\leq 0$ which is non-Hicks-Lucas. Actually, if $\partial x_2/\partial\rho>0$, $\partial l/\partial\rho<0$.

In the remainder of this section we shall be assuming that the solution is interior at some $\sigma(\cdot)>0$. This means that the labor-supply function is Hicks-Lucas at least around $\rho=-1$. Then $\rho>-1$ and the first-order conditions in (33), expressed in terms of $(q_2,\ q_3,\ \beta_1)$ and the tax rates (t_w, t_n), are

$$(36a) \quad l_2 t_w + x_{22} t_n = -(x_2 - \beta_2) < 0$$

$$(36b) \quad l_3 t_w + x_{23} t_n = l > 0$$

$$(36c) \quad l_\beta t_w + x_{2\beta} t_n < 1$$

PROPOSITION B5: *Either t_w or t_n or both are positive.*

Since $l_3>0$ and $x_{23}>0$, $t_n\leq 0$ implies $t_w>0$ from (36b).

PROPOSITION B6: *If $l_2\geq 0$ at the optimum, then $t_n>0(\rho<n)$.*

Assume the contrary, that $t_n\leq 0$. Then $t_w<0$ when $l_2\geq 0$ by (36a). But $t_w>0$ if $t_n<0$ by (36b)—Proposition B5. Hence $t_n>0$ if $l_2\geq 0$.

To the same end, we may solve (36a) and (36b) for t_n and t_w:

$$(37)\qquad t_n=\frac{ll_2+(x_2-\beta_2)l_3}{l_2x_{23}-l_3x_{22}}$$

$$t_w=\frac{-(x_2-\beta_2)x_{23}-lx_{22}}{l_2x_{23}-l_3x_{22}}$$

If $l_2\geq 0$, then the common denominator is positive and so is t_n. But t_w may apparently be of either sign.

The presence of anti-Hicks-Lucas effect $l_2\geq 0$ at the optimum rules out $\rho>n$ as we have just seen. One might expect also that it precludes a solution where σ is backward-bending: that is, $\partial\sigma/\partial\rho<0$ or $\sigma_2>0$. But we can prove only the conditional statement:

PROPOSITION B7: *Should the optimum be anti-Hicks-Lucas ($l_2\geq 0$), then $\sigma_2<0$ if optimal $t_w\geq 0$.* COROLLARY: *If $\sigma_2\geq 0$ at such optimum, then $t_w<0$.*

To prove, we note that

$$(38)\quad J_2=t_wl_2+\sigma_2-(1+n)^{-1}x_{22}=0$$

Then $t_w\geq 0$ and $x_{22}<0$ imply $\sigma_2<0$. The corollary is immediate and may be interpreted as follows: An increase of ρ within any backward-bending stretch of σ must, apart from its effect on labor supply, reduce $J(\cdot)$ because it must reduce $t_n(x_2-\beta_2)$ —certainly as long as $t_n>0$, as it must be when $l_2\geq 0$. If this increase is optimal to make when it would reduce labor supply ($\partial l/\partial\rho<0$), it must be that t_w is negative.

By the same reasoning, if $\partial l/\partial\rho>0$, pushing ρ into a backward-bending stretch of σ would seem to be plausible only if $t_w>0$, so that q_3l rises by less than $F(\cdot)$; but here we have to watch for the possibility that $t_n<0$ so that x_{22} is another "plus" in the case for high ρ. However, we can prove

PROPOSITION B8: *If $\sigma_2\geq 0$ at the optimum, then $l_2t_w<0$ so that t_w and $\partial l/\partial\rho$ are like-signed.*

From (38) it follows that $t_wl_2=-\sigma_2+(1+n)^{-1}x_{22}<0$.

The foregoing analysis may have left the false impression that l_2 can be of any sign *independently* of ρ and ω. In our model, the Hicks-Lucas effect $l_2<0$ requires that first-period leisure and second-period consumption are net substitutes, in the aggregate at any rate, and this substitution effect of a rise of ρ, which encourages less leisure, overcomes the income effect, which encourages more leisure. Our assumptions so far pose no bar to considering the possibility that at least for all $\rho<n$ the income effect is swamped by the substitution effect so that in that region $l_2<0$ (Hicks-Lucas). This leads to the result:

PROPOSITION B9: *If $l_2<0$ for all $\rho<n$, then optimal $l_2<0$. It is then possible that optimal $\rho>n$ and $t_w>0$.*

The proof is by contradiction. If optimal $l_2\geq 0$, optimal $\rho<n$ (Proposition B6). But, by supposition, $l_2<0$ at every $\rho<n$. Hence optimal $l_2\geq 0$ is impossible. As before, optimal $l_2<0$ leaves the signs of t_n and t_w indeterminable from (36a)–(36c). However, it is clear from (37) that $t_n>0$ for some $l_2<0$.

Suppose the optimum is indeed such that $t_n<0$ ("machines are subsidized") and $t_w>0$ ("toil is taxed"). Where is the justice in that? The only possible justification for raising ρ beyond the biologic rate n is its Hicks-Lucas stimulus to the supply of output, $F(k, l)$. Undoubtedly many critics of the "socialist" goal of propertylessness, and of the "humanist" opposition to interest income, have had this "model" implicitly in mind: Prohibitions against property holding and heavy taxation of interest *may* so dampen income-earning incentives as to be inefficient in the maximization of any Bergson-type social welfare function.

Finally, we state the following results although the proof will have to be omitted here:

PROPOSITION B10: *If first-period leisure and second-period consumption are "independent" goods or else net complements for all households at the optimum, then $t_r>0$ and $t_w>0$.*

C. *Optimal Taxes with Optimized Capital*

In the previous section, capital was fixed and the per capita debt is free to be set at its optimal stationary level. The latter is calculable by regarding d as endogenously determined by $(1+n)^{-1}\sigma(\cdot)-k$. In Act 2, as it were, the shoe is on the other foot. The debt per worker is fixed and capital per worker is allowed to realize its optimal level, determined by

$$(39)\quad k = (1+n)^{-1}\sigma(\cdot) - d \equiv \sigma^*(\cdot) - d$$

when evaluated at optimal q_2, q_3, β_1. It may be noticed that the notion of a welfare-maximizing steady-state k is just the Golden Rule exercise with the new wrinkle of a government deficit constraint. But it is with the optimal taxes in this steady state that we are concerned.

In the special case where $V_2^0=\beta_2-x_2^0=0$, our new problem may be formulated analogously to (32):

$$(40)\quad -nd = \max_{\{q_2, q_3\}} P(q_2, q_3, b(d); d)$$

in which $P(\cdot)$ is defined as the government's algebraic budgetary surplus per worker:

$$(41)\quad P(\cdot) = F[(\sigma^* - d), l] - q_2^{-1}(1-q_2)\sigma^* - q_3 l - \beta_2(1+n)^{-1} - \gamma - \beta_1$$

$P(\cdot)$ is to be evaluated at the maximum $\beta_1=b(d)$. The maximization in (40) is constrained by $\sigma^*-d=k\geq 0$ and $\sigma^*\geq 0$. In the following analysis it is assumed that an interior solution exists that makes both k and σ^* positive.

The first-order conditions for an interior maximum are

$$(42)\quad P_2 = 0, \quad P_3 = 0, \quad P_\beta \leq 0$$

Equations (41) and (42) yield

$$(43a)\quad l_2 t_w + x_{22}(1+r)(1+n)^{-1}t_r = -(1+r)(1+n)^{-1}(x_2 - \beta_2) < 0$$

$$(43b)\quad l_3 t_w + x_{23}(1+r)(1+n)^{-1}t_r = l > 0$$

$$(43c)\quad l_\beta t_w + x_{2\beta}(1+r)(1+n)^{-1}t_r \leq 1$$

where, recalling (30c), $t_r \equiv (r-\rho)(1+\rho)^{-1}\cdot(1+r)^{-1}$ and $r=F_k$.

A less cumbersome way to write (43) uses $\rho=q_2^{-1}(1-q_2)$ and $\omega=q_3$:

$$(44a)\quad \frac{\partial l}{\partial \rho}\cdot(F_l - \omega) + \frac{\partial \sigma^*}{\partial \rho}(F_k - \rho) = \sigma^*$$

$$(44b)\quad \frac{\partial l}{\partial \omega}\cdot(F_l - \omega) + \frac{\partial \sigma^*}{\partial \omega}(F_k - \rho) = l$$

$$(44c)\quad l_\beta\cdot(F_l - \omega) + \sigma_\beta^*(F_k - \rho) \leq 1$$

We state the following results with remarks on the proofs.

PROPOSITION C1: *Neither $\rho=-1$ nor $\omega=0$ is optimal.*

This follows from our assumptions that $\sigma^*>0$ and $\sigma^*-d>0$.

PROPOSITION C2: *If optimal $l_3<0$, then $t_r>0$.*

In (43b), $t_w l_3 - l<0$ if $l_3<0$ because $l+l_3 q_3>0$.

PROPOSITION C3: *Either $t_r>0$ or $t_w>0$ or both.*

If $l_3<0$, $t_r>0$. If $l_3\geq 0$, $t_w>0$ or $t_r<0$ by (43b). If $l_3<0$, nothing new follows if also $l_2\geq 0$. But

PROPOSITION C4: *If optimal* $l_2>0$, *then* $t_r>0$.

If $t_w\leq 0$, $t_r>0$ by C3. If $t_w>0$, and $l_2>0$, then $t_r>0$ by (43a).

Section B was greatly simplified by the result that $l_3>0$. It hinged on the result that $\partial[F(k, l)-\omega l]/\partial q_3$ is positive. In a symmetrical world, it would be true here that $\sigma_2<0$, that is, $\partial\sigma/\partial\rho>0$. The hitch is apparently that we do not *generally* have $\partial[F(k, l)-\omega l]/\partial q_2$ positive. Hence very low q_2 might be repaid from the labor end. Nevertheless, suppose optimal $t_w l_2\geq 0$. Then (43a) yields

$$(45)\quad [q_2-(1+r)^{-1}]x_{22}+x_2-\beta_2 = -l_2t_w(1+r)^{-1}(1+n)<0$$

Hence, from (17),

$$(46)\quad \sigma_2=-l_2t_w(1+r)^{-1}(1+n)+(1+r)^{-1}x_{22}<0$$

This leads to the result:

PROPOSITION C5: *If optimal* $l_2t_w\geq 0$, $\sigma_2<0$ *(non-backward-bending solution). If* $\sigma_2\geq 0$, *then* t_w *and* $\partial l/\partial\rho(\neq 0)$ *are like-signed.*

For completeness, we solve for $r-\rho$ and t_w from (44a) and (44b):

$$(47)\quad r-\rho=\frac{l\dfrac{\partial l}{\partial\rho}-\sigma^*\dfrac{\partial l}{\partial\omega}}{\dfrac{\partial l}{\partial\rho}\dfrac{\partial\sigma^*}{\partial\omega}-\dfrac{\partial l}{\partial\omega}\dfrac{\partial\sigma^*}{\partial\rho}}$$

$$t_w=\frac{\sigma^*\dfrac{\sigma\sigma^*}{\partial\omega}-l\dfrac{\partial\sigma^*}{\partial\rho}}{\dfrac{\partial l}{\partial\rho}\dfrac{\partial\sigma^*}{\partial\omega}-\dfrac{\partial l}{\partial\omega}\dfrac{\partial\sigma^*}{\partial\rho}}$$

A nominally identical "formula" for t_w can be obtained for the problems posed in Section A and Section B. Analysis of (47) adds nothing to what has preceded.

D. *Optimal Taxes with Optimized Capital and Debt*

When the debt is free, the problem of optimal taxation is to maximize V^0 subject to $J(\cdot)=nk$. Because $V^0(q_2, q_3, \beta_1, \beta_2)$ is not a function (directly) of k, the latter influences maximum $V^0(\cdot)$ only through its impact on the investment constraint. If $F_k-n>0$, an increase of k increases maximum feasible β_1 for any q_2, q_3, β_2. Hence at optimal k, $F_k=n$. If $F_k\neq n$, some change of k could permit higher β_1, hence greater $V^0(\cdot)$ for fixed q_2, q_3, and β_2, so that k could not be the best sustainable, steady k.

The argument is easily formalized for the special case using (27) of Section A. Optimized debt implies

$$(48)\quad 0=b_d(k, d)=\frac{\lambda(1+n)}{-J_\beta+\lambda\sigma_\beta}$$

Hence $\lambda=0$. Also optimal k implies

$$(49)\quad 0=b_k(k, d)=\frac{F_k-n+\lambda(1+n)}{-J_\beta+\lambda\sigma_\beta}$$

Therefore when both k and d are optimized

$$(50)\quad F_k(k, l)=n$$

The Golden Rule path, unconstrained by a nonoptimal debt requirement, has a technological characterization in terms of the *capital-labor ratio* k/l, similar to the Golden Rule path's characterization in *technocratic* models where no attention is paid to the institutions, fiscal or otherwise, needed to sustain that path and other parallel, steady-state paths.

Is this result surprising? In one respect, no. To maximize V^0 and β_1, we desire efficiency in production; the Golden Rule path is, in a sense, the *most* "efficient" of all sustainable paths. It leaves the largest slack for consumption per head, and thus for β_1 which "absorbs" this slack.

But in another respect it is surprising and a little misleading. One might have thought that a larger capital stock would impose larger "deadweight" frictional costs beyond the increase in steady-state investment nk; perhaps higher tax *rates* t_w and t_r eventually might be needed to ensure the increase in nk as k approached its technocratic level because effective labor per head l is smaller than what one supposes it would be if taxation were (optimally) lump sum. Actually, institutions do indeed affect the magnitudes of k and l on the Golden Rule path. But they do so equiproportionately. Thus they leave $F_k=n$ and $F_l=\Psi(F_k)=\Psi(n)$, the function Ψ describing the factor price frontier.

With regard to the Golden Rule debt level, if optimal σ^* is less than (greater than) Golden Rule k, then optimal d is negative (positive). To analyze the optimal tax mix, we use $n=r$ and (36a)−(36c) to obtain

$$l_2 t_w + x_{22} t_r = -(x_2 - \beta_2) < 0 \tag{51a}$$

$$l_3 t_w + x_{23} t_r = l > 0 \tag{51b}$$

$$l_\beta t_w + x_{2\beta} t_r = 1, \quad t_w = \Psi(n) - q_3 \tag{51c}$$

Obviously, all the propositions about $n-\rho$ in Section B hold here, and hold also for $r-\rho$. The propositions in Section C where applicable also hold. We consolidate these as follows.

PROPOSITION D1: *Neither* $\rho=-1$ *nor* $\omega=0$ *is optimum.*

PROPOSITION D2: *At the optimum,* $l_3>0$ *and max* $(t_r, t_w)>0$.

PROPOSITION D3: *If* $\sigma_2>0$ (*non-forward-rising*), $l_2\neq0$. *If* $l_2<0$, $t_w>0$. *If* $l_2>0$, $t_w<0$ *and* $t_r>0$. COROLLARY: *If* $l_2 t_w\geq0$, $\sigma_2<0$.

PROPOSITION D4: *If* $\sigma_2<0$, *then* $t_r>0$ *if* $l_2\geq0$.

PROPOSITION D5: *If* $l_2<0$ *for all* $\rho<n$, *which implies strong net substitutability, then optimal* $l_2<0$ *and* t_r *may be of either sign. If* $l_2\geq0$ *for all* $\rho>n$, *because of a strong income effect or weak enough substitutability, then optimal* $t_r>0$ *and* l_2 *of either sign.*

PROPOSITION D6: *If, for every household,* x_0^m *and* x_2^m *are either net complements or independent goods, then* $t_w>0$ *and* $t_r>0$.

III. Concluding Remarks

Less than the usual purpose would be served by a recapitulation of the results obtained in this paper. If our paper is important, it is because of the suggestiveness of the model constructed and the learning experience of analyzing it, not the results finally wrung from it.

The model studied here has some obvious limitations that need no comment: the closedness of the economy to international trade, investment, and migration, the perfectness of the capital market, the fixity of the technology, and the constancy of the population growth rate.

Another limitation is the imposition onto the model of steady-state behavior. Clearly our steady states are only long run. In the immediate present, at the outset of the just era, it cannot generally be supposed that the old wealth owners faced after-tax factor rewards equal to the rewards it will be optimal to offer the young generation—whether or not it is desired to maintain capital and public debt per worker. The kind of steady state studied here is thus to be understood as a rest point to which the system will gravitate once intergeneration justice is in force—at least under basically stationary conditions on the technology and population.

Yet our steady-state formulations are not quite suited to this notion of a just rest point. As a lesson in optimal taxation, the problem posed in Section A is underrestricted—for not *every* steady (k, d) state could be a just rest point. If $b_k(k, d) < 0$ or $b_d(k, d) > 0$, for example, one would work off the excess capital or make up the deficiency in debt. On the other hand, Sections B and C are misrestricted—for neither $b_d(k, d) = 0$ nor $b_k(k, d) = 0$ is an optimal rest point except in the singular Golden Rule case where both equalities hold. And it is perhaps obvious that Section D (the Golden Rule state) is too restrictive in one sense—for intergeneration justice should not be assumed to demand that society from every imaginable initial state should trudge its way to the Golden Rule state.

These propositions (or their analogues) are demonstrated in an as-yet preliminary paper by Phelps and John Riley that studies the dynamics of capital and wealth in an economy (similar to the one here save for homogeneity of the population) programmed for Rawlsian *inter*generational justice. However, in neglecting population heterogeneity that paper omits the matter of *intra*generational justice that is the focus here. Perhaps the union of these two papers will sometime bear fruit in a third paper, that one on taxation for Rawlsian justice within and across generations.

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INTRODUCTION

Nearly all of the papers in the other parts of this volume raise the question of the optimum volume of national saving. The papers in this part digress from that grand theme to take up, at least implicitly, a related subject: the efficient use of a given volume of saving in the choice among national investments. The former question is the distributional side, and the latter question the efficiency side, of the problem of optimal growth.

A simple version of the idea of efficiency in investment is this: The program of investments is efficient if no redirection of investment out of the given national saving in any period or periods could produce higher consumption in some future period without producing lower consumption in some other future period. If the technology is such that there exists a one-period social rate of return to each investment activity, or kind of capital, then an efficient investment program causes all the social rates of return within any period to be equalized.

The papers in the present group were written when three broad kinds of capital were commonly distinguished: (1) tangible capital mainly in the form of plant and equipment; (2) technological capital contained in manuals and programs; and (3) the human capital people acquire through education and experience. My papers on these kinds of investments were conceived with the expectation that they might eventually advance the estimation of the social rates of return to investment in each kind of capital, and thus identify any redirection of national investment needed for economic efficiency. Of course, one could hardly avoid recognizing that investment in human capital is a special case both because of its consumption value to the persons serving as vehicles for the investment and because of the role that education plays in the distribution of earnings. (We might draw back from asking a marginal worker to trade off his education, below some minimum necessary for normal functioning in society, for the gift of some government bonds.) Nevertheless a natural curiosity about the social rates of return remained, especially as regards investments in tangible versus technological capital.

My paper on some implications of the "vintage view" of the capital stock asks whether the role of plant and equipment as carriers of technological progress might justify a reemphasis upon tangible capital formation as an instrument for economic growth in preference, say, to greater technological research. After dispatching a few wrong reasons for restoring tangible investment to more favorable consideration, the paper finally adduces a new reason: The capital stock is smaller than we thought—because capital-embodied technological progress, like ordinary physical deterioration, generates economic depreciation of capital goods—and therefore capital's marginal product must likewise be greater than we thought. Robert Solow called to my attention, in time for the printed version, a significant (but not decisive) qualification: The addition to the rate of economic depreciation, namely the rate of obsolescence, must be subtracted when recalculating the net rate of return.

It was quite amazing, at least to me, to see how many complaints of conceptual error the paper drew. The sole persuasive objection I have seen is one raised by Robert Hall: The rate of depreciation used by me to calculate the surviving capital stock already contains an appropriate allowance for technological obsolescence, so it is double-counting to add to it a rate of obsolescence; the standard methods of measuring depreciation do not measure only physical deterioration but rather the whole loss of value of aging capital goods from all sources. That objection leads to the question: If the social (and private) rate of return is not as high as previously estimated after all, how can we explain the size of profits? Here I incline to the theory, which I was led to by my work with Sidney Winter, that much of the net returns to the firm represent "good will." The puzzle of a discrepancy between the social rate of return to tangible investment and the real rate of interest before tax rested on a false premise; there is little or no such discrepancy.

My paper on putty-clay is neither the first nor the last on that difficult subject. But it is the source of the simile of putty—as "putty in his hands"—turned to hard-baked clay. Mrs. Robinson scores a valid point in observing that putty turns hard by itself, so "clay" is redundant. Yet the phrase putty-clay is here to stay. Oddly, the substantive side of the paper is remembered more for its contribution to investment-demand theory than to its intended subject. The paper missed the result later found by Trevor Swan, that the elasticity of steady-state output with respect to the saving-income ratio is just as it was found to be in the putty-putty model with or without capital-embodied technical progress: namely, the ratio of capital's share to that of labor.

The next two papers take the view that technological change requires researchers to produce it, hence saving. However, the first of these papers takes the research requirements of technological progress to be negligible and focuses instead on the factor-saving direction of the technological change that is produced. Emanuel Drandakis, the paper's other author, had become interested in the induced-invention ideas of Charles Kennedy; I had forgotten that the factor-augmentation formulation of invention possibilities that we came up with had earlier been shown to me by Christian vonWeizsäcker. As it turned out, Paul Samuelson was working on a similar model at the same time. By a fortunate choice of "expedient" state-variables with which to formulate the stability problem, we were able to demonstrate a much wider tendency for stability of the growth rate and factor shares than had been inferred by Samuelson. I do not recall, though, our feeling like Watson and Crick having beaten Pauling. The pleasure must have been tempered by the thought that the model was built on a pretty wild abstraction—that the choice of the factor to "save" was to be a global one covering all possible activity-analysis processes.

The other essay on technical change, with its progress function, is a research memorandum that was later absorbed into my paper on the Golden Rule of Research. Subsequently the progress function was put to empirical use by Michael Lovell in describing how the number of journal pages produced per year depends upon the number of researchers and the stock of pages that they have to work with. The results were gratifying. Precisely as I had insisted against a chorus of doubters, the marginal product of paper in producing itself—the own-rate of interest on paper—is positive. Read these pages and see your productivity increase!

The third outlet for saving is investment in persons, so-called human capital formation. The paper with Richard Nelson on this subject was based on his thesis that the utility of education is a function of the flow of new information to be processed. As Jean Piaget put it, "the principal goal of education is to create people who are capable of doing new things, not simply of repeating what other generations have done." The paper presents two algebraic models, the first of them Nelson's and the other my slender contribution.

Perhaps my paper on population policy sits oddly in the present group; it could have gone alongside the paper on fiscal neutralism in Part III as well since it seems to contemplate the formulation of intergeneration policies without benefit of a formal ethical criterion. Still, the paper does fit here: Children are regarded as costly consumer durables, and hence constitute a kind of investment. And for the nation of adults as a whole, today's children, while costing us some consumption today, will tomorrow work the capital we have left in order to produce the output on which our consumption then will depend. Both this paper and my earlier essay on the Golden Rule of Procreation give attention to the consumer-durable side, and the present paper ponders also the producer-durable side.

The paper blends theoretical invention, analytical error, and empirical insight (in that order, as I recall); so it is understandably one of my own favorites. The invention was the imposition onto the factor-price plane of the iso-utility contours alongside the factor-price contour in a steady-state context. But the curvatures were wrong, so I drew back from announcing a new golden rule of population increase making $r = n$. Alas, not alerted by his assistant assigned to my paper, Paul Samuelson mistook the minimum for a maximum and so announced another golden rule.

My own error was in concluding that factor-price effects did not warrant from the standpoint of the self-interest of the current generation of young adults a subsidy to their having and raising children. My impression is that Guillermo Calvo, my long-time colleague, has settled the issues here with definitiveness.

The empirical insight in the paper is its realization that we would not be where we are today in science and the arts had our ancestors not made haste to populate the world, since each step of progress has needed human input. So the coming deceleration of the world's population will have a double effect on the global real rate of interest: Our capital accumulated over the future will have less labor to man it and that labor's productivity will have benefitted from fewer ideas. It turned out that this Mozart Proposition of mine, to use the code word suggested by William Nordhaus, had been discovered much earlier by Simon Kuznets. Maybe the irony of having been preceded in this very discovery is entirely fitting. The diminishing returns to research in the technological progress function is largely a reflection of the diminishing probability of a researcher's being first.

THE NEW VIEW OF INVESTMENT

In 1956 appeared the first in a series of papers[1] disputing the traditional thesis that capital deepening is the major source of productivity gains and conjecturing that we owe our economic growth to our progressive technology.

Thesis and antithesis were synthesized by 1960. Investment has been married to Technology.[2] In the new view, the role of investment is to modernize as well as deepen the capital stock. Now investment is prized as the carrier of technological progress.

No criticism is made here of this "new view" of the role of investment. Nor is the need for accelerated investment, public and private, questioned. This paper is concerned only with the logic of certain conclusions which the new view has shown a tendency to inspire. In what sense does its new role make investment more important? What are the prospects of modernizing the capital

* I am grateful to Edwin Mansfield, Arthur M. Okun and Robert M. Solow for their suggestions and comments on earlier drafts.

1. M. Abramovitz, "Resource and Output Trends in the United States since 1870," *American Economic Review*, XLVI (May 1956). O. Aukrust, "Investment and Economic Growth," *Productivity Measurement Review*, No. 16 (1959), pp. 35–53. S. Fabricant, *Basic Facts on Productivity Change*, National Bureau of Economic Research (New York, 1959). John W. Kendrick, "Productivity Trends: Capital and Labor," *Review of Economics and Statistics*, XXXVIII (Aug. 1956). B. Massell, "Capital Formation and Technological Change in United States Manufacturing," *Review of Economics and Statistics*, XLII (May 1960). T. W. Schultz, "Reflections on Agricultural Production, Output and Supply," *Journal of Farm Economics* XXXVIII (Aug. 1956). R. Solow, "Technical Change and the Aggregate Production Function," *Review of Economics and Statistics*, XXXIX (Aug. 1957).

2. PEP (Political and Economic Planning), *Growth in the British Economy* (London: Allen and Unwin, 1960). R. Solow, "Investment and Technical Progress," *Mathematical Methods in the Social Sciences, 1959* (Stanford: Stanford University Press, 1960). U. N. Economic Commission for Europe, *Economic Survey of Europe in 1958*, Chap. II (Geneva, 1959). U.S. Joint Economic Committee, "The American Economy in 1961: Problems and Policies," Council of Economic Advisers, *Hearings on the Economic Report of the President, 1961.*

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stock through increased thrift? Does the new view of investment present any new reasons — should added ones be needed — for faster capital accumulation? The analysis is confined largely to investment-thrift policies described by a fixed saving ratio. The final section presents estimates of the rate of return to investment as implied by certain new-view assumptions. The results of the inquiry are summarized at the conclusion of the paper.

The Basis of Investment Pessimism

The empirical work cited above spans a great variety of analytical methods and historical materials. One of the best known papers is that by Professor Solow.[3] A number of other investigators followed much the same approach.

That method postulates aggregate output, Q_t, to be a continuously differentiable function of capital, K_t, employment, N_t, and "time" (standing for the state of technology). If, further, technical progress is "neutral," then output is a separable function of time and the inputs, as follows:

$$Q_t = A(t)\,F(K_t, N_t). \tag{1}$$

Such a production function implies that technical progress is organizational in the sense that its effect on productivity does not require any change in the quantity of the inputs. Existing inputs are improved or used more effectively.

It follows that the growth rate of output is equal to the rate of technical progress plus a weighted average of the growth rates of the inputs. These weights are the elasticities of output with respect to capital and to labor. Assuming constant returns to scale, the weights add to one and we obtain

$$\frac{\dot{Q}_t}{Q_t} = \frac{\dot{A}_t}{A_t} + a_t\frac{\dot{K}_t}{K_t} + (1 - a_t)\frac{\dot{N}_t}{N_t} \tag{2}$$

where a_t is the capital elasticity of output, that is $\dfrac{F_K(K_t,N_t)K_t}{Q_t}$.

There are two unknowns in equation (2), the rate of technical progress and the capital elasticity. Solow, and later Massell,[4] relied on an "outside" estimate of the capital elasticity and proceeded to focus on the rate of technical progress. Solow took capital's relative share of national income in year t as a measure of a_t and Massell, who assumed a_t was constant over time, used the average share going to capital. It is not known how close such approxima-

3. Solow, "Technical Change and the Aggregate Production Function," *op. cit.*

4. *Op. cit.*

tions are. The practice presumes pure competition (which is not strictly implied by the model) as well as constant returns to scale.

The results of this approach produced a wave of investment pessimism. From a study of U.S. time series it was concluded that less than one-third of the average growth rate of output per worker in the last quarter century could be credited to the increase in capital per worker which occurred.[5]

Of course, it does not follow from this conclusion that capital deepening is ineffectual. It might mean only that over the time period investigated little capital deepening took place.[6] For policy purposes, the effectiveness of additional investment is of greater interest. On this score too, however, the approach outlined above produces some gloomy results.

Consider the effect of doubling the (net) investment-income ratio from .09 to .18. If the capital-output ratio is about 3, then this increase in the saving ratio would in a year increase the capital stock by about 3 per cent (beyond what it would have increased otherwise). Now capital's share in (net) national income is less than one-third. Therefore, according to equation (2), the 3 per cent increase in the capital stock would increase (net) output by less than 1 per cent (and it would increase output even less if the capital-output ratio rose).[7] Solow has remarked of such a calculation: "This seems like a meager reward for what is after all a revolution in the speed of accumulation of capital."[8]

5. From equation (2) it is easy to derive the proportion of the growth rate of output per worker which is attributable to capital deepening. It is

$$\frac{a_t(k_t - n_t)}{q_t - n_t} = \frac{a_t(k_t - n_t)}{r_t + a_t(k_t - n_t)}$$

where k_t, n_t, q_t and r_t denote the (relative) growth rates of capital, labor, output, and technology respectively, at time t. If there is no capital deepening, meaning $k_t = n_t$, then the proportion is equal to zero. If there is no technical progress, the proportion is equal to one.

The Solow-Massell result is easy to explain. In the U. S. time series they employed, capital and output grew at approximately the same rate. But if k_t equals q_t then the proportion equals a_t. Their factor share data put a_t at about one-third (or less).

6. The current alarm over the decline since the early twenties in the capital-output ratio rests on just such a counterinterpretation.

7. H. Stein and E. Denison's remarkably pessimistic paper for the President's Commission on National Goals is based on calculations of this kind. E. F. Denison and H. Stein, "High Employment and Growth in the American Economy," in *Goals for Americans*, report of the President's Commission on National Goals (Englewood Cliffs, N.J.: Prentice-Hall, 1960).

8. Solow, "Investment and Technical Progress," *op. cit.*

The New View

Just when the reputation of investment seemed at low ebb came the first signs of a new tide. Critics of the research described contended that new technologies generally require new kinds of capital goods. Therefore without positive (gross) investment productivity could hardly be expected to grow at all. Furthermore, it was argued, the higher the rate of gross investment, the newer and hence more modern and "efficient" will the capital stock become. Proponents of this new view of investment were apt to assign as much weight to capital modernizing as to capital deepening.[9]

In 1961 the new view of investment was embraced by the new administration. The President's Economic Message to Congress in January, 1961 stated:

> Expansion and modernization of the Nation's productive plant is essential to accelerate economic growth and to improve the international competitive position of American industry. Embodying modern research and technology in new facilities will advance productivity, reduce costs, and market new products.[1]

Expansion and modernization are put on equal footing and the latter is stressed. A statement by the Council of Economic Advisers before the Joint Economic Committee in March 1961 amplifies this view:

> One of the reasons for the recent slowdown in the rate of growth of productivity and output is a corresponding slowdown in the rate at which the stock of capital has been renewed and modernized. . . As has been confirmed by more recent research, the great importance of capital investment lies in its interaction with improved skills and technological progress. New ideas lie fallow without the modern equipment to give them life. From this point of view the function of capital formation is as much in modernizing the equipment of the industrial worker as in simply adding to it. The relation runs both ways: investment gives effect to technical progress and technical progress stimulates and justifies investment.[2]

To clarify the meaning of this new notion and to lay the basis

9. Two of the earliest documents taking the new view are *Economic Survey of Europe in 1958, op. cit.*, and *Growth in the British Economy, op. cit.* They argue that rapid labor force growth — contrast Britain and Germany — will raise output per worker by stimulating gross investment. The stimulation required is not spelled out.

1. Message from the President of the United States relative to a Program to Restore Momentum to the American Economy, *New York Times*, Feb. 2, 1961.

2. *Op. cit.*, p. 338.

for the analysis to follow, we turn now to an important theoretical paper by Solow which adopts the new view.[3] The purpose of that paper is to show that such neoclassical concepts as aggregate capital and the aggregate production function (containing aggregate capital) can be modified to accommodate the new view.

Solow postulates an index of technology, $B(t)$, which advances neutrally and exponentially at the rate λ. The nature of the technology so indexed is such that at every point of time it affects the efficiency only of new capital goods. Every capital good embodies the latest technology at the moment of its construction but it does not participate in subsequent technical progress. Thus "capital" becomes a continuum of heterogeneous vintages of capital goods.

The output rate at time t, $Q_v(t)$, of capital equipment of vintage v is assumed to be given by a Cobb-Douglas function,

(3) $$Q_v(t) = B_o e^{\lambda v} K_v(t)^{\alpha} N_v(t)^{1-\alpha}$$

where $K_v(t)$ denotes the amount of equipment (in physical terms) of vintage v surviving at time t and $N_v(t)$ denotes the amount of labor employed on that equipment. Since technical progress is neutral, the elasticity parameter α is the same for all vintages.

Solow then shows that if labor is allocated efficiently over the various vintages (by equalizing labor's marginal productivity on all equipment), aggregate output — the sum of the homogeneous outputs of the various vintages — is given by:

(4) $$Q_t = B_o J_t^{\alpha} N_t^{1-\alpha}$$

where

$$J_t = \int_{-\infty}^{t} e^{\frac{\lambda v}{\alpha}} K_v(t)\, dv.$$

The "J" variable might be called "effective capital." The integral adds up all the (surviving) capital goods like the conventional capital measure; but here the capital goods of older vintages (with their small v's) receive a smaller weight than new capital goods.

For comparison with the old-fashioned model, let us specialize (1) in the same way. If all technological progress is organizational, neutral and proceeding at the constant relative rate, then

(5) $$Q_t = A_o e^{\mu t} K_t^{\alpha} N_t^{1-\alpha}.$$

According to this classical view, old and new capital goods share alike in technical progress, so that "capital," K_t, is simply the sum of the homogeneous surviving capital goods. Hence (5) can be written:

3. Solow, "Investment and Technical Progress," *op. cit.*

$$(6) \qquad Q_t = A_o \left[\int_{-\infty}^{t} e^{\frac{\mu}{\alpha} t} K_v(t)\, dv \right]^{\alpha} N_t^{1-\alpha}.$$

The encouragement drawn from the new view — as represented by (4) — as compared with the old view — represented by (6) is illustrated by the following example.

Suppose that existing machines are of just two vintages, v_1 (old) and v_2 (new), and that there are an equal number of machines of the two vintages.

According to (6) a 2 per cent increase in the number of machines of the current vintage, v_2, will bring about a 1 per cent increase in the value of K and of the bracketed expression in (6); we are weighting a 2 per cent and a zero increase equally.

Consider the case in equation (4). J is the weighted sum of the machines of the two vintages with the weight for the contemporary machines, namely $e^{\lambda v_2}$, being greater. Consequently a 2 per cent increase in the number of machines of current vintage will produce an increase of J in excess of 1 per cent. Here current investment increases output per man partly through affecting the average modernity of the capital stock.

What if we lengthen our view and ask what happens as the program of capital accumulation continues? Pretty soon we will be confronted by a changed situation; large investments today will present us with a large amount of old equipment in the future. Investment must grow in order to maintain a constant average age of capital. And as we shall see, there is (under certain plausible assumptions) an average age of capital such that no smaller average age is tenable for long. The modernizing effects of expanded investment are limited.

Suffice it to say that the long-run consequences of a change in investment policy are not so clear as the immediate effect, and both are deserving of study. True, in the long run we are dead but our children will have to live in it. Can we control to an important degree the modernity of the capital stock they will inherit? Do we owe the modernity (such as it is) of our present stock to our ancestors' thrift? What significance has the new view of investment for the long run? This is examined now.

A Simple Model of Growth

We shall confine our analysis to the implications for output growth and productivity of investment policies which make gross investment a fixed proportion of gross output. The choice of an investment policy is thus a matter of selecting the investment-output ratio s. Hence, where $I(t)$ denotes the rate of investment at t:

(7) $I(t) = s\,Q(t)$.

Second, we assume that the labor force grows at the constant relative rate n:

(8) $N_t = N_o e^{nt}$.

Finally we assume that all capital goods depreciate exponentially at the rate δ per annum. Hence

(9) $K_v(t) = I(v)\, e^{-\delta(t-v)}$.

If we think of δ as a mortality rate, then the average lifetime of capital goods is $\frac{1}{\delta}$ years.

Now our purpose is to compare the relation between investment and growth under the new and old view. We can do this by comparing a pure new-view model with a pure old-view model. But the simplest approach is to examine a single model which, by a variation of parameters, can be made to represent either pure or a mixture of both.

Thus we shall work with the following "general" production function which is simply a blend of (4) and (6):

(10) $Q_t = B_o\, e^{\mu t} J_t^{\alpha} N_t^{1-\alpha}$

where, as before

$$J_t = \int_{-\infty}^{t} e^{\frac{\lambda}{\alpha}v} K_v(t)\, dv$$

or, by virtue of (9)

$$J_t = e^{-\delta t} \int_{-\infty}^{t} e^{(\frac{\lambda}{\alpha}+\delta)v} I(v)\, dv.$$

When we compare the new view to the old view we are comparing the behavior of the model with $\lambda > 0$, $\mu = 0$ against the behavior when $\mu > 0$, $\lambda = 0$. And if one believes in both kinds of technological progress then he can let $\lambda > 0$, $\mu > 0$ simultaneously.[4] (In that

4. It may be (and has been) objected that it cannot be assumed that the other parameters, α, δ and so forth, are invariant to the nature of the technology (i.e., whether it is the λ-type or μ-type). But we find no implication in the new view that the nontechnological parameters differ from their supposed or implied values under the old view. That is, "$\lambda > 0$, $\mu = 0$" implies nothing about δ and α; to the contrary, the postulate that the embodied or λ-type technical progress is "neutral" implies that α is independent of λ. Whether empirical estimates of α and δ would be affected depends upon the method of estimation. Under neoclassical conditions it is common practice to take capital's relative share as an estimate of α; this procedure is equally

case the efficiency of all capital goods may be said to rise at the rate $\frac{\mu}{a}$ except the efficiency of new capital goods which rises at the rate $\frac{\lambda+\mu}{a}$.)

Differentiating Q_t in (10) with respect to time yields (omitting the t subscript):

$$(11) \qquad \dot{Q} = \mu Q + aBe^{\mu t} J^{a-1} N^{1-a}\left[e^{\frac{\lambda}{a}t} I - \delta J\right]$$
$$(1-a)\ Be^{\mu t} N^{-a} J^{a} \dot{N}$$

where we have used the relation $\dot{J}_t = e^{\frac{\lambda}{a}t} I - \delta J$ by virtue of (9). Using (7), (8) and (9) (to express J^{a-1} in terms of Q and N) we obtain the following differential equation governing the growth path of output:

$$(12) \qquad \dot{Q} = c_1 Q + c_2 Q^{c_3} e^{c_4 t}$$

where
$$c_1 = \mu - a\delta + (1-a)n$$
$$c_2 = a\, s\, B^{\frac{1}{a}} N_o^{\frac{1-a}{a}}$$
$$c_3 = \frac{2a-1}{a}$$
$$c_4 = \frac{\lambda+\mu+(1-a)n}{a}.$$

This equation can be solved for the path of output.[5] In the next section the long-run or asymptotic behavior of output will be considered.

Investment and Productivity in the Long Run

These models have the convenient property[6] that, starting from the initial position, the path of growth will be asymptotic to a balanced-growth, "golden-age" equilibrium growth path along which path production, consumption, investment, and the capital stock

appropriate on the two views. One's assumptions about λ and μ would affect B_o, the technology index at $t = 0$; we return to this in a footnote *infra*.

5. For the solution we are indebted to a regrettably unpublished paper by Dernburg and Quirk, which analyzes an old-view Cobb-Douglas growth model. T. Bernburg and J. Quirk, "Per Capita Output and Technological Progress," Institute of Quantitative Research in Economics and Management, Purdue University (1960).

6. Dernburg and Quirk, *op. cit.* R. Solow, "A Contribution to the Theory of Economic Growth," this *Journal*, LXX (Feb. 1956). T. Swan, "Economic Growth and Capital Accumulation," *Economic Record*, XXXII (Nov. 1956).

(of all ages) all grow exponentially at the same rate. This "equilibrium" output path is denoted $\bar{Q}(t)$.

The limiting or asymptotic solution to equation (13) or (15) is

$$(13) \qquad \bar{Q}(t) \approx \bar{Q}_0 \, e^{\frac{c_4}{1-c_3}t} .$$

Thus the growth rate, g, tends in the long run to the constant $\frac{c_4}{1-c_3}$. In terms of the original parameters:

$$(14) \qquad g = \frac{\lambda + \mu}{1 - a} + n.$$

It will be noticed that the limiting growth rate is independent of the investment ratio. This is a well-known property of old-style Cobb-Douglas models.[7] It is not surprising to find this same property in the "new model," which allows $\lambda > 0$. Associated with this exponential growth pattern is a certain unchanging age distribution of capital. Capital which is $(t - v)$ years old will grow at the rate g like most everything else; the proportion of capital which is $(t - v)$ years old or less is constant over time. The fact that capitals of different vintages get different technical weights is immaterial in the determination of the exponential equilibrium growth rate.

Note also that the long-run growth rate depends only upon the total rate of technical change, say, $\Delta = \lambda + \mu$, not upon the nature of the change. The reason is that the efficiency of capital $(t - v)$ years old will, in exponential equilibrium, improve at the rate Δ in either (pure or any mixed) case.

What then is the relation between investment and productivity in the long run? The higher the investment ratio that society chooses the larger will be its capital stock (at every point of time) in the long run. Thus the *level* of the "equilibrium" exponential growth path which the economy approaches is a function of the investment ratio. In short, $\bar{Q}_0$, the equilibrium value of Q at "time zero" (chosen arbitrarily), is a function of s. This value is to be distinguished from the actual output at time zero, Q_0; the two will be equal only if the initial capital-output ratio happens to equal that ratio which the chosen s would have brought about.[8]

7. Solow, "A Contribution to the Theory of Economic Growth," *op. cit.*, and Swan, *op. cit.*

8. On the equilibrium path the "conventional" capital-output ratio is constant; both K and Q grow at the rate g. But if $\lambda > 0$, "effective" capital grows at the rate $g + \frac{\lambda}{a}$ and so the effective capital-output ratio rises.

The solution for the long-run growth path is:

$$(15) \qquad \bar{Q}_0 = \left[\frac{(1 - c_3)\, c_2}{c_4 - (1 - c_3) c_1} \right]^{\frac{1}{1-c_3}}$$

or, in terms of the original parameters

$$(16) \qquad \bar{Q}_0 = s^{\frac{a}{1-a}} \left[\frac{(1-a)\, B^{\frac{1}{a}} N_0^{\frac{1-a}{a}}}{\mu + \frac{\lambda}{a} + (1 - a)\,(n + \delta)} \right]^{\frac{a}{1-a}}.$$

What significance, we ask again, has the new view in relation to investment and productivity in the long run? From (16) one can see immediately that the elasticity of Q_0 with respect to s is $\frac{a}{1-a}$, independent of λ and μ. Whether one takes the new view or the old, it follows from this model that, in the long run, a 1 per cent increase in the investment ratio will yield asymptotically an output rate which is $\frac{a}{1-a}$ per cent in excess of what asymptotically it would otherwise have been (i.e., had the orginal investment ratio prevailed).

This result seems at first to contradict the little example of increased investment presented at the end of the "The New View" section. The explanation of the puzzle lies in the behavior of the average age — or more precisely, the age distribution — of capital. It has apparently been overlooked that, in exponential growth, the age distribution of capital depends upon the rate of growth and the rate of depreciation and upon nothing else. Since both rates are, in the long run, independent of the investment ratio, a once-for-all change in that ratio can have no permanent influence on the age distribution of capital. Consequently, in the long run, any increase in thrift must rely for its effectiveness upon the prosaic mechanism of capital deepening — of an equiproportionate deepening of capital of every age.

This is easily proved. Suppose the economy has been growing smoothly at the rate g, along the growth path corresponding to the chosen fixed investment ratio, for quite some time. If, say at $t = 0$ (for convenience only), we were to look at the distribution of capital equipment by age we could summarize our findings by the exponential curves in Figure I. In order to obtain the amount of capital of vintage v still in use at $t = 0$, $K_v(0)$, we have to multiply $I(v)$ by $e^{\delta v}$. This gives the lower curve.

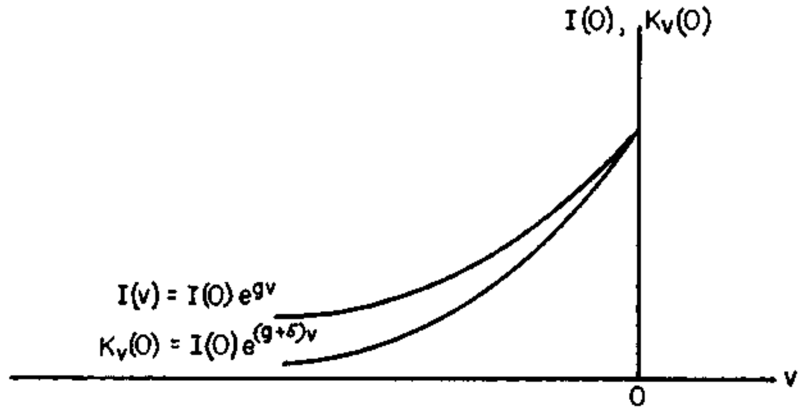

Figure I

To obtain the mean age and the other moments of the age distribution of capital, it is necessary to normalize the curve so that its area will equal 1. This requires dividing $K_v(0)$ by $I(0)/(g+\delta)$ for all v.[9] Thus we obtain the formula for the proportion of equipment of age v:

(17) $$f(v) = (g+\delta)\, e^{(g+\delta)v}.$$

It is clear that all the moments of the equilibrium age distribution of capital are independent of the quantity of capital and the rate of investment. The equilibrium mean age of capital, for example, is simply $\frac{1}{g+\delta}$[1]

Given the investment ratio, the mean age of capital depends in the long run only upon the rate of depreciation and the limiting rate of growth, and neither of these depend upon the investment ratio in this model.

9. $I(0)/(g+\delta)$ is the total area under the $K_v(0)$ curve, by the familiar "capitalization" formula.

1. Proof:

$$-\bar{v} = \int_{-\infty}^{0} (g+\delta)\, e^{(g+\delta)v}\,(-v)\, dv$$

$$= \left.\frac{-v(g+\delta)e^{(g+\delta)v}}{g+\delta}\right]_{-\infty}^{0} + \int_{-\infty}^{0} \frac{(g+\delta)e^{(g+\delta)v}}{(g+\delta)}\, dv$$

$$= 0 + \left.\frac{e^{(g+\delta)v}}{g+\delta}\right]_{-\infty}^{0}$$

$$= \frac{1}{g+\delta}$$

Therefore a once-for-all rise in the investment ratio can significantly "modernize" the capital stock only temporarily. Ultimately the average age (or modernity) of capital must settle back toward its equilibrium level. A permanent modernization of the capital stock (starting from equilibrium) would require the investment ratio to increase without limit, a policy which is not feasible (without foreign assistance at any rate).

Of course, actual economies are never found in dynamic long-run equilibrium because of fluctuations in investment. An upswing in investment is usually associated with a downswing in the mean age of capital. But it should be understood that when the mean age of capital exceeds its equilibrium value, a decline in the mean age is bound to occur eventually no matter what investment ratio society elects to adopt.[2] This leads us to digress briefly on the present mean age of capital in the United States and the direction in which it may be expected to move.

The Terborgh-Knowles estimates of the average age of capital, which end at 1957, together with the experience of the past five

2. When the economy is out of equilibrium, the basic model is likely to forecast a different limiting growth path (corresponding to a given investment ratio) for every different value of λ we should assign. If, for example, the mean age is below its equilibrium value then a new-view forecast, taking the eventual equilibrating increase in mean age into account, would predict a lower equilibrium path (whatever the investment ratio) than would an old-view forecast because the latter would attribute no significance to the eventual rise in the mean age of capital. This fact in no way invalidates the conclusions of this section concerning the long-run growth rate, the "investment elasticity" and the equilibrium mean age of capital, these relations being independent of the level of the equilibrium growth path.

A special case of some interest is that in which the economy has always traveled along the equilibrium path corresponding to the prevailing investment ratio. In this case the mean age of capital is in equilibrium and the value of λ will not affect the predicted equilibrium growth path corresponding to any investment ratio, because μ adjusts to satisfy (14) and B to satisfy (16).

Note that high λ implies high B. Let n, δ and s be recorded from direct observation and let a be estimated from relative shares. Then Δ can be estimated simply from (14). If we believe some of this Δ is λ then $\frac{\lambda}{a} + \mu$ rises so we have to make an upward adjustment of B in (16) in order that the model be able to explain the actual level of output $Q_0 = \overline{Q}_0$.

The adjustment of B makes sense because the implied old-view estimate of B is actually an estimate of the average level of technology embodied in all capital goods while the new view implies that the current level of technology is superior to the average. At time zero, the current (or best-practice) level of technology is measured by B.

years suggest that the mean age today is about 17.5 years.[3] Thus the postwar vintages comprise half the nation's capital. In 1975 the postwar investment boom will be working against a modern capital stock. Then all capital of vintages 1957 and earlier — particularly the heavy investments of 1946–57 — will be older than 17 years. Between now and 1975 we apparently require an increase in investment comparable to the postwar increase in order to avert an increase in the mean age of capital.

Yet such an acceleration of investment is not unlikely, even without an increase of the investment ratio. Due to the expected rise in the rate of increase of the labor supply, many observers anticipate full-employment growth at 4½ per cent or more over the next decade — about 1 percentage point better than the postwar experience (in output and investment) to date.[4] Therefore if investment should keep pace with output over the future, the mean age of capital may well hold steady or even fall.

Still, the major impression drawn from a study of the Terborgh-Knowles series is the remarkable stability of the mean age of capital. It took a depression and a war to raise the average age from 16.5 (in 1930) to 21.2 years (in 1945). This suggests that, given the technical and demographic factors which determine the limiting growth rate, it would be very difficult to reduce by means of investment the mean age of capital by more than 3 or 4 years. And, as we have seen, according to the model here this gain could not be indefinitely maintained. Eventually the mean age would slip back up to its natural long-run level.

Investment and Productivity in the Short Run

The foregoing analysis has some significance for "positive economics." For example, a sustained improvement in the modernity of the capital stock of a country should be ascribed (proximately) not to the level (rate) nor to the rate of growth of its investment but to a rise of the rate of growth of investment. The improvement can be expected to be permanent only if there have been (or will be) technical and demographic changes causing a rise in the limiting growth rate of output (thus averting a future deceleration of investment).

3. See James Knowles, "The Potential Economic Growth in the United States," Study Paper 20, *Employment, Growth and the Price Level* (U. S. Congress, Joint Economic Committee, 1959), p. 26.

4. See, for example, *The Economic Report of the President, 1962* (Washington, U. S. Government Printing Office, 1962.)

The implications of the analysis for investment policy depend, of course, on the decision rules used by the policymaker. Many (all?) sensible rules will involve, among other things, the responsiveness of output *in the short run* to a policy of greater thrift and investment. The short run assumes considerable importance when we observe that our model economy approaches its limiting path only asymptotically. Even to get close to that path may take considerable time. It is worthwhile therefore to inquire into the speed with which the economy adjusts to a change in the equilibrium path brought about by a change in the investment ratio. It will be seen that the new view forecasts a faster transition from the old to the new equilibrium path.

This task requires the full solution of the differential equation in (12), for which we are indebted to the paper by Dernburg and Quirk.[5] The complete solution is:

$$(18)\qquad Q(t) = \left[\left(Q_0^{1-c_3} - \bar{Q}_0^{1-c_3}\right)e^{c_1(1-c_3)t} + \bar{Q}_0^{1-c_3}\, e^{c_4 t}\right]^{\frac{1}{1-c_3}}$$

where $\bar{Q}_0$ is given in equation (16).

Equation (18) implies that output will "approach" its equilibrium path, in the sense that $\dfrac{Q^t}{\bar{Q}^t} \to 1$ as $t \to \infty$, if and only if $c_1(1-c_3) - c_4 < 0$, in which case the model is said to exhibit *absolute stability*.[6] This stability condition can be seen more clearly if we look at equation (18) in the form

$$(18\text{a})\qquad Q(t) = \bar{Q}(t)\left\{1 + \left[\left(\frac{Q_0}{\bar{Q}_0}\right)^{1-c_3} - 1\right] e^{[c_1(1-c_3)-c_4]t}\right\}^{\frac{1}{1-c_3}}.$$

The condition $c_1(1-c_3) - c_4 < 0$ means $\mu + \dfrac{\lambda}{a} + (1-a)(n+\delta) > 0$ which is assumed here. Thus the latter expression determines the rate of approach to equilibrium. Given n, δ and a, the larger $\mu + \dfrac{\lambda}{a}$ the faster is the approach. How does the new view affect it?

It is clear that if one were to start with a pure old-view model, with its rate of approach determined by $\mu + (1-a)(n+\delta)$, and

5. *Op. cit.*

6. If only the limiting growth rate span (and not also the limiting path) is independent of initial conditions then the model possesses only "relative stability."

then proceeded to add $\frac{\lambda}{a}$ to this expression, as if λ measured a neglected source of technical progress, the result would be a faster implied rate of approach to equilibrium.

But λ and μ are not additive. An alteration of the model does not change the world but only the conception and estimation of its parameters. Suppose a pure old-view adherent, if one could be found, and a pure new-view supporter were dispatched out into the world to estimate the relevant parameters. Using conventional estimation procedures (based on neoclassical assumptions), they would return with identical estimates except for Δ (the rate of technical progress) and B (the level of technology), the latter fortunately being irrelevant to the question at hand.

A little reflection will indicate that the new view estimate of λ will exceed the old view estimate of μ if the mean age of capital has been steadily increasing and will fall short of the old-view estimate if the mean age has been steadily falling. A fluctuating mean age complicates the picture. In any event, the estimates could not be presumed to be equal unless the economy had happened to be in long-run equilibrium.[7]

Therefore, if the mean age of capital had been falling sharply, the estimate of λ might be smaller than the estimate of μ by a factor of a or more.[8] In this event, the old-view model would paint a more dynamic and adaptable economy than would the new-view model. It would predict a higher limiting growth rate (μ being larger than λ); and, with respect to the question posed in this section, it would imply a capacity to close a given disequilibrium more quickly and therefore to make the transition faster from a low equilibrium path to a higher one.

Circumstances are conceivable, therefore, in which a permanent increase in the investment ratio would appear — at least for a while — more attractive on the old view of the economy than on the new view. But it must be noted that when the economy has been out of equilibrium — and this is an essential part of those circumstances — the two models will imply different absolute levels of the equilibrium path corresponding to the prevailing investment ratio, and they will

7. For details of this argument, see the extended footnote of the section above concerning the long run.

8. It would be better to represent the new view by a mixed model allowing $\lambda > 0$. Then the μ estimate would be compared with the estimated sum of $\mu + \frac{\lambda}{a}$ in the new model.

imply different limiting growth rates. A rational investment policy may well take these factors into account together with the transition speed. Further, a wide discrepancy between the λ and μ estimates could only be temporary. As the economy approached long-run equilibrium, they would have to come together.

This last observation reminds us once again that the mean age of capital does not move sharply and that estimates of λ and μ (in alternative pure models utilizing the same data) do not differ much. Actually, estimates of λ tend slightly to exceed estimates of μ in the United States because of the secular upward trend in the mean age of capital in this country. In point of fact, then, a permanent increase in thrift does appear to be more effective in the new view than it does in the old view.

The Rate of Return on Current Investment

This paper has studied the effect on the path of output of a once-for-all increase in the investment ratio under alternative models. Presumably the purpose of such an increase would be to raise the time path of consumption (public and private). A higher consumption rate could be sustained in all future years.

But what if it were desired to increase only the consumption of a single future period? In this case clearly and perhaps more generally, the rate of return on investment would be a desideratum of investment policy.

The marginal productivity of investment, in the sense of $\frac{\partial Q}{\partial I}$, is determined in the old model by

$$(19) \qquad \frac{\partial Q}{\partial I} = a\frac{Q}{K} \qquad \text{(from (5))}$$

and the new model by

$$(20) \qquad \frac{\partial Q}{\partial I} = a\frac{Q}{J}\frac{\partial J}{\partial I} = a\frac{Q}{J}e^{\left(\frac{\lambda}{a}\right)t} \qquad \text{(from (4)).}$$

Since the assignment of the zero point is arbitrary, we can consider $\frac{\partial Q}{\partial I}$ only at $t = 0$ without loss of generality. Since all $v < t$ are then negative, old (surviving) capital goods will be assigned weights $e^{\frac{\lambda}{a}v}$ smaller than unity in J while all (surviving) capital goods receive unit weights in K. Hence $J < K$ (if there is any old capital) and the marginal product of investment is higher in the new view.

Some rough new-view estimates of the marginal product of investment in the United States may be of interest. The President's Council of Economic Advisers has compiled several time series of the fixed reproducible tangible capital in the business sector (excluding shelter and the output of government-owned enterprises) corresponding to different quality improvement rates of the λ type. A little manipulation of these data and the addition of corresponding inventory estimates by Goldsmith yield the following table: [9]

TABLE I

EFFECTIVE CAPITAL AND POTENTIAL OUTPUT IN THE U. S. BUSINESS SECTOR 1954
(in billions of dollars)

	Definition	Excluding Inventories	Including Inventories
K	No improvement	543	643
J′	2% improvement rate[1]	412	512
J″	3% improvement rate[1]	369	469
Q	4% unemployment	301	301

Source: Council of Economic Advisers and R. W. Goldsmith, *op. cit.*

[1] The improvement rate corresponds to $\frac{\lambda}{\alpha}$ in equation (4). Hence, if $\alpha = .4$, 3% improvement implies $\lambda = 1.2\%$. This is small but not so implausible when it is recalled that the 3% is a correction for quality improvement not already reflected in the conventional K series. The competition by producers of outmoded equipment tends to depress the capital goods price index below its appropriate level for K calculations to the extent that old equipment will no longer be produced when inventories of them are depleted. Also, some quality improvements are usually taken into account in the deflation of investment expenditures. In addition, econometric models of the mixed variety will normally show some technical progress of the μ type.

Making use of (19), the following estimates of the marginal productivity of 1954 investment can be computed:

TABLE II

POTENTIAL MARGINAL PRODUCTIVITY OF 1954 INVESTMENT

	$\alpha = .15$	$\alpha = .25$	$\alpha = .40$
K:	7.0%	11.7%	18.7%
J′:	8.8%	14.7%	23.5%
J″:	9.6%	16.0%	25.7%

9. Raymond W. Goldsmith, *The National Wealth of the United States in the Postwar Period* (Princeton: Princeton University Press, 1962), Statistical Appendix, Table A-39, Columns (2), (3) and (4).

The a values are for illustrative purposes only. They denote the elasticity of gross final potential business output with respect to effective business capital. Relative gross factor share data indicate that business before-tax quasi-rents as a ratio to business product at high levels of activity is somewhere between .25 and .40. This fact is a rough guide as to the value of a.

These marginal productivity estimates are equivalent to "gross earning" rates as defined by quasi-rents, aQ, divided by the market (equals replacement) value of the capital stock, J. This concept is gross of obsolescence and physical depreciation. To figure the *net* (social and private) *rate of return* to investment, we must deduct the annual proportionate decline in the real market value of the investment due to these causes.

By the net rate of return on investment we shall mean the marginal rate of transformation between next year's and this year's consumption *minus* one, subject to constancy of consumption possibilities in all subsequent periods. That is, let $-\dfrac{\partial C_1}{\partial C_0} - 1$, subject to $C_2, C_3, \ldots$ held constant, define the net rate of return on investment, where C_t denotes consumption t periods in the future.

Normalizing conveniently, and denoting the annual improvement factor by $\iota = \dfrac{\lambda}{a}$, we can write

$$P_0 = C_0 + I_0 = F(J_0, L_0)$$
$$P_1 = C_1 + I_1 = (1 + \mu)\, F\,(J_1, L_0) = F\Big(J_0(1 - \delta) + I_0, L_0\Big)$$
$$P_2 = C_2 + I_2 = (1 + \mu)^2 F\,(J_2, L_2) = F\Big(J_1(1 - \delta) + (1 + \iota) I_1, L_0\Big)$$

where δ, μ and λ measure simple annual rates. For simplicity we have assumed a constant labor supply. By $F(J,L)$ we mean the Cobb-Douglas function but we use this notation for convenience.

The consumption possibilities beginning two periods hence will be unchanged by this year's and next year's investment (consumptions) if and only if capacity output two periods hence, P_2, is constant. But this requires that J_2 be constant. Hence we have the constraint:

$$J_2 = [J_0(1 - \delta) + I_0]\,(1 - \delta) + (1 + \iota)\, I_1 = \text{constant.}$$

Now if we consume less this year in order to invest more, we can consume more in the future for two reasons: We will get more P_1.

And we will need less I_1 — to the extent the I_0 does not wear out — to meet our fixed J_2 goal. Algebraically,

$$-\frac{\partial C_1}{\partial C_0} = \frac{\partial P_1}{\partial I_0} = \frac{\partial I_1}{\partial I_0}$$

$$= \frac{\partial P_1}{\partial I_0} + \frac{1-\delta}{1+\iota}.$$

Finally we obtain the net rate of return:

$$-\frac{\partial C_1}{\partial C_0} - 1 = \frac{\partial P_1}{\partial I_0} - \frac{\iota+\delta}{1+\iota}.$$

$\frac{\partial P_1}{\partial I_0}$ is the marginal productivity of base year investment (here 1954). Evidently it is necessary to deduct from this the rates of "obsolescence," $\frac{\iota}{1+\iota}$, and "effective" depreciation, $\frac{\delta}{1+\iota}$, to obtain the net rate of return on investment. By applying this result to the marginal productivity estimates of Table II (and neglecting the lag between I_0 and the increase of capacity it creates) we obtain the illustrative rates of return in Table III.

TABLE III

Potential Rate of Return on 1954 Investment
Net of Obsolescence and 3% Physical Depreciation

	$a = .15$	$a = .25$	$a = .40$
$K(\iota = 0)$	4.0%	8.7%	15.7%
$J'(\iota = 2\%)$	3.9%	9.8%	18.6%
$J''(\iota = 3\%)$	3.8%	10.2%	19.9%

The table shows that, with respect to the higher and more "reasonable" values of a, the new view yields higher estimates of the net rate of return on investment in the United States around 1954. But it is interesting and possibly important to note that this implication of the new view could not have been taken for granted on a priori grounds. Suppose that the 1954 capital stock had been so up-to-date that the J's differed little from K. Or suppose we believed that a was only .15 or less because we thought quasi-rents as a ratio to final output, while in the neighborhood of 25–40 per cent, contained a very large element of monopoly profit. In either case — Tables II and III verify the second case — the alternative marginal productivity estimates corresponding to different improvement factors would be much smaller and would *differ so little* among themselves that the *net* rates of return would be smaller the larger is the assumed rate of obsolescence!

Summary

A growth model has been constructed which accommodates two types of technical progress. The first type can be implemented by existing capital while the second type needs to be embodied in new kinds of capital goods. Comparison of the solutions of the model corresponding to these two types reveals that:

(1) the limiting long-run growth rate depends on the rate of technical progress, not the type of progress;

(2) the elasticity of the limiting exponential growth path with respect to the investment ratio depends only on the capital elasticity of output, which is independent of the type of technical progress;

(3) no permanent, finite modernization of the capital stock can be achieved by increased thrift; in the model constructed here, the limiting equilibrium age distribution of capital depends only on the long-run rates of growth and depreciation and neither of these is affected by the fraction of income saved;

(4) the anticipated rise in the labor force growth rate in the United States will lead to a more modern stock, given a fixed investment ratio;

(5) normally, but not necessarily, the new-view model — which represents the second type of technical change — will paint a more adaptable economy, one faster to make the transition to the equilibrium growth path corresponding to a higher level of thrift;

(6) however, empirical estimates of the rate of technical progress (and other parameters of the model) might differ depending on which type of technical progress was assumed; in this event the two variants of the model will predict different limiting growth paths (corresponding to any investment ratio) and different limiting growth rates; this complicates at least the answer to the question of which "view" of technical progress offers the larger investment incentive;

(7) finally, the new view implies a higher estimate than does the old view of the "potential" net rate of return to 1954 United States business investment; however, this result need not hold in all countries nor in this country at all times; if the capital stock is sufficiently up-to-date or capital's income share sufficiently small, then the new-view implication that additional investment today would satisfy future capital requirements which would be "cheap" to fill with the super-investments of tomorrow will operate to reduce our estimate of the rate of return to present investment.

Yale University

SUBSTITUTION, FIXED PROPORTIONS, GROWTH AND DISTRIBUTION

TWO OPPOSING CONCEPTS of capital pervade contemporary models of economic growth. The fixed-proportions school treats the labor requirements of capital goods as rigid, not subject to choice. The neoclassical school imagines that capital is like putty; it can be continuously reshaped to accommodate any supply of labor. There may be some truth in both concepts. This paper presents a model which incorporates elements of both.

In the present model, only new capital is putty. Before their installation, machines can be designed to utilize any desired amount of labor. But once this putty takes shape, it turns to hard-baked clay. The labor requirements of machines are fixed forever at the time of construction. The utilization of these machines may change over time, but that is a different matter.

One of the products of this model is a theory of the operating life and labor intensity of capital goods. A machine is retired here when rising wages have absorbed all its revenues. Therefore, a machine will operate longer the smaller its labor intensity. The labor intensity of the optimal type of new machine depends upon the anticipated course of wages and the rate of interest.

These relationships introduce a new dimension to the connection between investment and the growth of productivity. An increase in thrift lowers the rate of return on capital, reduces the labor intensity of new machines, and thus ultimately lengthens the operating life of all machinery. We call this process "capital lengthening" to distinguish it from capital deepening, which denotes here the multiplication of machines without any change of their longevity. Increased thrift affects productivity through both the lengthening and deepening of capital.

It follows that an increase in thrift, far from modernizing the capital stock, except temporarily, must eventually increase the average age of machinery. But is maturity of the capital stock a bad thing? It is shown that the capital lengthening effect acts, on balance, to

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[1] The author is grateful to Edwin Mansfield and T.N. Srinivasan for discussions with them of this subject.

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reinforce the capital deepening effect of increased thrift upon productivity so long as there is so little capital that the rate of interest exceeds the long-run rate of growth. This condition appears to be the rule at least in technologically progressive economies. Thus investment may be a more effective growth agent in these economies than it has been judged to be on the basis solely of its capital deepening effect.

Another product of the model is a theory of factor shares. Since labor's relative share of aggregate output is a weighted average of its relative share of the output of every machine and that share is normally higher at old machines than new, the average age of machines is seen to affect aggregate shares. The average age will tend to be greater and the share of wages (quasi-rents) in total output smaller (larger) the thriftier is the economy. This may help to explain the positive relation observed between saving and profits (as shares of income) across economies.

In neoclassical models, capital's share equals the capital elasticity of output. Solow and others have used the former as an estimate of the latter in order to assess the importance of investment as a source of productivity growth in the American economy.[2] In the present model, capital's share falls short of its neoclassical level, so the procedure indicated would understate the importance of investment, wherever thrift is insufficient to drive the interest rate down to (or below) the rate of growth.[3] In the United States, therefore, where this condition is satisfied, it may be that the effectiveness of capital deepening has been underestimated (quite apart from the matter of capital longevity discussed above). Thus the results here encourage greater optimism about investment as a means to increase productivity.

Finally, some brief acknowledgments of the related literature: The notion that labor can be combined with new investment in variable proportions but with existing capital only in fixed proportions was introduced by Johansen.[4] A model along similar lines recently appeared by Massell.[5] This paper owes much to their ideas. This paper

[2] Robert M. Solow, "Technical Change and the Aggregate Production Function," *Review of Economics and Statistics*, XXXIX (August, 1957), 312-20.

[3] In the model here, "capital" and the "capital elasticity" do not exist, but an appropriate substitute is the investment elasticity of additions to capacity. The sentence above states that capital's share is smaller than this elasticity under the condition given.

[4] Leif Johansen, "Substitution versus Fixed Production Coefficients in the Theory of Economic Growth: A Synthesis," *Econometrica*, XXVII (April, 1959), 157-76.

[5] Benton F. Massell, "Investment, Innovation and Growth," *Econometrica*, XXX (April, 1962), 239-52.

differs from theirs in the treatment of the longevity of machinery as a dependent variable rather than as a parameter.

A third paper that takes a point of view nearer the one here is by Solow.[6] His paper is directed toward capital theory rather than growth theory, as here. However it has been useful at a number of places in the present paper, especially in Section 3.

1. THE SETTING

The setting is an economy or industry producing a single good by means of two scarce inputs, machinery and labor time.[7] While there is an infinite variety of machines with respect to their labor requirements, their durability cannot be varied: All capital lasts forever.

A basic notion of the model is the "capacity" of a machine. This is defined as the maximum output rate obtainable from it by means of increasing the amount of labor employed on it. $\bar{Q}(v, t)$ shall denote the capacity output at time t of all machines built at time v.

To produce their capacity output at time t, machines of vintage v require a certain (minimum) amount of labor time, denoted $\bar{N}(v, t)$. All investment consists of the purchase of new machines. Existing machines cannot be modified in any way.

While certain patterns of efficiency loss through wear and tear of equipment would be easy to introduce, it is simplest to assume that the capacity of a machine remains constant throughout its life. The labor requirement at capacity is also assumed to be fixed. Thus

$$\bar{Q}(v, t) = \bar{Q}(v, v) \qquad \text{for all } t \geqq v\,, \tag{1.1}$$

$$\bar{N}(v, t) = \bar{N}(v, v) \qquad \text{for all } t \geqq v\,. \tag{1.2}$$

Next we suppose that the producer neglects any possibility of underutilization of capacity when he buys a new machine. He assumes he can sell whatever output he can produce at the ruling market price. Pure competition prevails.

Further, an inverse relation between capacity utilization and unit variable (labor) costs is postulated, making it optimal (profit maximizing) for the machine to produce at capacity if it is preferable to produce

[6] Robert M. Solow, "Substitution and Fixed Proportions in the Theory of Capital," *Review of Economic Studies*, XXIX (June, 1962), 207–18.

[7] The "economy" interpretation raises the question of the source of machinery. One can think of the single good as homogeneous putty of unchanging quality in consumption. At the same time, engineers find increasingly efficient ways to shape this putty into machines. We ignore the machine-producing sector by assuming that no scarce inputs are required to mold putty into machinery.

at all.[8] It is optimal, therefore, to produce at capacity if revenues cover capacity labor costs with some quasi-rent left over; otherwise it is optimal to shut down the machine. Thus

$$Q^*(v, t) = \begin{cases} \bar{Q}(v, v) & \text{if } w(t)\bar{N}(v, v) < \bar{Q}(v, v)\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{1.3}$$

where $w(t)$ denotes the wage rate at t and $Q^*(v, t)$ denotes the optimal output rate of vintage v machines at time t.

Under these conditions the producer-investor who buys a new machine will expect to operate it at capacity so long as he expects it to be profitable to operate it at all. The producer is supposed to predict the future course of the wage rate with complete confidence and to predict that the wage will rise at a constant relative rate. Under these conditions he will expect to operate the machine continuously (at capacity) up to the date on which he expects the machine to cease to earn positive quasi-rents. He will expect to retire the machine permanently at that time.

Let $\hat{w}(u, t)$ denote the wage rate expected at time t to prevail at time $u, u \geqq t$. Of course, $\hat{w}(t, t) = w(t)$. And let $\omega(t)$ denote the constant relative rate of increase in the wage rate which is expected by producers at time t. Then

$$\hat{w}(u, t) = w(t) \exp\{\omega(t)(u - t)\}\,. \tag{1.4}$$

Thus a machine with capacity $\bar{Q}(t, t)$ and labor requirement $\bar{N}(t, t)$ would be expected when new—that is, at t—to produce and yield quasi-rent for $\hat{z}(t)$ years, where $\hat{z}(t)$ is determined by the relation

$$\hat{w}(t + \hat{z}(t), t)\bar{N}(t, t) = \bar{Q}(t, t)\,, \tag{1.5}$$

which, using (1.4), reduces to

$$w(t)\bar{N}(t, t) \exp\{\omega(t)\hat{z}(t)\} = \bar{Q}(t, t)\,. \tag{1.6}$$

It is evident that the prospective lifetime of new machines may vary through time so that $\hat{z}(t)$ is not a constant. Variations in $\omega(t)$, for example, will clearly produce changes in $\hat{z}$.[9] In Section 4, a growth model is presented in which $\omega(t)$ is made a dependent variable instead of a parameter.

[8] This assumes the absence of escapable overhead (labor) costs. All wage costs are variable costs, and all variable costs are wage costs.

[9] Also, note that it is only when we ignore gestation periods, as we do, that the initial wage rate that a new machine owner would have to pay, $w(t)$, can be taken as a datum. If a gestation period were introduced, the investor would choose a machine for time $u, u > t$, on the basis of the wage rate expected to prevail then, $\hat{w}(u, t)$.

In order to select the optimal type of new machinery the producer needs a discount rate to compare prospective quasi-rents occurring at different times. Let $\hat{r}(u, t)$ denote the rate of return which the firm at time t expects to prevail at time u. The firm knows at any time t the rate of interest, $r(t)$, currently prevailing. It will be assumed that producers expect the rate of interest to remain at its current level even though past rates of interest may have differed from the present rate. Thus

$$\hat{r}(u, t) = r(t) \qquad \text{for all } u > t \,. \tag{1.7}$$

Of course, producers may be wrong and $r(t)$ may in fact change through time.

Producers have to determine the scale and labor inensity of new machines in the light of these expectations and the technological possibilities before them. To eliminate the scale decision we take as exogenous the constant-dollar level of expenditures on new machinery at time v, $I(v)$, and derive the implied $r(v)$. With $I(v)$ given, the problem reduces to the question of "labor intensity": Shall those I dollars be spent on a type of machinery requiring much or little labor (at capacity)? Presumably an engineer who is hired to design a machine costing I dollars can offer one having greater capacity the greater is the amount of labor which the machine can utilize. Labor-using machinery which does not produce more than labor-saving machinery is obviously inefficient and would never be used. We shall assume that the relations among capacity, labor requirement and cost of new (efficient) plants at time v are given by the familiar Cobb-Douglas function,

$$\bar{Q}(v, v) = B(v)I(v)^{\alpha}\bar{N}(v, v)^{\beta} \,, \qquad \begin{aligned} &B'(v) > 0 \,, \\ &0 < \alpha < 1 \,, \\ &0 < \beta < 1 \,. \end{aligned} \tag{1.8}$$

The function $B(v)$ indicates the state of the technology at time v. The assumption of competition requires that $\alpha + \beta \leqq 1$ and through much of the paper it is required that $\alpha + \beta = 1$ (constant returns to scale)..[10]

2. THE LABOR INTENSITY OF NEW MACHINERY

Since the labor requirement (for capacity output) of any type of

[10] On the economy interpretation, $I(v)$ can be measured in the same units as $Q(t, t)$, e.g., pounds of putty. On the industry interpretation, (1.8) effectively assumes that the prices of all machine types move equiproportionately so that investment outlays can be deflated without any index number problem arising.

machinery is immutable, once chosen, it is impossible to buy a machine now which uses the optimal amount of labor at all times during its lifetime. The machine having a labor requirement such that it yields the greatest possible flow of quasi-rent in the near future will not be the machine yielding the greatest possible flow of quasi-rent later when the wage rate is higher. The optimal machine type at time t has the labor requirement $\bar{N}(t, t)$ which maximizes the sum, denoted by U, of the expected discounted quasi-rents over its expected lifetime,

$$U = \int_t^{t+\hat{z}(t)} R(u, t) \exp\{-r(t)(u - t)\} du \,, \tag{2.1}$$

where $R(u, t)$ is the flow of quasi-rent expected as of t to accrue at time u.

By the assumptions made above,

$$R(u, t) = \bar{Q}(t, t) - w(t)\, \bar{N}(t, t) \exp\{\omega(t)(u - t)\} \,. \tag{2.2}$$

Equations (2.1) and (2.2) yield

$$\begin{aligned} U = {} & \bar{Q}(t, t) \int_t^{t+\hat{z}(t)} \exp\{-r(t)(u - t)\} du \\ & - w(t)\bar{N}(t, t) \int_t^{t+\hat{z}(t)} \exp\{[\omega(t) - r(t)](u - t)\} du \,, \end{aligned} \tag{2.3}$$

which is to be maximized with respect to $\bar{N}(t, t)$, subject to the production function (1.8) and the operating lifetime function (1.6).

To find the optimal $\bar{N}(t, t)$ we take the derivative $\partial U/\partial \bar{N}$, letting both $\hat{z}(t)$ and $\bar{Q}(t, t)$ vary with $\bar{N}(t, t)$, and equate it to zero. This yields

$$\begin{aligned} & \frac{\partial \bar{Q}}{\partial \bar{N}} \int_t^{t+\hat{z}(t)} \exp\{-r(t)(u - t)\} du \\ & - w(t) \int_t^{t+\hat{z}(t)} \exp\{[\omega(t) - r(t)](u - t) du \\ & + \frac{\partial \hat{z}}{\partial \bar{N}} [\bar{Q}(t, t) - w(t)\bar{N}(t, t) \exp\{\omega(t)\hat{z}(t)\}] \exp\{-r(t)\hat{z}(t)\} = 0 \,. \end{aligned} \tag{2.4}$$

The expression on the left has to be evaluated at the optimal $\hat{z}(t)$ if the solution of (2.4) for $\bar{N}(t, t)$ is to be optimal. Recalling (1.6) we note that the bracketed expression must equal zero. That a slightly smaller labor requirement will increase slightly the prospective lifetime of the plant is irrelevant to the choice of machinery because quasi-rent toward the end of the life of the plant is zero.

Since the integrands in (2.4) are exponential functions, an additional simplification is possible, and we obtain

$$\frac{\partial \bar{Q}(t, t)}{\partial \bar{N}(t, t)} = c_t(\hat{z})w(t) , \tag{2.5}$$

where

$$c_t(\hat{z}) \equiv \frac{r(t)}{r(t) - \omega(t)} \cdot \frac{1 - \exp\{-[r(t) - \omega(t)]\hat{z}(t)\}}{1 - \exp\{-r(t)\hat{z}(t)\}} .$$

Thus the marginal product of the capacity labor requirement is equated to the current wage rate multiplied by some constant $c_t(\hat{z})$, which is a reflection of expectations.

It is shown in Appendix A that $c_t(0) = 1$ and that $c_t(\hat{z})$ is monotonically increasing in $\hat{z}(t)$, for all $\hat{z}(t) > 0$.

Combining the marginal productivity formula from the production function (1.8),

$$\frac{\partial \bar{Q}(t, t)}{\partial \bar{N}(t, t)} = \beta \frac{\bar{Q}(t, t)}{\bar{N}(t, t)} , \tag{2.6}$$

with (2.5), we obtain

$$\frac{\bar{Q}(t, t)}{\bar{N}(t, t)} = \frac{c_t(\hat{z})w(t)}{\beta} . \tag{2.7}$$

Equations (2.7) and (1.6), which can be written

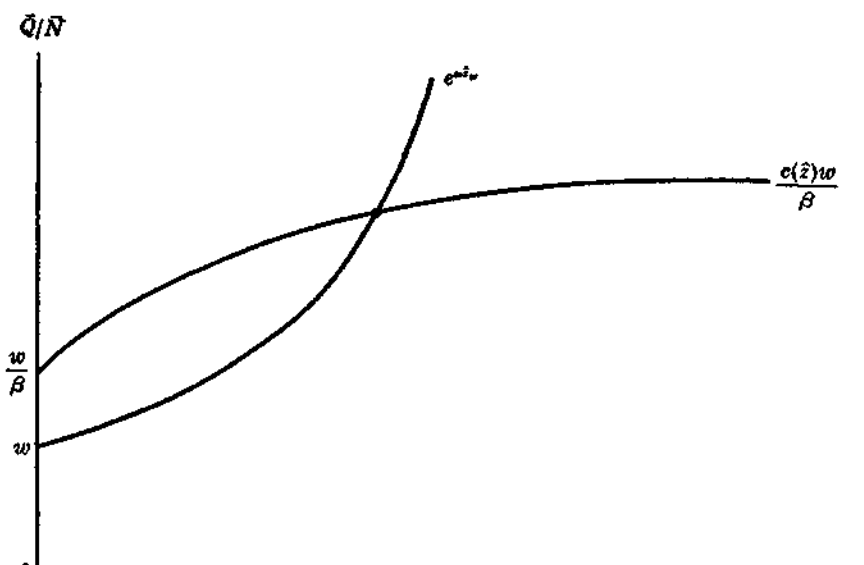

FIGURE 1

$$\text{(1.6a)} \qquad \frac{\bar{Q}(t,t)}{\bar{N}(t,t)} = w(t) \exp\{\omega(t)\hat{z}(t)\} ,$$

constitute two equations in two unknowns: the prospective life of the new machine, $\hat{z}(t)$, and the machine's optimal "labor intensity", as defined by the ratio $\bar{N}(t,t)/\bar{Q}(t,t)$. These two equations are graphed in Figure 1. The intersection of the curves marks the optimal values of $\bar{Q}/\bar{N}$ and $\hat{z}$.[11]

The values of $\bar{N}(t,t)$ and $\bar{Q}(t,t)$ depend upon the amount to be invested, $I(t)$, which we take as exogenous. The production function (1.8) and (2.7) imply

$$\text{(2.8)} \qquad \bar{Q}(t,t) = \beta^{\beta/(1-\beta)}\left[\frac{B(t)}{(c_t(\hat{z})w(t))^{\beta}}\right]^{1/(1-\beta)} I(t)^{\alpha/(1-\beta)}$$

and

$$\text{(2.9)} \qquad \bar{N}(t,t) = \left[\frac{\beta B(t)}{c_t(\hat{z})w(t)}\right]^{1/(1-\beta)} I(t)^{\alpha/(1-\beta)} .$$

The equilibrium rate of interest remains to be determined. On the assumption stated earlier that the rate of interest at t, $r(t)$, is expected to remain at its current level, producers will invest at t an amount such that the marginal rate of return to investment just equals the rate of interest. Hence the equilibrium competitve rate of interest must satisfy the same relation which defines the marginal rate of return,

$$\int_t^{t+\hat{z}} \frac{dR(u,t)}{dI(t)} \exp\{-r(t)(u-t)\}du = 1 .$$

This states that the surplus over wages arising from the marginal investment is absorbed by interest costs.[12] If the left hand integral were greater (smaller) than unity, there would be a "pure" profit (loss) on the marginal investment. Since we take $I(t)$ as exogenous, it is the wage rate and interest rate which must bring about the equality.

[11] From equations (2.7) and (1.6a) we have $c_t(\hat{z}) \exp\{-\omega(t)\hat{z}(t)\} = \beta$. Denote the left hand expression by $g(\hat{z})$. Appendix D proves that $g(0) = 1$, $g'(\hat{z}) < 0$ and $g(\infty) = 0$. Also $0 < \beta < 1$. Therefore, there is just one value of $\hat{z}$, $0 < \hat{z} < \infty$, and hence, by (1.6a), just one value of $\bar{Q}/\bar{N}$ which satisfies these equations.

[12] Under constant returns to scale, all surplus is absorbed by interest costs, with no surplus going to a fixed factor like land. In this case, the rate of interest also satisfies the relation

$$\int_t^{t+\hat{z}} R(u,t) \exp\{-r(t)(u-t)\}du = I(t) .$$

From (2.2) we have

$$\frac{dR(u,t)}{dI(t)} = \frac{\partial R(u,t)}{\partial I(t)} + \frac{\partial R(u,t)}{\partial \bar{N}(t,t)}\frac{d\bar{N}(t,t)}{dI(t)}$$
$$= \frac{\partial \bar{Q}(t,t)}{\partial I(t)} + \left[\frac{\partial \bar{Q}(t,t)}{\partial \bar{N}(t,t)} - w(t)\exp\{\omega(t)\}(u-t)\}\right]\frac{d\bar{N}(t,t)}{dI(t)}$$

for $u - t \leqq \hat{z}(t)$. Hence,

$$\int_t^{t+\hat{z}} \frac{dR(u,t)}{dI(t)} \exp\{-r(t)(u-t)\}du$$
$$= \int_t^{t+\hat{z}} \frac{\partial \bar{Q}(t,t)}{\partial I(t)} \exp\{-r(t)(u-t)\}du$$
$$+ \frac{d\bar{N}}{dI}\int_t^{t+\hat{z}}\left[\frac{\partial \bar{Q}(t,t)}{\partial \bar{N}(t,t)} - w(t)\exp\{\omega(t)(u-t)\}\right]\exp\{-r(t)(u-t)\}du$$
$$= \frac{\partial \bar{Q}(t,t)}{\partial I(t)}\left[\frac{1-\exp\{-r(t)\hat{z}(t)\}}{r(t)}\right] + \frac{d\bar{N}(t,t)}{dI(t)}\cdot\frac{\partial U}{\partial \bar{N}(t,t)}$$

from (2.4). But $\partial U/\partial \bar{N} = 0$ if $\bar{N}$ is optimal, so we obtain

$$\frac{\partial \bar{Q}(t,t)}{\partial I(t)} = \frac{r(t)}{1-\exp\{-r(t)\hat{z}(t)\}} \tag{2.11}$$

which upon differentiating the production function (1.8) yields

$$\frac{I(t)}{Q(t,t)} = \alpha\left[\frac{1-\exp[-r(t)\hat{z}(t)]}{r(t)}\right]. \tag{2.12}$$

Equations (2.8) and (2.12) determine $r(t)$. Of course, $r(t)$ depends upon $w(t)$. If there are constant returns to scale ($\alpha + \beta = 1$), $r(t)$ is independent of the extent of current investment, $I(t)$. But $w(t)$ will depend upon the total past investment.

Equations (2.7) and (2.12) yield the least-cost capital-labor ratio. If there are constant returns, they imply

$$\frac{I(t)}{\bar{N}(t,t)}\cdot\frac{\beta}{\alpha} = w(t)\left[\frac{1-\exp\{-[r(t)-\omega(t)]\hat{z}(t)\}}{r(t)-\omega(t)}\right]. \tag{2.13}$$

The right hand side is the marginal rate of substution. In the neoclassical case, in which capital is continuously reshaped, the future drops out and the marginal rate of substitution is equated simply to $w(t)/r(t)$. Here the future has to be taken into account.

3. EQUILIBRIUM AGGREGATE OUTPUT AND EMPLOYMENT

The equilibrium or "capacity" output of the entire industry (or economy), denoted $Q^*(t)$, is defined as the sum of the optimal outputs of the constituent firms. $N^*(t)$ shall denote the associated level of emploment. By (1.3), $Q^*(t)$ is equal to the sum of the capacities of those plants which are currently profitable to operate.

As indicated by (1.3), the vintages of those existing machines which are profitable are those for which $w(t)\bar{N}(v, v) < \bar{Q}(v, v)$. Denote the set of such vintages by $V(t, w(t))$. Then aggregate equilibrium output is[13]

$$Q^*(t) = \int_{v \in V(t,w(t))} \bar{Q}(v, v)dv \,, \tag{3.1}$$

and aggregate equilibrium employment is

$$N^*(t) = \int_{v \in V(t,w(t))} \bar{N}(v, v)dv \,. \tag{3.2}$$

Substituting (2.8) into (3.1) and (2.9) into (3.2), we obtain

$$Q^*(t) = \beta^{\beta/(1-\beta)} \int_{v \in V(t,w(t))} \left[\frac{B(v)}{(c_v(\hat{z})w(v))^\beta} \right]^{1/(1-\beta)} I(v)^{\alpha/(1-\beta)} dv \,, \tag{3.3}$$

and

$$N^*(t) = \beta^{1/(1-\beta)} \int_{v \in V(t,w(t))} \left[\frac{B(v)}{c_v(\hat{z})w(v)} \right]^{1/(1-\beta)} I(v)^{\alpha/(1-\beta)} dv \,. \tag{3.4}$$

Equation (3.4) provides a demand function for labor. At a sufficiently high current wage no existing machine can cover wage costs, so none operate and the amount of labor demanded is zero. As the wage rate falls, eventually the least labor intensive machines become profitable. The amount invested in that vintage determines the amount of labor demanded by these machines as shown by (2.9). The smaller the wage, the larger is the set of machines (or vintages) that can operate and earn quasi-rents (up to the point where all machines are operating). Thus $N^*(t)$ is a decreasing function of $w(t)$.

If the industry or economy being modeled is small, the real wage might reasonably be treated as a parameter. If we want to make the wage an endogenous variable, the simplest way to determine its level is to assume that the wage rate equates labor demand, $N^*(t)$, to a perfectly wage-inelastic supply of labor, denoted $L(t)$. Then, in equilibrium

$$L(t) = \beta^{1/(1-\beta)} \int_{v \in V(t,w(t))} \left[\frac{B(v)}{c_v(\hat{z})w(v)} \right]^{1/(1-\beta)} I(v)^{\alpha/(1-\beta)} dv \,. \tag{3.5}$$

[13] This notion of the set of profitable vintages is borrowed from the related paper by Solow, *op. cit.*

(3.5) determines how large $V(t, w(t))$ must be, and thus how low $w(t)$ must be, in order to employ the available labor supply. We assume that a nonnegative equilibrium wage rate exists.[14]

With $V(t, w(t))$ known, aggregate output can be computed from (3.3). A point of some methodological interest is the absence of $L(t)$ from the output equation. In Appendix B it is shown that it is not possible in general to express output as a function of labor and "capital." The reason appears to be that, loosely speaking, labor is not allocated over machines in any systematic way (according to a set of rules which apply at all times, e.g., "equalize labor's marginal productivity on all machines") but rather according to historical accident. Certain special histories do admit a production function. If the wage rate is stationary and expected to be so ($\omega(t) = 0$), aggregate output can be written as a Cobb-Douglas function of aggregate labor and capital. If the wage rate is constant, it makes no difference whether capital is putty or clay.

Summarizing: Given the history of investment and employment—thus an inventory of machines and their labor requirements—we can determine potential output and employment at various wage rates. Given the current labor supply, the equilibrium wage rate, output and employment are determinate. From the wage rate and the current rate of investment we can determine the labor requirement and capacity output of new machines, provided we know the expected rate of increase in the real wage. Thus the whole future course of output and the wage rate is determined.

As a model of growth, the above is incomplete in that it treats investment and wage expectations as exogenous. There is however one special case—the famous "golden age" of exponential growth—in which these problems can be solved simply if not entirely satisfactorily. These simplifications suggest certain short cuts which might be taken in practical application of the model. In particular the set $V(t, w(t))$ can be characterized quite easily. Also, to the degree that the exponential case approximates actual experience, the analysis may aid in the understanding of long-term growth and distribution.

4. EXPONENTIAL GROWTH

Exponential or golden-age growth may be defined as an equilibrium in which labor, investment, and output all grow at constant relative rates, with the latter two growth rates equal.

In many models this equilibrium will be approached asymptotically

[14] This is Solow's "nonredundancy assumption", *op. cit.*

upon the following conditions:

$$I(t) = sQ^*(t) \,, \qquad 0 < s < 1 \,, \tag{4.1}$$

$$B(t) = B_0 e^{\lambda t} \,, \qquad \lambda > 0 \,, \tag{4.2}$$

$$L(t) = L_0 e^{\gamma t} \,, \qquad \gamma > 0 \,. \tag{4.3}$$

That is presumably true of the model here, but we are unable to show the necessity or inevitability of this exponential equilibrium. We are able to find a golden-age solution to these equations. The difficulty lies in showing that it is the only asymptotic solution possible.

The solution found has the following properties. First, output and investment grow exponentially. Thus

$$Q^*(t) = e^{gt} Q^*(0) \,. \tag{4.4}$$

Second, there is an age level $z(t)$, such that all machines at time t which are older than $z(t)$ are too labor intensive to be profitable to operate while all newer machinery is profitable to operate; and this age level is constant through time. Thus

$$z(t) = z \,. \tag{4.5}$$

Of course, the values of of g, $Q^*(0)$, and z have to be determined.

Our procedure for showing that this is a solution to the model is to adopt (4.4) and (4.5) as assumptions. Then it is shown that associated with this trial solution is a time path of the real wage and interest rate which will sustain this equilibrium.

The first step in finding the long-run golden-age solution is to find the limiting distribution of employment over the operating vintages of machines. In Appendix C it is shown, by virtue of the equation (derived from (3.2), (4.2) and (4.3)),

$$L_0 e^{\gamma t} = \int_{t-z}^{t} \bar{N}(v, v) dv \,, \tag{4.6}$$

that, in long-run equilibrium, the amount of labor assigned to new plants (and also to plants of any age $x < z$) must grow exponentially at the same rate γ. Therefore, the "equilibrium" plant-age distribution of labor requirements is "exponential." The labor assigned to new machines is related to the total labor supply as follows:

$$\bar{N}(t, t) = L(t)\left[\frac{\gamma}{1 - e^{-\gamma z}}\right] = L_0 e^{\gamma t}\left[\frac{\gamma}{1 - e^{-\gamma z}}\right] . \tag{4.7}$$

Substituting into the production function, (1.8), the expression for

$\bar{N}(t, t)$ in (4.7), $I(t)$ as given in (4.1), and $B(t)$ as given (4.2), we obtain

$$\bar{Q}(t, t) = B_0 e^{(\lambda+\beta\gamma)t} s^{\alpha} Q^*(t)^{\alpha} L_0^{\beta}\left[\frac{\gamma}{1 - e^{-\gamma z}}\right]^{\beta} . \tag{4.8}$$

Noting that, by the constant z assumption,

$$Q^*(t) = \int_{t-z}^{t} \bar{Q}(v, v)dv , \tag{4.9}$$

we obtain

$$Q^*(t) = B_0 s^{\alpha} L_0^{\beta}\left[\frac{\gamma}{1 - e^{-\gamma z}}\right]^{\beta}\int_{t-z}^{t} e^{(\lambda+\beta\gamma)v} Q^*(v)^{\alpha} dv . \tag{4.10}$$

Little seems to be known about the asymptotic behavior of the solution(s) of nonlinear integral equations like (4.10). One solution is the exponential growth of $Q^*(t)$.

Retreating behind the exponential growth assumption, (4.4), we can write

$$Q^*(t) = B_0 s^{\alpha} L_0^{\beta}\left[\frac{\gamma}{1 - e^{-\gamma z}}\right]^{\beta}\int_{t-z}^{t} e^{(\lambda+\beta\gamma+\alpha g)v} Q^*(0)^{\alpha} dv . \tag{4.11}$$

It follows easily that $Q^*(t)$ grows at the rate $\lambda + \beta\gamma + \alpha g$; but also at the rate g by definition. Equating these we find that $Q^*(t)$ grows at the rate $(\lambda + \beta\gamma)/(1 - \alpha)$.

Next, solving for the "level," at some arbitrary $t = 0$, of the exponential time path of output, $Q^*(0)$, we find

$$Q^*(0) = s^{\alpha/(1-\alpha)}\left\{B_0 L_0^{\beta}\left[\frac{1 - e^{-gz}}{g}\right]\left[\frac{\gamma}{1 - e^{-\gamma z}}\right]^{\beta}\right\}^{1/(1-\alpha)} , \tag{4.12}$$

where $g = (\lambda + \beta\gamma)/(1 - \alpha)$.

Solutions resembling (4.12) have been obtained by Johansen, *op. cit.*, but our model differs from his in that the operating life of machines, $z(t)$, is a variable decided by economic considerations instead of a fixed parameter.

We turn now to the remaining unknowns. Capital's relative share, its operating life, and its rate of return have to be solved simultaneously. One of the links between the wage and the operating life of machinery is the *ex-post* analogue to the expectational equation (1.5),

$$w(t)\bar{N}(t - z, t - z) = \bar{Q}(t - z, t - z) . \tag{4.13}$$

If z is constant and $\bar{Q}(v, v)$ and $\bar{N}(v, v)$ grow exponentially at rates g and γ respectively, then $w(t)$ must grow exponentially at the rate

$g - \gamma$.

Recalling the exponential growth relation between $\bar{N}(t, t)$ and $L(t)$ in (4.7) and the output analogue

$$\bar{Q}(t, t) = Q^*(t)\left[\frac{g}{1 - e^{-gz}}\right], \tag{4.14}$$

we find the equilibrium wage rate as a function of productivity

$$w(t) = b(z)\, e^{-(g-\gamma)z} \frac{Q^*(t)}{L(t)}, \tag{4.15}$$

where

$$b(z) = \left[\frac{g}{\gamma} \cdot \frac{1 - e^{-\gamma z}}{1 - e^{-gz}}\right].$$

This equation is essentially (1.6a) of Section 2, and it is the first of three equations we need.

Another equation necessary for determining the operating life of machinery involves the rate of return on new investments. We suppose that investors are able in golden-age equilibrium to predict accurately the rate of return, the rate of increase of the wage ($\omega(t) = g - \gamma$) and the lifetime of new machinery ($\hat{z}(t) = z$). Then from the golden-age relations (4.1), (4.7), (4.14), and equation (2.12) it follows that

$$\alpha/s = f(z),$$

where

$$f(z) = \left[\frac{r}{g} \cdot \frac{1 - e^{-gz}}{1 - e^{-rz}}\right]. \tag{4.16}$$

Since g, α, and s are constants, r (and $f(z)$) must be constant through time if z is constant.

The third and last equation for determing distribution and the operating life of machinery recognizes that the labor intensity of the economy's productive processes (whence also the operating life of machinery) is the product of investor decisions and is thus a function of the rate of return and the real wage. Turning back to (2.7) and combining this with (4.7) and (4.14) we obtain

$$\frac{Q^*(t)}{L(t)} = \frac{w(t)}{\beta} \frac{c(z)}{b(z)}, \tag{4.17}$$

where

$$c(z) = \left[\frac{r}{r - \omega} \cdot \frac{1 - e^{-(r-\omega)z}}{1 - e^{-rz}}\right].$$

Of course, $c(z)$ is $c_s(z)$ of Section 2 without the time subscript. Since r and ω are constant over time, the function $c(z)$ is independent of time.

The three equations (4.15), (4.16) and (4.17) contain three unknowns, the operating life of machinery, the rate of return and the ratio of the wage rate to output per unit of labor (i.e., labor's relative share).

To solve for r and z, write

$$\text{(4.15a)} \qquad \frac{W(t)}{Q^*(t)} = b(z)\, e^{-(g-\gamma)z}$$

and

$$\text{(4.17a)} \qquad \frac{W(t)}{Q^*(t)} = \beta \frac{b(z)}{c(z)}\,,$$

where $W(t)$ denotes the wage bill, $w(t)L(t)$.

Equating the two expression, we have

$$\text{(4.18)} \qquad c(z) = \beta\, e^{(g-\gamma)z}\,.$$

(4.18) together with (4.16) constitute two equations in two unknowns, r and z. These equations are graphed in the upper quadrant of Figure 2 below. The diagram shows that a solution exists for all values of s. Readers who wish to pursue this further may consult the accompanying footnote.[15]

[15] The relation between z and r in (4.18) can be derived from the lower quadrant of Figure 2. As r is increased, e.g., from r_2 to r_1, $c(z)$ pivots upward around the vertical intercept at $c(0) = 1$. This causes the intersection of $c(z)$ and $\beta e^{(g-\gamma)z}$ to move upward and leftward along the latter curve, thus reducing z. As r approaches infinity z tends to a positive lower limit $\bar{z}$, where $\beta e^{(g-\gamma)\bar{z}} = 1$. On the other hand, as r falls and approaches zero, $c(z)$ shifts downward causing the intersection with $\beta e^{(g-\gamma)z}$ to move upward and rightward without limit. Thus a finite z requires $r > 0$. The relation between z and r in (4.18) is therefore inverse and asymptotic to these two lower limits.

(4.16) also implies an inverse relation if $s < \alpha$. Then $r > g$ so that, relying once again on Appendix A, $f(z)$ is increasing in z with upper limit r/g. As r is increased, $f(z)$ pivots upward around the vertical intercept where $f(0) = 1$. This moves the intersection of $f(z)$ with $\alpha/s > 1$ to the left, thus reducing z. As r approaches infinity, z tends to zero. As r falls and approaches $(\alpha/s)g$, $f(z)$ pivots downward, approaching horizontality, so that the intersection with α/s moves to the right without limit, so that z approaches infinity. Since $(\alpha/s)g > g > 0$, the inverse relation between r and g implied by (4.16) must intersect the relation implied by (4.18).

If $s = a$, $r = g$ independently of z.

If $s > \alpha$ then $r < g$, and $f(z)$ is decreasing in z with lower limit r/g so that r and z are positively related rather than inversely. As r approaches zero, so does z. As r approaches $(\alpha/s)g$, z approaches infinity. Since this upper limit to r, $(\alpha/s)g$, from (4.16) exceeds the lower limit to r, which is 0, from (4.18), the curves must cross and there is a solution.

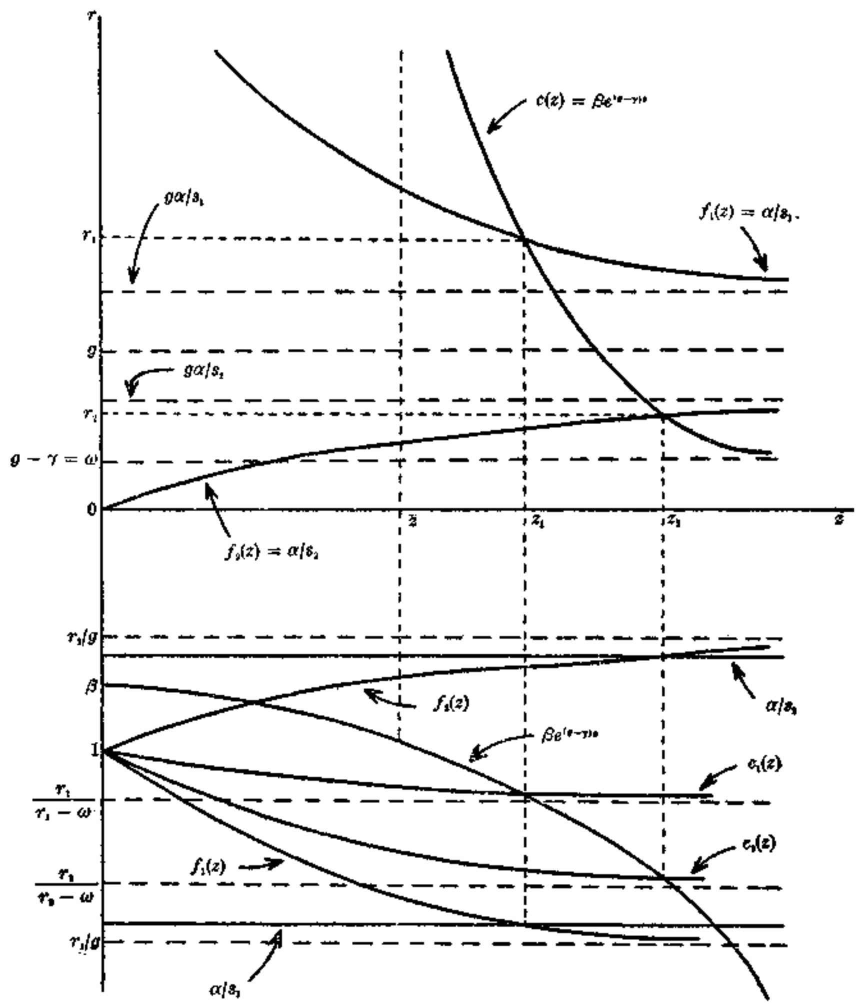

FIGURE 2

The graph indicates the effect of a change of the investment ratio from $s_1 < \alpha$ to $s_2 > \alpha$. The rate of return is decreased from $r_1 > g$ to $r_2 < g$, and the operating life of machinery is increased from z_1 to z_2.

What effect has a change in thrift upon labor's share of output? (4.15a) indicates that thrift influences labor's share through the operating life of capital. Appendix D shows that the right hand side of (4.15a) is decreasing in z and approaches zero as a lower limit.

(See Figure 3.) Increased thrift, therefore, by increasing the operating life of machines, decreases the share of wages in total output.

What is the relation of labor's share, thus determined, to the share labor would earn in this model's neoclassical analogue in which capital is putty? In the neoclassical version labor receives a share equal to β, the labor elasticity of output, of the produce of every machine throughout its operating life. Then labor's aggregate share is also equal to β. There is no such tendency in the present model where labor receives its marginal product on a given machine only for an instant during its operation.

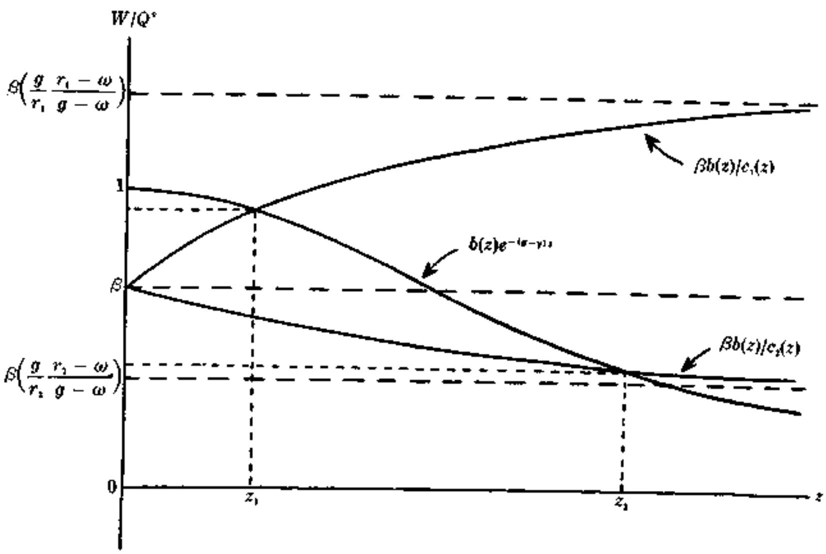

FIGURE 3

Figure 3, which graphs equations (4.15a) and (4.17a), shows the relation between $W(t)/Q^*(t)$ and β. If $s = \alpha$, then $r = g$ and $b(z)/c(z) = 1$ for all z, in which case $W(t)/Q^*(t) = \beta$. The number of "young" machines yielding a share to labor below β is balanced by the number of "old" machines on which labor earns a share exceeding β. If $s < \alpha$, $r > g$ and z is smaller, and hence $W(t)/Q^*(t) > \beta$. The "average" machine is more labor intensive as reflected in the smaller z and higher r. Obversely, if $s > \alpha$, $r < g$ and z is larger, so that $W(t)/Q^*(t) < \beta$. In this highly capital intensive case it takes a long time for a machine to grow old and yield labor a share of its output greater than β; the relative scarcity of old machines depresses labor's share below β.[16]

5. PRODUCTIVITY AND THRIFT

These latter results cast new light on the question of the historical importance and future utility of investment in raising productivity. The degree of importance, by almost any measure, is a function of the capital elasticity of output α. This might better be called the investment elasticity of new capacity, gross of retirements. Recent practice has been to take the relative share of capital income in total output as a measure of this elasticity. This is correct under the neoclassical assumption that old and new capital are both putty. But if old capital is brittle (and presumably if only old capital is less malleable than new capital), capital's share underestimates the capital elasticity in those economies where the rate of return exceeds the rate of growth. As a rule, progressive and industrialized economies do exhibit a growth rate well under the rate of return. In the U.S.A., for example, the latter might plausibly be put anywhere between 8 percent and 20 percent (averaging over all capital goods), but it is surely greater than 4 percent, roughly the American secular growth rate.

This finding is encouraging because if α is larger than has been thought, then so too are the opportunities for higher productivity through increased thrift. This follows from the solution for output as a function of s and z in (4.12).

The relation between thrift, as measured by s, and the equilibrium exponential output path, Q^*, is interesting. First there is the *direct* (capital-deepening) effect upon productivity of an increase in the investment ratio. A one percent increase in s will increase by one percent the number of machines of every age in the new equilibrium. The magnitude of this direct effect upon productivity is measured by the partial elasticity (holding z fixed) of Q^* with respect to s in (4.12). It equals $\alpha/(1-\alpha)$ which is increasing in α; for this reason a high α is favorable. Second, there is an additional *indirect* (capital-lengthening) effect of an increase in the investment ratio. As s increases so does z, in the manner described by Figure 2.

How does a change in the operating life of machinery affect productivity? One can imagine an economy in which the operating life of machinery is so small that productivity suffers from the crowding

[16] By (4.17a) if $g = \gamma$ (or $\omega = 0$), labor's share equals β independently of g and z. Labor earns its marginal product on all machines at all times and the neoclassical result is obtained. But the economy of the model is progressive ($\lambda > 0$), so that $g - \gamma = \omega > 0$. The example suggests, however, that the scope for possible divergence between β and labor's share is greater the more progressive is the economy.

of labor around brand-new machines. If the operating life were increased, productivity would rise because the available labor would have more, albeit less modern, machines on which to work. Continued lengthening of machinery's operating life would increase productivity without limit were it not that the progressively older machines being dusted off and assigned a portion of the labor force are progressively less "efficient" (more labor using) than the competing new machines.[17] Eventually a finite operating life is reached—call it $\bar{z}$—such that any further lengthening produces a net decline in productivity: The effect on productivity of spreading workers over more (already existing) machines is more than offset by the resulting decline in the average modernity and "efficiency" of machinery.

What is the typical position of the progressive economy with respect to this indirect effect? If thrift is sufficiently little that $z < \bar{z}$, then the indirect effect of an increase in thrift will reinforce the direct effect, both working to raise productivity. If thrift is so great that $z > \bar{z}$, the capital lengthening effect works against the capital deepening effect of increased thrift. We show now that progressive economies are typically in the former situation.

To find the algebraic sign of the indirect effect we take the partial derivative of the logarithm of Q^* with respect to z in (4.12). Assuming constant returns to scale, this equals $g/(e^{gz} - 1) - \beta\gamma/(e^{\gamma z} - 1)$, which is positive for $z < \bar{z}$ and negative for $z > \bar{z}$. A little manipulation shows that the derivative is positive if $b(z)e^{-(g-\gamma)z} > \beta$, in the notation of the previous section.[18]

The familiar left hand expression is none other than labor's relative share, W/Q^* (see (4.15a)). Therefore, the indirect effect is possitive if labor's share exceeds β. As argued earlier, the latter condition is the rule in progressive economies.

It follows that the indirect effect of an increase in thrift supports the direct effect up to the point where s reaches α, whence $r = g$ and $W/Q^* = \beta$. Further increases in s will cause $W/Q^* < \beta$, so that the indirect capital lengthening effect will work against the capital deepening effect. Thus there are increasing, then decreasing, marginal returns to thrift.

The proper objective of investment policy is not maximum productivity, but an optimal path of consumption. The area of investment

[17] Could a machine be found sufficiently old to absorb the whole labor force? No, because the labor force was never as large as now, so no such machine would have been built.

[18] The result was first obtained by Massell *op. cit.*

policy is well beyond the scope of the present paper. It may be of interest, however, to many readers that the policy of equating investment to profits (quasi-rents) which was proved to be a quasi-optimal policy for certain neoclassical models is also a quasi-optimal policy here.[19]

The investment ratio corresponding to the highest attainable exponential time path of consumption, $C(t)$, will be said to be quasi-optimal. Along this maximal path the total derivative of $C = (1-s)Q^*$ with respect to s will therefore be zero. This derivative is the sum of a direct and indirect effect

$$\frac{\partial}{\partial s}[(1-s)Q^*] + \frac{\partial z}{\partial s}\frac{\partial}{\partial z}[(1-s)Q^*] = 0$$

If the same value of s should happen to equate both $\partial C/\partial s$ and $\partial C/\partial z$ to zero this value would be quasi-optimal, for it would make the dervative zero independently of $\partial z/\partial s$. In fact such a solution occurs.

First,

$$\frac{\partial C}{\partial s} = -Q^* + (1-s)\frac{\partial Q^*}{\partial s} = 0$$

implies

$$\frac{s}{1-s} = \frac{\partial Q^*}{\partial s}\frac{s}{Q^*} = \frac{\alpha}{1-\alpha}\,.$$

Hence, $\partial C/\partial s = 0$ if $s = \alpha$.

But if $s = \alpha$, $W/Q^* = \beta$ so that $\partial C/\partial z = 0$ simultaneously. Therefore $s = \alpha$ is quasi-optimal. This policy equates investment to quasi-rents.

6. CONCLUDING REMARKS

Undoubtedly the reader can think of many desirable generalizations and modifications of the model.

The Cobb-Douglas production function, for example, was adopted out of convenience rather than any evidence of its validity. On the other hand, there is no convincing evidence against it. A more serious restriction may be the assumption that existing machines cannot be renovated. A renovation function applying to old machines is needed alongside the production function which applies to new machines. The investor then has to allocate abstract new capital between new and old machines. Moreover, he must consider the

[19] Edmund Phelps, "The Golden Rule of Accumulation: A Fable for Growthmen," *American Economic Review*, LI (September, 1961), 638–43.

extent to which renovations can extend the lifetime of old machines.

This is a "pure obsolescence" model in which the choice of physical durability of capital goods is elided. If less durable machines cost less, contrary to the assumption here, it might pay to buy machines that "expire" before becoming completely obsolete. Exponential decay is easy to introduce.

Finally, it may be important to introduce a second, machine building sector. Then an increase of the rate of saving will shift resources to this sector, changing the structure of the economy. This can be expected to introduce many complications and possibly to change some results.

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APPENDIX A

Let

$$f(z) = \frac{a}{b} \cdot \frac{1 - e^{-bz}}{1 - e^{-az}} ,$$

with $a > 0, b \leqq a$. Then

$$\lim_{b \to 0} f(z) = \frac{az}{1 - e^{-az}} ; \quad \lim_{z \to 0} f(z) = 1 ;$$

$$\lim_{z \to \infty} f(z) = \begin{cases} a/b & \text{if } b > 0 , \\ \infty & \text{if } b \leqq 0 . \end{cases}$$

To prove: that $f'(z) > 0$ for $z > 0$.

Case i: $b > 0$. Then

$$f'(z) = \frac{e^{-(a+b)z}}{(1 - e^{-az})^2} p(z) ,$$

where $p(z) = be^{az} - ae^{bz} + a - b$. The algebraic sign of $f'(z)$ is the sign of $p(z)$. Note that $p(0) = 0$. Therefore, if $p'(z) > 0$ for all $z > 0$ then $p(z) > 0$ and hence $f'(z) > 0$ for $z > 0$. Differentiating, we have

$$p'(z) = ab(e^{az} - e^{bz}) .$$

$p'(z)$ is positive for $z > 0$ if $a > b > 0$.

Case ii: $b < 0$. Rewrite $f'(z)$ in the form

$$f'(z) = \frac{-ae^{-az}}{b(1-e^{-az})^2}\, s(z) \ ,$$

where $s(z) = -be^{(a-b)z} - (a-b)e^{-bz} + a$. We have $s(0) = 0$ and

$$s'(z) = -(a-b)b[e^{(a-b)} - e^{-bz}] > 0$$

for $z > 0$. Hence $f'(z) > 0$ for all $z > 0$.

Case iii: $b = 0$. Then

$$f'(z) = \frac{a}{1-e^{-az}} - \frac{za^2e^{-az}}{(1-e^{-az})^2} = \frac{a}{(1-e^{-az})^2}\, q(z) \ ,$$

where $q(z) = 1 - e^{-az}(1 + az)$. We have $q(0) = 0$ and $q'(z) = a^2ze^{-az} > 0$ for $z > 0$. Hence $f'(z) > 0$ for all $z > 0$.

APPENDIX B

The total labor requirement of the economy is

$$\text{(B.1)} \qquad N^*(t) = \int_{v \in V(t,w(t))} \bar{N}(v, v)dv \ .$$

(B.1) together with (2.9) of the text yield

$$\text{(B.2)} \qquad N^*(t) = \beta^{1/(1-\beta)} G^*(t) \ ,$$

where

$$G^*(t) = \int_{v \in V(t,w(t))} \left[\frac{B(v)}{c_v(\hat{z})w(v)}\right]^{1/(1-\beta)} I(v)^{\alpha/(1-\beta)} dv \ .$$

Therefore, (2.9) can be written

$$\text{(B.3)} \qquad \bar{N}(v, v) = \frac{N^*(t)}{G^*(t)} I(v)^{\alpha/(1-\beta)} \left[\frac{B(v)}{c_v(\hat{z})w(v)}\right]^{1/(1-\beta)} ,$$

which, together with the production function (1.8), implies

$$\text{(B.4)} \qquad \bar{Q}(v, v) = B(v)I(v)^{\alpha} \frac{N^*(t)^{\beta}}{G^*(t)^{\beta}} I(v)^{\alpha\beta/(1-\beta)} \left[\frac{B(v)}{c_v(\hat{z})w(v)}\right]^{\beta/(1-\beta)} ;$$

this simplifies to

$$\text{(B.4a)} \qquad \bar{Q}(v, v) = \left[\frac{B(v)}{(c_v(\hat{z})w(v))^{\beta}}\right]^{1/(1-\beta)} I(v)^{\alpha/(1-\beta)} N^*(t)^{\beta} G^*(t)^{-\beta} \ .$$

Integrating to obtain aggregate output, we find

$$\text{(B.5)} \qquad Q^*(t) = H^*(t) G^*(t)^{-\beta} N^*(t)^{\beta} \ ,$$

where

$$H^*(t) = \int_{v \in V(t, w(t))} \left[\frac{B(v)}{(c_v(\hat{z}) w(v))^{\beta}} \right]^{1/(1-\beta)} I(v)^{\alpha/(1-\beta)} dv \,.$$

It is apparent that $G^*(t)$ and $H^*(t)$ are capital-like variables. Unfortunately they differ in the exponent over $(c_v(\hat{z})w(v))$. Therefore, they cannot be merged for all time paths of $I(v)$ unless $c_t(\hat{z})w(t)$ is constant over time. Then $G^*(t) = H^*(t)$, and output can be written as a function of "effective capital" $J^*(t)$

$$Q^*(t) = J^*(t)^{1-\beta} N^*(t)^{\beta} \,,$$

where

$$J^*(t) = \int_{v \in V(t, w(t))} \left[\frac{B(v)}{cw} \right]^{1/(1-\beta)} I(v)^{\alpha/(1-\beta)} dv \,.$$

This is essentially the production function obtained by Solow in his extension of the neoclassical model to the case of investment-embodied technical progress.[20]

It is interesting to notice that we can simply sum investments to obtain "capital" only if $\alpha + \beta = 1$. If $\alpha/(1-\beta) = 2$ (increasing returns), investments must be squared before being summed.

APPENDIX C

Differentiating the expression

$$L(t) = \int_{t-z}^{t} \bar{N}(v, v) dv \,, \tag{C.1}$$

we obtain, for all v,

$$\bar{N}(v, v) = \dot{L}(v) + \bar{N}(v - z, v - z) \,. \tag{C.2}$$

Therefore,

$$\bar{N}(v, v) = \dot{L}(v) + \dot{L}(v - z) + \dot{L}(v - 2z) + \cdots + \dot{L}(v - nz + z) + \bar{N}(v - nz, v - nz) \,. \tag{C.3}$$

Since $\bar{N}(v, v) \leqq L(v)$ by (C.1) and since $L(v)$ vanishes as $v \to -\infty$ by virtue of its exponential growth, $\bar{N}(v - nz, v - nz)$ goes to zero as n goes to infinity. Therefore, as n approaches infinity, we obtain

$$\bar{N}(v, v) = \dot{L}(v)\{1 + e^{-\gamma z} + e^{-2\gamma z} + \cdots\} \,, \tag{C.4}$$

whence, writing $\dot{L}(v) = \gamma L(v)$,

$$\bar{N}(v, v) = L(v) \left[\frac{\gamma}{1 - e^{-\gamma z}} \right] . \tag{C.5}$$

[20] R.M. Solow, "Investment and Technical Progress" in *Mathematical Methods in the Social Sciences*, ed, K.J. Arrow, S. Karlin, and P. Suppes (Stanford: Stanford University Press, 1960), 89-104.

APPENDIX D

Let

$$g(z) = f(z)e^{-(a-b)z} = \frac{a}{b}\frac{e^{bz} - 1}{e^{az} - 1},$$

where $f(z)$ is defined in Appendix A and $a > 0, b \leqq a$. Then

$$\lim_{b \to 0} g(z) = \frac{az}{e^{az} - 1}; \quad \lim_{z \to 0} g(z) = 1; \quad \lim_{z \to \infty} g(z) = 0.$$

To prove: that $g'(z) < 0$ for all $z > 0$.

There are three cases: $b > 0, b < 0$ and $b = 0$. The proofs that $g'(z) < 0$ in each of these cases parallel those for $f'(z) > 0$ in Appendix A.

INDUCED INVENTION, GROWTH AND DISTRIBUTION

ONE of the Great Ratios of contemporary economics is the ratio of wages (and of profits) to national income. Notwithstanding some correlation with slack in the economy and perceptible trends in some countries, distributive shares have been remarkably constant in most western economies. Yet the modern economist has almost ceased to wonder at Bowley's Law. He is familiar with the kind of growth model[2] in which a path of *golden age* growth is approached from most or all initial states. On a golden age path every variable changes, if at all, at a constant proportionate rate so that output, consumption, investment and capital all grow at the same rate. Then factor shares are constant, their magnitudes being a function of the parameters determining the particular golden age path that the economy will approach: the saving–income ratio, the population growth rate and the technological parameters.

But this kind of growth model does not really solve the puzzle of factor-share constancy. For a golden age growth path can exist only if technical progress is Harrod neutral along that path;[3] and Harrod neutrality entails that progress has a special factor-saving character. Further, in demonstrating the stability of the equilibrium golden age path—the tendency of the economy to approach that golden age path corresponding to the parameters of the model—it is usually postulated[4] that progress is Harrod neutral for *all* capital–labour ratios, and therefore that progress can be expressed as (purely) *labour augmenting*.[5] Thus, after scrutinising this growth model one is led to ask why progress should be assumed to be Harrod neutral, either in or out of a golden-age equilibrium.[6]

[1] The work on this paper was supported by a National Science Foundation grant to Yale University, and by a Ford Foundation grant to Massachusetts Institute of Technology and Yale University. The authors are indebted to D. Cass of Yale University for his valuable comments.

[2] See especially R. M. Solow, "A Contribution to the Theory of Economic Growth," *Quarterly Journal of Economics*, Vol. 70 (February 1956), pp. 65–94, and H. Uzawa, "Neutral Inventions and the Stability of Growth Equilibrium," *Review of Economic Studies*, Vol. 28 (February 1961), pp. 117–24.

[3] Progress is said to be Harrod neutral if, when the marginal product of capital is constant, the average product of capital and hence capital's share is also constant.

[4] See Uzawa, *op. cit.*

[5] Technical progress is said to be *factor augmenting* if the production $F(X_1, X_2, \ldots X_n; t)$ can be put into the form $G[A_1(t)X_1, A_2(t)X_2, \ldots A_n(t)X_n]$, where the X_i's are inputs and t denotes time. In effect, progress "augments" the inputs. The proportionate rate of increase of $A_i(t)$ is said to be the "rate of augmentation of the *i*th input." Progress is "purely *i*th factor augmenting" when only $A_i(t)$ increases over time, all other $A_j(t)$ coefficients constant.

[6] It is not only golden-age equilibria that can exhibit constant-factor shares. For example, if the saving–income ratio is exponentially declining there may exist an equilibrium growth path on which shares are constant, capital and output grow exponentially (at different rates) and consumption grows non-exponentially. But the existence of such a growth path, like a golden-age path, requires that progress have a special factor-saving character.

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We conclude that a satisfactory model of the evolution of factor shares—and such a model must be at the same time a model of economic growth—depends on a satisfactory theory of the factor-saving character of technical progress. This conclusion is not new. Hicks, Fellner and others,[1] observing the "constancy" of factor shares, presuming that the aggregate elasticity of substitution is less than one and deducing that progress is labour saving (in the Hicksian sense), have asked whether there is some market mechanism which slants progress in the labour-saving direction (in the Hicksian sense). Hicks asserted that there is, without specifying the mechanism. Fellner has argued that competitive firms will lean towards a relatively labour-saving invention only if they expect wages to rise faster than capital-good rentals—but even then, Fellner argued, the optimal invention may be capital saving. To our knowledge, this was as far as the theory of induced invention had gone until the publication of Charles Kennedy's thought-provoking paper on the subject.[2]

Kennedy takes a great step forward by introducing what we shall call the *invention possibility frontier*.[3] Previous writers failed to specify in their models, to conjecture as it were, the family of alternative new technologies (isoquants) which inventors can produce and from which firms must choose, given the original technology. It was primarily for this reason that the theory lacked any very useful results. Kennedy's postulated frontier characterises the alternative new isoquants that are producible, given the original isoquant. He combines this frontier concept with a maximisation postulate that may be a good first approximation, namely that firms seek to maximise, subject to the frontier, the current rate of cost reduction (hence, the current rate or intensity of technical progress), taking no interest in the factor-saving time of technical progress *per se*.

On these and other postulates, Kennedy shows that there may exist golden-age equilibria, all of which yield the same factor shares. In these equilibria, he shows, factor shares depend only upon the shape of the invention possibility frontier (at a particular point), not upon relative factor sup-

[1] J. R. Hicks, *The Theory of Wages* (London: Macmillan and Co., 1932), Chapter 6; W. J. Fellner, "Two Propositions in the Theory of Induced Innovations," ECONOMIC JOURNAL, June 1961, pp. 305–8, and "Does the Market Direct the Relative Factor-saving Effects of Technological Progress?" in the National Bureau of Economic Research volume, *The Rate and Direction of Inventive Activity* (Princeton, N.J.: Princeton University Press, 1962).

[2] C. M. Kennedy, "Induced Innovation and the Theory of Distributive Shares," ECONOMIC JOURNAL, September 1964, pp. 541–7. Earlier unpublished work by Christian von Weizsäcker was remarkably similar. Kennedy generously suggests (p. 547) that his concept of the "innovation possibility frontier" can be derived from Kaldor's "technical progress function." This is not generally true.

[3] Kennedy calls it the "innovation possibility function" and prefers generally to speak of induced "innovation." By "innovation" we are accustomed to think of the introduction of known techniques not previously utilized; the costs of whose discovery have already been incurred. On that definition, there is no problem of choosing among innovations; the firm should accept all innovations that reduce unit costs. It seems to us more reasonable to speak of induced *invention*, as Hicks first had it.

plies (hence not upon the saving-income ratio) nor the elasticity of substitution as conventional growth theory holds. Thus, we are given a new theory of distributive shares in golden-age equilibrium.

But more needs to be done. Kennedy failed to show the stability of the factor-share equilibrium. If factor shares do not approach their equilibrium values for many initial conditions the new theory of equilibrium shares is uninteresting. Further, a constant saving–income ratio is implicit in Kennedy's golden-age analysis. We believe it is useful to consider the behaviour of the model under a broader saving postulate. Finally, Kennedy worked with a fixed-proportions production model, in which the principle by which the real wage-rate is determined at any moment of time is not specified.

We present here a model of induced invention, based on an interpretation of Kennedy's invention possibility hypothesis. In this model technical progress is factor augmenting. The invention possibility frontier indicates the maximum rate of labour augmentation corresponding to a given rate of capital augmentation. Production and distribution at any moment of time are governed by the customary neoclassical principles. We then investigate, under certain postulated saving behaviour, the existence, uniqueness and stability of a growth equilibrium (not necessarily a golden age) in which factor shares are constant. A brief summary of the principal findings and some suggestions for improving the model conclude the paper.

I. BASIC CONCEPTS AND RELATIONS

We shall consider a one-sector economy composed of identical, purely competitive firms. The firm's production function, which is also the aggregate production function for the economy, is supposed to be twice differentiable (smooth marginal productivities), homogeneous of the first degree in capital and labour (constant returns to scale), with positive marginal products and diminishing marginal rate of substitution everywhere:

$$Q = F(K, L; t) \tag{1}$$

where Q denotes the rate of output, K the stock of capital, L the labour force and t is time. While capital and labour are each homogeneous—there is no capital-embodied or labour-embodied technical progress—there is technical progress of the " disembodied " kind if $\frac{\partial F}{\partial t} = F_t$ is positive.

Two characteristics of technical progress that are important to a neoclassical analysis of the evolution of factor shares are the *rate* or *intensity* of progress, R, and the factor-saving *bias* or *direction* of progress, D. We define the rate of progress as the (proportionate) rate of growth of output for fixed inputs:

$$R = \frac{F_t}{F} \tag{2}$$

Our measure of bias, in the Hicksian sense, is the proportionate rate of change of the ratio of the marginal product of capital (F_K) to the marginal product of labour (F_L), *i.e.*, the marginal rate of substitution, for a given capital–labour ratio:

$$(3) \qquad D = m_K - m_L$$

where
$$m_K = \frac{F_{Kt}}{F_K}, m_L = \frac{F_{Lt}}{F_L}$$

Both R and D are functions of the capital–labour ratio and time.[1] At a particular capital–labour ratio and at a particular time, technical progress is labour-saving in the Hicksian sense if $D > 0$, Hicks neutral if $D = 0$ and capital-saving if $D < 0$.

Another important concept is the elasticity of substitution:

$$(4) \qquad \sigma = -\frac{d(K/L)}{d(F_K/F_L)} \cdot \frac{F_K/F_L}{K/L}$$

The substitution elasticity may vary with the capital–labour ratio and time; we shall ultimately restrict (1) to be such that the quantity $\sigma - 1$ is of constant algebraic sign for all capital–labour ratios and time.[2]

Following Amano[3] and Diamond,[4] we can now derive an equation for the growth of capital's competitive share. This share, denoted a, is

$$(5) \qquad a = \frac{F_K K}{Q}$$

whence, letting $\hat{x}$ denote the proportionate rate of change of a variable x, $\frac{\dot{x}}{x}$,

$$(6) \qquad \hat{a} = \hat{F}_K + \hat{K} - \hat{Q}$$

Amano and Diamond have shown, from equation (1)–(4), that

$$(7) \qquad \hat{F}_K = m_K - \frac{1-a}{\sigma}(\hat{K} - \hat{L})$$

[1] These measures are in the same spirit as those defined by W. E. G. Salter in his *Productivity and Technical Change* (Cambridge: Cambridge University Press, 1960), Chapter 3. He defined the rate of progress as the proportionate rate of reduction of unit costs for given factor prices. His measure equals ours if there are constant returns to scale and factor prices equal the respective factors' marginal products. His measure of bias is the proportionate rate of increase of the least-cost capital–labour ratio for fixed factor prices and output. His measure is equal to ours times the elasticity of substitution; thus, the two measures are of the same algebraic sign. The measures employed here, which are the most convenient for the purposes at hand, have been used before. See J. C. H. Fei and G. Ranis, *Development of the Labor Surplus Economy* (Homewood, Illinois: Irwin, 1964); A. A. Amano, "Neoclassical Biased Technical Progress and a Neoclassical Theory of Economic Growth," *Quarterly Journal of Economics*, Vol. 78 (February 1964), pp. 129–38; and P. A. Diamond, "Disembodied Technical Change in a Two-Sector Model," *The Review of Economic Studies*, Vol. XXXII (April 1965), pp. 161–8.

[2] The "constant elasticity of substitution" production function of K. J. Arrow, H. B. Chenery, B. S. Minhas and R. M. Solow is an example of such a function. See their paper, "Capital–Labour Substitution and Economic Efficiency," *Review of Economics and Statistics*, Vol. 43 (August 1961), pp. 225–50.

[3] Amano, *op. cit.* [4] Diamond, *op. cit.*

Also,

(7a) $$\hat{F}_L = m_L + \frac{a}{\sigma}(\hat{K} - \hat{L})$$

Total differentiation of (1) with respect to time shows that

(8) $$\hat{Q} = a\hat{K} + (1 - a)\,\hat{L} + R$$

By constant returns to scale,

(9) $$Q = F_K K + F_L L$$

Partial differentiation of this with respect to time, holding K and L constant, yields

(10) $$R = a\, m_K + (1 - a)\, m_L$$

Namely, the rate of progress equals a share-weighted average of the rates of increase of the marginal productivities for fixed K and L.

Upon substituting (7), (8) and (10) into (6), one obtains the required equation for the growth of capital's share as a function of the bias, the substitution elasticity and the rates of increase of the factors:

(11) $$\dot{a} = a(1 - a)\left[D - \frac{1 - \sigma}{\sigma}(\hat{K} - \hat{L})\right]$$

This equation confirms a familiar proposition in distributive share theory. Constancy of non-zero shares ($\dot{a} = 0$) in the face of, say, a rising capital–labour ratio ($\hat{K} > \hat{L}$) requires that technical progress be labour saving ($D > 0$) if $\sigma < 1$, Hicks neutral if $\sigma = 1$, and capital saving if $\sigma > 1$.

If D, σ, $\hat{K}$ and $\hat{L}$ are exogenous or functions only of factor shares, then these functions and (11) are all that is required for the analysis of the evolution of capital's share.

We suppose that $\hat{L}$ is exogenous and constant:

(12) $$L(t) = L_0 e^{\gamma t} \text{ or } \hat{L} = \gamma \geqslant 0,\ L_0 > 0$$

Concerning K, we assume a constant or exponentially declining saving–income ratio:[1]

[1] The case of an exponentially declining saving–income ratio seems to be supported by the experience of the United States Economy. Thus, *e.g.*, S. Kuznets, *Capital in the American Economy: Its Formation and Financing* (Princeton, N.J.: Princeton University Press, 1961), summarising the trends in total capital formation in the United States in pp. 395–400, observes that the ratio of gross capital formation to gross national product, if measured in constant prices, is declining through time from 1869–88 to 1946–55, although if it is measured in current prices it appears as rather stable. He also observes that the ratio of net capital formation to national income, which is the relevant ratio for our purposes, shows a distinct downward trend over the same period.

Moreover, the findings of J. E. La Tourette, " Potential Output and the Capital–Output Ratio in the United States Private Business Sector, 1909–1959," *Kyklos*, Vol. XVIII (1965), pp. 316–32, and of P. A. David and Th. van de Klundert, " Biased Efficiency Growth in the U.S.," *The American Economic Review*, Vol. LV (June 1965), pp. 357–94, can be cited in support of our assumption. La Tourette finds a downward trend in the capital–output ratio during the entire period 1909–59 and also during the 1946–59 period. David and Klundert find a positive average rate of capital augmentation during the period 1899–1960. We will show below that an economy moving along a long-run equilibrium path should exhibit both these properties if $\eta > 0$.

$$(13) \qquad K(t) = \theta e^{-\eta t} Q(t) \quad \text{or} \quad \hat{K}(t) = \theta e^{-\eta t} \frac{Q(t)}{K(t)}$$
$$0 < \theta < 1, \eta \geq 0, K(0) > 0$$

As we have said above, it will be sufficient for our purposes to assume that $\sigma - 1$ is of constant algebraic sign.

Finally, we shall show that our Kennedy-based theory of induced invention makes the rate and bias of progress functions only of factor shares, hence $D = D(a)$, $R = R(a)$.

From (11), (12) and the relation $D = D(a)$ we obtain the differential equation

$$(14) \qquad \dot{a} = a(1 - a)\left[D(a) - \frac{1 - \sigma}{\sigma}(\hat{K} - \gamma)\right]$$

$\hat{K}$ is an endogenous variable for which we need another differential equation. From (13) we have, upon differentiation,

$$(15) \qquad \frac{\dot{\hat{K}}}{\hat{K}} = -\eta + \hat{Q} - \hat{K}$$

where $\dot{\hat{K}} = \dfrac{d\hat{K}}{dt}$, the absolute time rate of change of the (proportionate) growth rate of capital. Substituting the formula for the growth rate of output, (8), into (15) and using the relation $R = R(a)$ to be derived yields the second differential equation required:

$$(16) \qquad \dot{\hat{K}} = \hat{K}[R(a) - \eta - (1 - a)(\hat{K} - \gamma)]$$

Equations (14) and (16) form a complete system for the analysis of our model. Our assumptions in (13) that $\theta > 0$ and initial $K(0) > 0$ guarantee that $\hat{K} > 0$ for all t. Hence an " equilibrium " means a growth path such that $\hat{K}(t) = \hat{K}^* > 0$, $a(t) = a^*$, $0 < a^* < 1$ for all t. Such an equilibrium is defined by the following equations, derived from setting the bracketed expressions in (14) and (16) equal to zero:

$$(17) \qquad D(a^*) = \frac{1 - \sigma}{\sigma}(\hat{K}^* - \gamma)$$

$$(18) \qquad R(a^*) - \eta = (1 - a^*)(\hat{K}^* - \gamma)$$

We shall consider now a model of induced invention that provides a theory of the rate and bias of technical progress.

II. The Induced Invention Hypothesis

First, we suppose that technical progress is factor augmenting, meaning that the production function (1) is of the form[1]

$$(1') \qquad Q(t) = F[B(t)K(t), A(t)L(t)]$$

[1] We exclude the Cobb–Douglas function ($\sigma = 1$ everywhere), for, by the multiplicative character of that function, $A(t)$ and $B(t)$ are then not defined. Moreover, factor shares are then always constant, being equal to the factor elasticities of the function.

where the coefficients $B(t)$ and $A(t)$ are constants at any moment of time *i.e.*, independent of the capital–labour ratio.

Over time, firms can contrive to increase B or A or both by employing exogenously supplied inventors. The rates of factor augmentation, $\hat{B}(t)$ and $\hat{A}(t)$, are endogenous variables, being subject to choice by firms; the production function (1′) merely records the implications for output growth of the paths of $B(t)$ and $A(t)$ selected by firms.

Second, we suppose the existence of an invention possibility frontier, known to firms, that gives the maximum rate of labour augmentation obtainable for a given rate of capital augmentation.[1] This frontier, illustrated in Fig. 1, is postulated to be constant over time and invariant to the capital–labour ratio (hence to factor prices and shares). The frontier is strictly concave; ever-increasing amounts of labour augmentation must be sacrificed to obtain equal successive increments of capital augmentation. Positive rates of factor augmentation are feasible. Following Kennedy, we also assume that, after a certain point, a further increase in the rate of capital (labour) augmentation is necessarily accompanied by a negative rate of labour (capital) augmentation. Finally, we assume that the highest rates of factor augmentation, which can be achieved at any point of time, cannot exceed certain upper bounds, denoted by $\bar{\hat{B}}$ and $\bar{\hat{A}}$, respectively.[2]

Summarising mathematically we have:

(19) $\hat{A} = \Phi(\hat{B})$ with $\hat{B} \leq \bar{\hat{B}}$ and $A \leq \bar{\hat{A}}$, $\Phi(0) > 0, \theta'(\hat{B}) < 0, \Phi''(\hat{B}) < 0$

The last part of the present hypothesis is that firms will choose $\hat{A}(t)$ and $\hat{B}(t)$ so as to maximise the current rate of technical progress (because they wish to maximise the current rate of cost reduction) subject to the frontier (19).

Differentiation of (1′) partially with respect to time shows that the rate of progress is a share-weighted average of the rates of augmentation:

$$R = a\hat{B} + (1 - a)\hat{A} \tag{20}$$

Using (19) we can express the firms' maximisation problem as

$$\max_{\hat{B}} R = a\ \hat{B} + (1 - a)\ \theta\ (\hat{B}) \tag{21}$$

For any a, $0 < a < 1$, a unique interior maximum exists in which the derivative with respect to $\hat{B}$ of the maximand in (21) equals zero:

$$\begin{aligned} \frac{\partial R}{\partial \hat{B}} &= a + (1 - a)\ \theta'(B) = 0 \\ \theta'(\hat{B}) &= \frac{-a}{1 - a} \end{aligned} \tag{23}$$

[1] Such a frontier was postulated by Christian von Weizsäcker in unpublished work in 1962–63. The model he formulated was different from the model studied here.

[2] For the record, we mention that the present formulation of the invention frontier is slightly different from that followed in our earlier draft, Cowles Foundation Discussion Paper No. 189 (July 1965), in which we did not allow the frontier to extend to the negative quadrants. The change was prompted by the simplicity in the analysis that the present formulation permits.

This solution is illustrated in Fig. 1 by the tangency of the frontier with a straight line of slope $\frac{-a}{1-a}$.

One can see immediately from Fig. 1 that an increase of a increases the

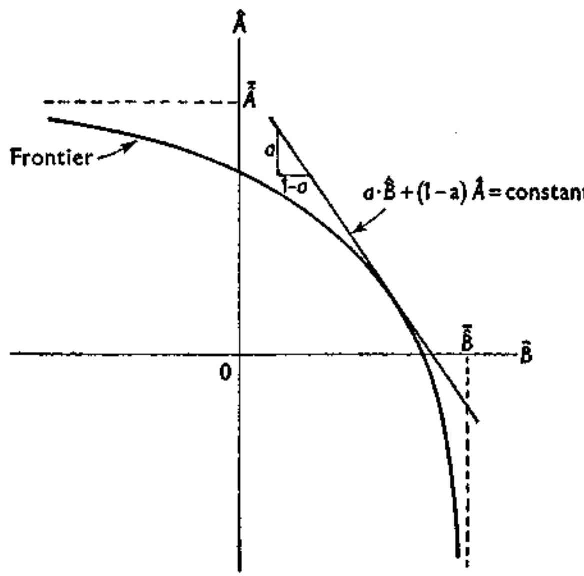

FIG. 1. Invention Possibility Frontier.

optimal $\hat{B}$ and decreases the optimal $\hat{A}$. In fact, total differentiation of (22) with respect to a confirms this:

$$(23)\quad \begin{aligned} \hat{B}'(a) &\equiv \frac{d\hat{B}}{da} = \frac{-1}{(1-a)^2\theta''(\hat{B})} > 0 \text{ since } \theta'' < 0 \\ \hat{A}'(a) &\equiv \frac{d\hat{A}}{da} = \theta'(\hat{B})\,\hat{B}'(a) < 0 \quad \text{since} \quad \theta' < 0 \end{aligned}$$

Thus, technical progress will be more capital augmenting on balance ($\hat{B} - \hat{A}$ larger) the larger is capital's share.

Summarising, the optimal $\hat{B}$ and $\hat{A}$ are continuous functions of a such that,

$$(24)\quad \begin{aligned} &\hat{A}'(a) < 0 \text{ for } 0 < a < 1, \lim_{a\to 0} \hat{A}(a) = \bar{\hat{A}}, \lim_{a\leftarrow 1} \hat{A}(a) = -\infty \\ &\hat{B}'(a) > 0 \text{ for } 0 < a < 1, \lim_{a\to 0} \hat{B}(a) = -\infty, \lim_{a\to 1} \hat{B}(a) = \bar{\hat{B}} \end{aligned}$$

We need now to express the bias of progress in terms of $\hat{B}$ and $\hat{A}$. In the appendix we derive the following formulas for the rates of increase of marginal productivities, holding factors fixed:

$$
(25) \qquad \begin{aligned} m_K &= \hat{B} - \frac{1-a}{\sigma}(\hat{B} - \hat{A}) \\ m_L &= \hat{A} + \frac{a}{\sigma}(\hat{B} - \hat{A}) \end{aligned}
$$

From these formulae and the definition (3) we obtain the bias:

$$
(26) \qquad D = \frac{1-\sigma}{\sigma}(\hat{A} - \hat{B})
$$

This result implies that technical progress which is labour augmenting on balance ($\hat{A} > \hat{B}$) will be labour saving ($D > 0$) or capital saving ($D < 0$) according as σ is less than or greater than one. Similarly, predominantly capital-augmenting progress will be capital saving if $\sigma < 1$ and labour saving if $\sigma > 1$. If $\hat{B} = \hat{A}$, progress is Hicks neutral for all σ.[1] Before proceeding now to the analysis of the model, we may provide a clearer understanding of the mechanism of induced invention hypothesised here by noting the following: We have seen that an increase (decrease) of capital's share increases (decreases) the rate of capital augmentation on balance; and that an increase (decrease) of capital augmentation on balance will make technical progress more (less) capital saving if $\sigma < 1$. But as (14) shows, an increase (decrease) of " capital savingness " depresses (raises) capital's share. Hence, if $\sigma < 1$ the mechanism of induced investment tends to stabilise capital's share around some equilibrium value. The subsequence analysis demonstrates, among other things, that $\sigma < 1$ everywhere is sufficient for the global stability of the factor-share equilibrium point.

III. The Behaviour of Factor Shares

Let us now examine the behaviour of factor shares and of the growth rate of capital under our induced invention hypothesis. From (14), (16), (20) and (26) we get the two differential equations:

$$
(27) \qquad \dot{a} = a(1-a)\frac{(1-\sigma)}{\sigma}[\hat{A}(a) - \hat{B}(a) - (\hat{K} - \gamma)]
$$

$$
(28) \qquad \hat{K} = \hat{K}(1-a)\left[\hat{A}(a) + \frac{a}{1-a}\hat{B}(a) - \frac{\eta}{1-a} - (\hat{K} - \gamma)\right]
$$

which, given (24), govern the behaviour of the share and growth rate of capital.

From (27) and (28) we see that an equilibrium, with $\hat{K}^* > 0, 0 < a^* < 1$, must satisfy

$$
(29) \qquad \hat{K}^* = \hat{A}(a^*) - \hat{B}(a^*) + \gamma
$$

[1] Kennedy, *op. cit.*, postulated a zero elasticity of factor substitution so that an increase of the rate of labour augmentation—or the rate of reduction of the unit-output labour requirement—implied greater labour-savingness of technical progress.

and

$$(30) \qquad \hat{K}^* = \hat{A}(a^*) + \frac{a^*}{1-a^*}\hat{B}(a^*) - \frac{\eta}{1-a^*} + \gamma$$

Equating the right-hand sides of (29) and (30) yields

$$(31) \qquad \hat{B}(a^*) = \eta$$

which determines a^*. Recalling that $\hat{A} = \Phi(\hat{B})$ and substituting (31) into (29), we obtain

$$(32) \qquad \hat{K}^* = \Phi(\eta) - \eta + \gamma$$

Thus (31) and (32) characterise an equilibrium of this system, if one (or more) exists. We shall now investigate the existence, uniqueness and stability of this equilibrium by geometric analysis.

In Fig. 2 (and also Fig. 3) there are two curves, labelled $\dot{a} = 0$ and $\dot{\hat{K}} = 0$. The first of these curves is the locus of points $(a, \hat{K})$ which make a constant, $0 < a < 1$. The equation of this curve, derived from (27), is

$$(33) \qquad \hat{K} = \hat{A}(a) - \hat{B}(a) + \gamma \equiv g(a)$$

Differentiating (33), we obtain

$$(34) \qquad g'(a) = A'(a) - B'(a)$$

By (24), $g'(a) < 0$ for $0 < a < 1$, $\lim_{a \to 0} g(a) = +\infty$, $\lim_{a \to 1} g(a) = -\infty$.

Thus, the $\dot{a} = 0$ curve is downward sloping for all values of capital's share a, approaching $+\infty(-\infty)$ as a approaches 0 (or 1).

Now, by (27) and (33), if $\sigma < 1$, then $\dot{a} < 0$ (a decreasing) when $(a, \hat{K})$ lies above this curve and $\dot{a} > 0$ (a increasing) when $(a, \hat{K})$ lies below this curve. The opposite is true if $\sigma > 1$. Supposing that $\sigma < 1$, we therefore show in Fig. 2 (a) arrows pointing in some easterly direction below the $\dot{a} = 0$ and westerly above. Fig. 2 (b) illustrates the case $\sigma > 1$.

The $\dot{\hat{K}} = 0$ curve is the locus of points $(a, \hat{K})$ which make $\hat{K}$ constant, $\hat{K} > 0$. The equation of this curve, derived from (28) is

$$(35) \qquad \hat{K} = \hat{A}(a) + \frac{a}{1-a}\hat{B}(a) - \frac{\eta}{1-a} + \gamma \equiv h(a)$$

Differentiating (35), we obtain[1]

$$(36) \qquad h'(a) = \hat{A}'(a) + \frac{a}{1-a}\hat{B}'(a) + \frac{\hat{B}(a) - \eta}{(1-a)^2} = \frac{\hat{B}(a) - \eta}{(1-a)^2}$$

Equations (35) and (36), together with (24), give the information required to plot the $\dot{\hat{K}} = 0$ curve.

[1] The last step in (36) follows from (22) and (23). Since

$$\frac{\partial R}{\partial \hat{B}} = 0, \hat{A}'(a) + \frac{a}{1-a}\hat{B}'(a) = \left[\Phi'(\hat{B}) + \frac{a}{1-a}\right]\hat{B}'(a) = 0 \text{ for all}$$

$a, 0 < a < 1$.

Let us first assume that $\eta = 0$, namely, a constant saving–income ratio. This is the case illustrated in Fig. 2. By (36) the $\hat{K} = 0$ curve is downward sloping for all values of capital's share a smaller than a^* for which $\hat{B}(a^*) = 0$, has a zero slope at a^* and is upward sloping thereafter. We observe that

$$h(a^*) = \hat{A}(a^*) + \gamma > 0, \text{ since } \hat{A}(a^*) > 0$$

and thus the $\dot{\hat{K}} = 0$ curve lies wholly in the upper quadrant.

If we examine the difference

$$g(a) - h(a) = -\frac{1}{1-a}\hat{B}(a)$$

we see that

$$\hat{B}(a) \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 0 \quad \text{implies} \quad g(a) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} h(a)$$

namely, that the $\dot{\hat{K}} = 0$ curve lies below the $\dot{a} = 0$ curve while it is downward sloping, it intersects the $\dot{a} = 0$ curve at a^*, having a zero slope at the intersection, and it lies above the $\dot{a} = 0$ curve for $a > a^*$. As a approaches 0 or 1, the difference $g(a) - h(a)$ becomes infinite.

Similarly, as a approaches 0 or 1 the slope of the $\dot{\hat{K}} = 0$ curves approaches $-\infty$ or $+\infty$, respectively, by virtue of (36) and (24). Finally, we can also show that the $\dot{\hat{K}} = 0$ curve itself starts at $\bar{\hat{A}} + \gamma$ when $a = 0$, and it approaches $+\infty$ as a approaches 1.[1]

If $(a, \hat{K})$ lies above the $\dot{\hat{K}} = 0$ curve, then $\dot{\hat{K}} < 0$ ($\hat{K}$ decreasing); if $(a, \hat{K})$ lies below this curve, $\dot{\hat{K}} > 0$ ($\hat{K}$ increasing); this is independent of σ, since (28) does not contain σ. Thus, the arrows above the $\dot{\hat{K}} = 0$ curve point south and the arrows below point north.

Fig. 2 shows that there exists in this case a unique equilibrium $(a^*, \hat{K}^*)$, with $0 < a^* < 1$ and $\hat{K}^* > 0$. a^* and $\hat{K}^*$ are determined by (31) and (32), respectively, which since $\eta = 0$, are given by

$$\hat{B}(a^*) = 0 \text{ and } \hat{K}^* = \hat{A}(a^*) + \gamma$$

Both a^* and $\hat{K}^*$ are positive, and this equilibrium is a golden age: output, capital, investment and consumption *all* grow at the "natural" rate $\hat{A}(a^*) + \gamma$, which is higher than γ since $\hat{A}(a^*)$ is positive. If the economy follows this golden-age path factor shares are constant, the rate of profit is constant and the wage-rate is increasing at the rate $\hat{A}(a^*)$. Of course, technical progress along this path is Harrod neutral, or in other words purely labour augmenting, since $\hat{A}(a^*) > 0$, $\hat{B}(a^*) = 0$.

Fig. 2 (a) also shows that if $\sigma < 1$, then this golden-age equilibrium is globally stable. The arrows showing the direction of the path of $(a, \hat{K})$, starting at any point in the upper quadrant of Fig. 2 (a) confirm this. At

[1] $\hat{A}'(a)$ and $\hat{B}'(a)$ must be assumed to remain finite as a approaches 0 or 1.

worst, $(a, \hat{K})$ can begin a counter-clockwise cycle around the equilibrium, landing in the zone bounded by the two curves in the interval $0 < a < a^*$. The arrows show that this zone traps $(a, \hat{K})$ and leads it to the equilibrium.

But if $\sigma > 1$, then as Fig. 2 (*b*) shows, the equilibrium is not stable even in the neighbourhood of equilibrium: $\dot{a}$ has the wrong sign for $(a, \hat{K})$ off the

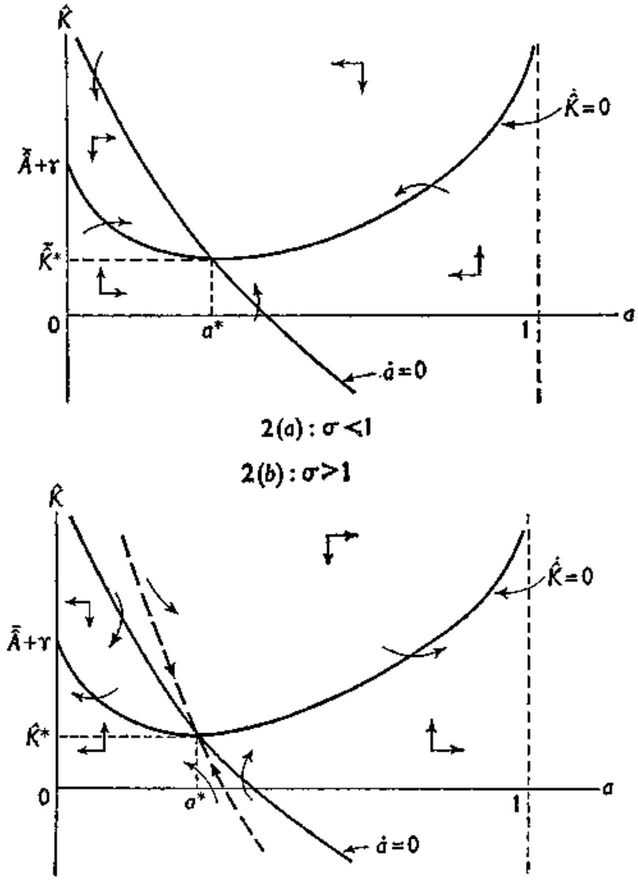

FIG. 2. Behaviour of the Share and Growth Rate of Capital if $\eta = 0$.

$\dot{a} = 0$ curve, with the result that a approaches zero or one asymptotically. However, Fig. 2 (*b*) shows that there exists a single path of $(a, \hat{K})$ leading to $(a^*, \hat{K}^*)$. Thus, if the initial position of the economy was such that at time zero $(a, \hat{K})$ lies in this path, then $(a^*, \hat{K}^*)$ will be approached asymptotically. Thus, if $\sigma > 1$ the equilibrium point is a saddle point.

Let us now consider the case $\eta > 0$, namely, the case of an exponentially declining saving–income ratio. We easily see that only the $\hat{K} = 0$ curve is thereby affected. Actually, this curve shifts downwards with an increase of η. It starts at $\bar{\bar{A}} + \gamma - \eta$, and it lies below the $\dot{a} = 0$ curve while it is downward sloping, namely as long as $\hat{B}(a) < \eta$ as it can be seen from (36) and from

$$g(a) - h(a) = \frac{\eta - \hat{B}(a)}{1 - a}$$

If $\eta < \hat{B}$, then there exists a value of capital's share a^*, $0 < a^* < 1$, for which $\hat{B}(a^*) = \eta$, and thus the $\hat{K} = 0$ curve intersects the $\dot{a} = 0$ curve at a^*, having a zero slope there, and it lies above the $\dot{a} = 0$ curve for $a > a^*$. The intersection of the two curves will occur in the upper quadrant if, and only if,

$$\Phi(\eta) - \eta + \gamma > 0$$

as it can be seen from (32). If thus η is not too large, then $\hat{K}^* > 0$, and the situation is not different from that in Fig. 2; a unique equilibrium point $(a^*, \hat{K}^*)$, with $0 < a^* < 1$ and $\hat{K}^* > 0$, exists and is globally stable if $\sigma < 1$.

However, we must point out that this equilibrium, although it exhibits constant factor shares, is *not* a golden-age equilibrium. Capital grows at a constant rate, but output grows at a higher constant rate, namely $\hat{Q}^* = \hat{A}(a^*) + \gamma > \hat{K}^*$. Also investment grows at a constant rate, namely, $\hat{I}^* = \hat{K}^*$, but consumption does *not* grow at a constant rate, since

$$\hat{C}^* = \hat{Q}^* + \frac{\eta\theta}{e^{\eta t} - \theta}$$

as it can be seen from (13). If the economy moves along this equilibrium path the growth rate of consumption is always higher than that of output, but it is declining through time. Finally, technical progress along this path is not Harrod neutral, since $\hat{B}(a^*) = \eta > 0$.

In Fig. 3, on the other hand, we analyse a case where there is no equilibrium because $\eta > \Phi(\eta) + \gamma$; the curves intersect in the lower quadrant.[1] However, this intersection is no longer relevant because an economically meaningful initial position of the system of equations (27) and (28) can only be in the upper quadrant, and the system cannot move to the lower quadrant. Actually, we can see from (28) that the $\hat{K} = 0$ curve which is relevant here consists of the curve $\hat{K} = h(a)$ for all a for which $h(a) \geq 0$, and of the horizontal axis $\hat{K} = 0$.[2] Thus, in this case there is what might be called a quasi-equilibrium state $(a^o, \hat{K}^o)$, given by the intersection of the $\dot{a} = 0$ curve with the horizontal axis. This quasi-equilibrium satisfies the equations

$$\hat{K}^o = 0 \tag{37}$$

$$\hat{A}(a^o) - \hat{B}(a^o) + \gamma = 0 \text{ [by (27)]} \tag{38}$$

[1] The curves may not even intersect, which will happen if $\eta \geq \bar{B}$. In this case the $\hat{K} = 0$ curve is everywhere downward sloping with infinite slope as $a \rightarrow 1$.

[2] In the extreme case where η is so large that $\bar{A} + \gamma < \eta$, only that horizontal axis is the relevant $\hat{K} = 0$ curve.

This state is not a feasible initial state—since $\theta > 0$ and $K(0) > 0$ imply that $\hat{K}(t) > 0$ for all t—but it can be approached asymptotically. It will always be approached (*i.e.*, it is globally stable) if $\sigma < 1$. The arrows in Fig. 3 indicate the direction of $(a, \hat{K})$ when $\sigma < 1$. The zone bounded by the $\dot{\hat{K}} = 0$ curve, the horizontal axis and the $\dot{a} = 0$ curve forces $(a, \hat{K})$ towards $(a^o, \hat{K}^o)$. As the arrows show, either $(a, \hat{K})$ goes directly to $(a^o, \hat{K}^o)$ or it cycles into this zone and thence to $(a^o, \hat{K}^o)$. Of course, as with

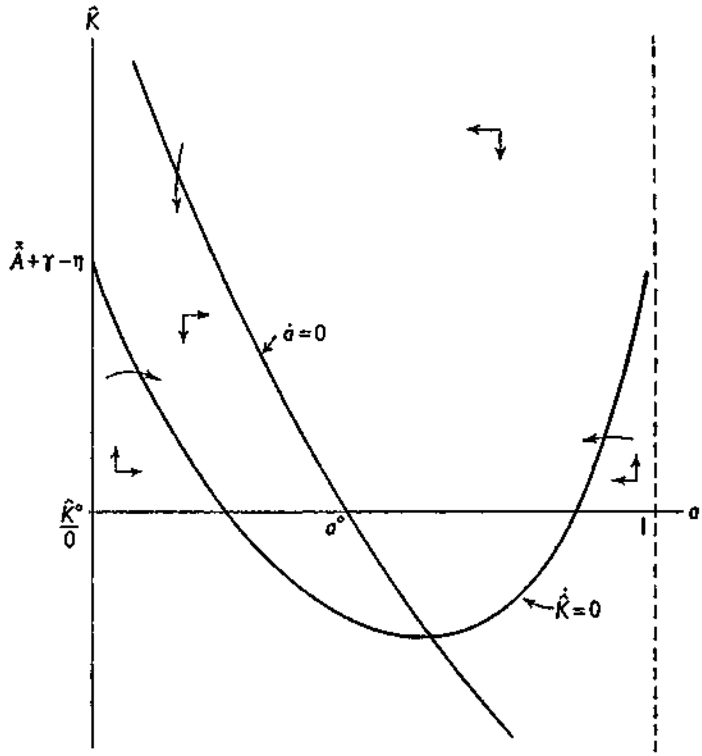

FIG. 3. Behaviour of the Share and Growth Rate of Capital if $\sigma < 1$ and $0 < \eta$, $\eta > \Phi(\eta) + \gamma$.

the approach to a true equilibrium, $\hat{K}$ approaches zero only asymptotically for as $\hat{K} \to 0$, $K \to 0$ by (31). The situation for $\sigma > 1$ is analogous to that of Fig. 2 (*b*). In this case $(a^o, \hat{K}^o)$ is a saddle point.

Finally, we wish to consider briefly some of the comparative static properties of the equilibrium point $(a^*, \hat{K}^*)$. If such an equilibrium exists, namely, if $0 \leq \eta < \Phi(\eta) + \gamma$, then (31) and (32) show that, given the invention possibility frontier, a^* and $\hat{K}^* - \gamma$ are functions only of η. Since $\hat{B}(a)$ is increasing in a, a^* is increasing in η. And since $\Phi(\eta) - \eta$ is decreasing in η, $\hat{K}^* - \gamma$ is decreasing in η, as expected. Since $\hat{Q}^* = \hat{A}(a^*) + \gamma$ [by (8), (20) and (29)], and $\hat{A}(a^*) = \Phi(\eta)$ is decreasing in η, $\hat{Q}^*$ is also decreasing in η. Finally, the rate of profit (wage-rate) is increasing faster (more slowly) as η increases, since $\hat{F}_K = \hat{B}(a^*) = \eta$ and $\hat{F}_L = \hat{A}(a^*) = \Phi(\eta)$. Thus, whenever $\eta > 0$, the rate of profit does not remain constant, even if the economy moves along the equilibrium path. It may also be of interest to note that since σ can, by (1′), be a function only of the ratio of augmented capital to augmented labour, BK/AL, and this is constant in equilibrium [by (32)], σ will be constant in equilibrium. Finally, an interesting point suggested to us by Professor Samuelson is that if the invention frontier is symmetrical, then $a^* = \frac{1}{2}$ when $\hat{K}^* - \gamma = 0$; for then $\hat{A}(a^*) = \hat{B}(a^*)$ and the a^* which makes the optimal $\hat{A}$ equal the optimal $\hat{B}$ is $\frac{1}{2}$. But by the same reasoning, when $\hat{K}^* - \gamma > 0$, then $a^* < \frac{1}{2}$. Hence the theory of long-run distributive shares which is offered here is consistent, on the assumption of a symmetrical frontier, with the observed tendency for capital's share to be less than one-half, since $\hat{K}^* - \gamma = \Phi(\eta) - \eta > 0$ for sufficiently small η.

IV. Concluding Remarks

By way of summary, the following results stand out. If the saving–income ratio is constant there will exist a unique equilibrium with Harrod-neutral technical progress, constant shares and golden-age growth. If the saving ratio is exponentially declining, but not too fast, again there will exist a unique factor-share equilibrium with exponential growth of capital and output. In both cases equilibrium shares depend only upon the shape of the invention possibility frontier and the equilibrium rate of capital deepening (which depends in turn upon the rate of decrease of the saving ratio)—not upon the initial level of the saving ratio nor the elasticity of substitution. However, the substitution elasticity was found to be critical for the stability of equilibrium. In particular, $\sigma < 1$ was found to be sufficient for the global stability of this equilibrium.

We note that, if the saving–income ratio is constant, technical progress is Harrod neutral in equilibrium, although it is not necessarily so along the actual path that the economy follows. Actually, technical progress is not Harrod neutral, even in equilibrium, if the saving–income ratio is exponentially declining. Also, if the invention frontier is assumed to be symmetrical the model will predict capital's share to be less than one-half in an economy enjoying steady capital deepening (capital outpacing labour).

We have many reservations about the model presented here. The vehicle for our analysis, a non-vintage, one-sector model of production, is undoubtedly unrealistic. Inventors produce new hardware, hence *capital embodied* progress, so that a " vintage " model of production is appropriate. In a multi-sector model one could study the problem of the optimal allocation of inventive effort among sectors. But the greatest weakness lies in the formulation of the invention hypothesis, and especially the decision rule followed by the firms of the economy.

Maximisation of the current rate of technical progress (or rate of cost reduction) is only a crude approximation to an optimal invention policy. Such maximisation may be shortsighted for two reasons. First, even if the invention frontier be stationary and even if expected wages and rentals are stationary, maximisation of current cost reduction may alter the shares of labour and capital in future unit costs and thereby, possibly, diminish the maximum rate of cost reduction attainable in the future. Second, even if there is to be just one invention, so that one does not need to consider the effect of current invention on the pay-off to optimal future invention, the firm will want to evaluate alternative inventions not only at current but also at future expected prices.[1] (The appropriate maximand under certainty and a perfect credit market is the reduction of the present discounted value of the stream of expected unit costs.)

We must also mention that the present formulation of the invention frontier, and particularly its extension to the negative quadrants, has certain weaknesses: *e.g.*, suppose that at a certain moment of time the value of capital's share is such that negative rates of capital or labour augmentation are chosen by the firms. Then if one considers the " old " and " new " unit-isoquants he would see that, at least for some types of production functions like the C.E.S. ones, the new isoquant may cross the old at some capital–labour ratio (or equivalently at a capital's share) *different* from the actual one on which the firms based their decisions. Equivalently, the rate of technical progress evaluated at a different capital's share may even become negative, if a larger weight is now assigned to the negative augmentation rate of one of the factors. Thus, if the invention frontier is as in Fig. 1 it may happen that an old isoquant may prove to be—at a later day—more profitable than the new one. Since we insist that the firms always use the new production processes which they have selected, we are in effect assuming that they " forget " the old technology as soon as they decide on the new one. This is surely very unrealistic, but the simplicity that it permits in the analysis is substantial, and it is probably worth the loss of realism. Actually, it is the myopic decision rule which is mainly responsible for these undesirable features. For even if the frontier did not extend outside of the positive orthant, it should be expected that the newly introduced technology would not in general be selected if the firms could take into account the effects of their current actions.

Most critical of all is the postulate of a stationary invention frontier of the form $\phi(\hat{A}, \hat{B}) = 0$. The postulate of factor augmenting progress is very restrictive. Second, such a function should vary with research effort and the latter should be endogenous in the model. Finally, a controversial objection to the stationality of the frontier has been raised by Professor Gary

[1] See Fellner, " Two Propositions," *op. cit.*

Becker. He suggests that if the rate of labour augmentation has been "abnormally" high for a long time maintenance of such a rate of labour augmentation will become increasingly expensive in terms of capital augmentation—that inventors might even exhaust (temporarily?) the possibilities for further labour augmentation. Clearly the notion, if accepted, that there are "normal" paths of the augmentation coefficients $A(t)$ and $B(t)$ from which the actual paths cannot far deviate, while not fatal to a mechanism of partially induced invention, may severely limit the scope of such a mechanism for explaining the behaviour of factor shares.

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Appendix I

Let

$$F(K, L;\ t) = G(BK, AL) \tag{1}$$

Then

$$F_K = G_1 B;\ a = \frac{F_K K}{F} = \frac{G_1 BK}{G};\ 1 - a = \frac{G_2 AL}{G} \tag{2}$$

$$\sigma = \frac{F_K F_L}{F\, F_{KL}} = \frac{G_1 B\, G_2 A}{G\, A\, G_{12} B} = \frac{G_1\, G_2}{G\, G_{12}} \tag{3}$$

where the subscripts denote partial derivatives. Hence

$$\begin{aligned} m_K = \frac{\partial F_K}{\partial t} \cdot \frac{1}{F_K} &= \hat{B} + \frac{\partial G_1}{\partial t} \frac{1}{G_1} \\ &= \hat{B} + (G_{11}\, K\, \dot{B} + G_{12}\, L\, \dot{A})\frac{1}{G_1} \\ &= \hat{B} + [(G_{11} BK)\hat{B} + (G_{12}\, AL)\hat{A}]\frac{1}{G_1} \end{aligned} \tag{4}$$

But $G_{11}BK = -G_{12}AL$ since G_1 is homogeneous of degree zero in BK and AL. Hence

$$\begin{aligned} m_K &= \hat{B} + \frac{G_{12} AL}{G_1}(\hat{A} - \hat{B}) \\ &= \hat{B} + \frac{G_2 AL\, G\, G_{12}}{G\, G_1 G_2}(\hat{A} - \hat{B}) \\ &= \hat{B} + \frac{1 - a}{\sigma}(\hat{A} - \hat{B}). \end{aligned} \tag{5}$$

The formula for m_L is derived similarly.

Appendix II

In the text of the article we have examined the system of differential equations (27) and (28) and showed that the solution approaches asymptotically the equilibrium given by (29) and (30) [or by (37) and (38)] provided that $\sigma < 1$. The solution, of course, depends on the initial conditions of the system, namely, $a_0 = a(0)$ and $\hat{K}_0 = \hat{K}(0)$. We have observed that, since $0 < a_0 < 1$ and $\hat{K}_0 > 0$, the solution remains always in the upper quadrant in Fig. 2 and 3.

However, a_0 and $\hat{K}_0$ are derived from the initial position of the economy, which is given by the values of K, L, B and A at time zero. We must establish the relation between these two sets of initial conditions, in order mainly to ascertain whether any point in the upper quadrant in Figs. 2 and 3 can in fact be an initial position of the economy.

We establish below that with any value of a and $\hat{K}$, $0 < a < 1$ and $\hat{K} > 0$, we can associate a unique positive value of B and x, where $x = \frac{BK}{AL}$ is the capital–labour ratio in " efficiency units." Thus, we see that *any* point in the upper quadrant in Figs. 2 and 3 can be an initial position for the system (27) and (28).

We can immediately see from (27) that $\hat{a} = -(1-a)\frac{1-\sigma}{\sigma}\hat{x}$, or

$$\frac{da}{dx} = -\frac{a(1-a)}{x}\frac{1-\sigma}{\sigma} \tag{1}$$

since

$$\hat{x} = \hat{B} + \hat{K} - \hat{A} - \hat{L}\,[1]$$

Now, if the elasticity of factor substitution, which itself is a function of x, is different from 1 for all x, then a is a strictly monotonic function of x and the inverse function exists. Thus, for any a, $0 < a < 1$, we determine a unique x, $x > 0$.

Moreover, from (13) we have for $t = 0$.

$$\hat{K} = \Theta\frac{Q}{K} = \Theta B\frac{g(x)}{x} \tag{2}$$

where $G(BK, AL(= AL\,g(x)$.

Given a, x and $\frac{g(x)}{x}$ are determined. Then, for any value of $\hat{K} > 0$ we can select values of B and A such that (2) is satisfied and x remains equal to its value determined by a.

Consequently, for any $(a_0, \hat{K}_0)$, $0 < a_0 < 1$ and $\hat{K}_0 > 0$, unique x_0, B_0 and $A_0 = \frac{B_0}{x_0}\frac{K}{L}$ are determined, provided that $\sigma \neq 1$ for all x, $x > 0$ (or equivalently for all a, $0 < a < 1$). It is, of course, quite natural that A should be given in terms of K/L.

[1] Of course, this can be shown directly if one expresses the production function and all other relations in terms of x.

ON THE INCREASE OF TECHNOLOGY

I shall explore three processes of technologic improvement in which new technical knowledge is some function of the number of researchers employed in that endeavor. These models bear on the questions, does there exist a "natural" rate of growth and is it positive only if population growth is positive?

1. Let $T(t)$ denote the index of technology at time t, and $R(t)$ the amount of research done at t. Then the simplest hypothesis may be the following:

$$T(t) = \int_0^t \Phi(R(v))\, dv \tag{1.1}$$

where $\Phi(R)$ is a continuous, increasing and concave function.

This hypothesis implies that T can grow without limit even if R does not. To see this set $R(t) = c$ in (1.1). Yet the *relative* rate of technologic increase,

$$\frac{\dot{T}(t)}{T(t)} = \frac{\Phi(R(t))}{T(t)} \tag{1.2}$$

approaches zero if $R(t)$ is constant or approaches some constant.

2. The preceding formulation fails to make the productivity of researchers, $\Phi(R)/R$, a function of the technology already in existence. Suppose that this productivity is a function of the index of technology w length of time earlier. The "retardation," w, might be interpreted as the "publication lag." Then

$$T(t) = \int_0^t H(R(v), T(v - w))\, dv \tag{2.1}$$

In this case,

$$\frac{\dot{T}(t)}{T(t)} = \frac{H(R(t), T(t - w))}{T(t)}, \tag{2.2}$$

which may behave in a variety of ways depending upon H as R is constant or growing.

3. As a special case, suppose that H is homogeneous of degree one in $T(v - w)$:

$$H(R(t), T(t - w)) = \Phi(R(t))T(t - w). \tag{3.1}$$

Then

$$\frac{\dot{T}(t)}{T(t)} = \Phi(R(t)) \frac{T(t - w)}{T(t)} \tag{3.2}$$

or

$$\dot{T}(t) = \Phi(R(t))T(t - w). \tag{3.2a}$$

If $R(t) = c$ then it may be simply verified that $\dot{T}(t)/T(t) = r$ in the transcendental equation

$$r = \Phi(c)e^{-rw} \tag{3.3}$$

Hence a constant rate of research permits a constant *relative* rate of increase of the technology index. As $w \to 0$, $r \to \Phi(c)$. For any w, r is increasing in c.

4. Alternatively it might be supposed that $H(R(t), T(t - w))$ is homogeneous of degree one in *both* $R(t)$ and $T(t - w)$. If technology has doubled we may still require twice the amount of research to double the time rate of new knowledge (current production of knowledge). In this case

$$\begin{aligned} \dot{T}(t) &= H(R(t), T(t - w)) \\ &= T(t - w)H\left(\frac{R(t)}{T(t - w)}, 1\right) \end{aligned} \tag{4.1}$$

Let $h(x) = H(x, 1)$. Then we have

$$\frac{\dot{T}(t)}{T(t)} = h\left(\frac{R(t)}{T(t - w)}\right) \frac{T(t - w)}{T(t)} \tag{4.2}$$

or

$$\dot{T}(t) = h\left(\frac{R(t)}{T(t - w)}\right)T(t - w) \tag{4.2a}$$

If $R(t)$ is a constant, $R(t) = c$, then $T(t)$ may increase without limit. But, as in the first model considered, $\dot{T}(t)/T(t) \to 0$ when $R(t)$ is constant if we make the natural restriction that $h(0) = 0$.

If T approached an upper limit then $\dot{T}/T$ would approach zero certainly. Suppose now that $T(t)$ increases without limit for $R(t) = c$. Then

$$\lim_{t\to\infty} \frac{\dot{T}(t)}{T(t-w)} = \lim_{t\to\infty} h\left(\frac{c}{T(t-w)}\right) = 0,$$

so

$$\lim \frac{\dot{T}(t)}{T(t)} \cdot \frac{T(t)}{T(t-w)} = \lim \frac{\dot{T}(t)}{T(t)} \cdot \lim \frac{T(t)}{T(t-w)} = 0$$

But if $T(t)$ increases without limit then

$$\lim \frac{T(t)}{T(t-w)} > 0$$

Hence

$$\lim \frac{\dot{T}(t)}{T(t)} = 0$$

Q.E.D.

On this second and slightly more plausible homogeneity hypothesis, therefore, rising research (hence rising population?) is necessary if there is to be steady productivity growth!

Suppose that $R(t)$ grows at a constant rate, $R(t) = R_0 e^{\rho t}$. Then, as may be easily verified, $T(t) = T_0 e^{\rho t}$ if R_0 and T_0 happen to satisfy the transcendental equation

$$\rho = h\left(\frac{R_0}{T_0} e^{\rho w}\right) e^{-\rho w}$$

It is conjectured that $T(t) \sim T_0 e^{\rho t}$ is the asymptotic solution for all initial states R_0, $T_0 > 0$.

INVESTMENT IN HUMANS, TECHNOLOGICAL DIFFUSION, AND ECONOMIC GROWTH

I. *Introduction*

Most economic theorists have embraced the principle that certain kinds of education—the three R's, vocational training, and higher education—equip a man to perform certain jobs or functions, or enable a man to perform a given function more effectively. The principle seems a sound one. Underlying it, perhaps, is the theory that education enhances one's ability to receive, decode, and understand information, and that information processing and interpretation is important for performing or learning to perform many jobs.

In applying this principle we find it fruitful to rank jobs or functions according to the degree to which they require adaptation to change or require learning in the performance of the function. At the bottom of this scale are functions which are highly routinized: e.g., running a power saw or diagnosing a malfunction in an automobile. In these functions, the discriminations to be made and the operations based on them remain relatively constant over time. In the other direction on this scale we have, for example, innovative functions which demand keeping abreast of improving technology. Even a highly routinized job may require considerable education to master the necessary discriminations and skills. But probably education is especially important to those functions requiring adaptation to change. Here it is necessary to learn to follow and to understand new technological developments.

Thus far, economic growth theory has concentrated on the role of education as it relates to the completely routinized job. In its usual, rather general form, the theory postulates a production function which states how maximum current output depends upon the current services of tangible capital goods, the current number of men performing each of these jobs, the current educational attainments of each of these jobholders, and time. To simplify matters, some analysts have specified a production function in which output depends upon tangible capital and "effective labor"; the latter is a weighted sum of the number of workers, the weight assigned to each worker being an increasing function of that worker's educational attainment. This specification assumes that highly educated men are perfect substitutes for less educated men (in the technical sense that the marginal rate of substitution between them is constant). Actually, it is possible that educated men are more sub-

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stitutable for certain capital goods than for other labor; they permit production with less complex machines. However, the exact specification of the production function does not concern us. The pertinent feature of this kind of production function is this: The "marginal productivity" of education, which is a function of the inputs and the current technology, can remain positive forever even if the technology is stationary. In the models we shall later introduce, education has a positive payoff only if the technology is always improving.

We shall consider now the importance of education for a particular function requiring great adaptation to change. We then propose two models which these considerations suggest.

II. *The Hypothesis*

We suggest that, in a technologically progressive or dynamic economy, production management is a function requiring adaptation to change and that the more educated a manager is, the quicker will he be to introduce new techniques of production. To put the hypothesis simply, educated people make good innovators, so that education speeds the process of technological diffusion.

Evidence for this hypothesis can be found in the experience of United States agriculture.[1] It is clear that the farmer with a relatively high level of education has tended to adopt productive innovations earlier than the farmer with relatively little education. We submit that this is because the greater education of the more educated farmer has increased his ability to understand and evaluate the information on new products and processes disseminated by the Department of Agriculture, the farm journals, the radio, seed and equipment companies, and so on.[2] The better educated farmer is quicker to adopt profitable new processes and products since, for him, the expected payoff from innovation is likely to be greater and the risk likely to be smaller; for he is better able to discriminate between promising and unpromising ideas, and hence less likely to make mistakes. The less educated farmer, for whom the information in technical journals means less, is prudent to delay the introduction of a new technique until he has concrete evidence of its profitability, like the fact that his more educated friends have adopted the technique with success.

This phenomenon, that education speeds technological diffusion, may take different forms outside of agriculture. In large, industrial corpora-

[1] See E. M. Rogers, *Diffusion of Innovations* (Free Press, 1962), especially Chap. 6.

[2] To be sure, some of the correlation described between education and diffusion may be spurious. Some farmers are undoubtedly both progressive and educated because they come from progressive and prosperous farming families that could afford to give them an education. But there is no question that educated farmers do read technical, innovation-describing literature more than do less educated farmers—and presumably because they find it profitable to do so.

tions, in which there is a fine division of labor, the function of keeping abreast of technological improvements (though perhaps not the ultimate responsibility for innovation) may be assigned to scientists. In this case, their education is obviously important; but so too is the education and sophistication of top management which must make the final decisions.[3]

So much for our broad hypothesis and the evidence supporting it. We shall consider now two specific models of the process of technological diffusion and the role of education.

III. *Two Models of Technological Diffusion*

We shall adopt a postulate about the factor-saving character of technical progress which permits us to speak meaningfully about the "level" or "index" of technology. Specifically, we suppose that technical progress is Harrod-neutral everywhere (i.e., for all capital-labor ratios), so that progress can be described as purely labor-augmenting. This means that if output, Q, is a function of capital, K, labor, L, and time, t, the production function may be written

$$Q(t) = F[K(t), A(t)L(t)] \tag{1}$$

In (1), the variable $A(t)$ is our index of technology in practice. If we interpret (1) as a vintage production function in which $K(t)$ is the quantity of currently purchased capital, $L(t)$ the labor working with it, and $Q(t)$ the output producible from it, then $A(t)$ measures the best-practice level of technology, the average technology level "embodied" in the representative assortment of capital goods currently being purchased. Alternatively, we could suppose that all technical progress is wholly "disembodied" and that (1) is the "aggregate" production function for the firm, industry or economy and $A(t)$ is the average index of technology common to all vintages of capital, old and new.

In addition to this concept, we introduce the notion of the theoretical level of technology, $T(t)$. This is defined as the best-practice level of technology that would prevail if technological diffusion were completely instantaneous. It is a measure of the stock of knowledge or body of techniques that is available to innovators. We shall suppose that the theoretical technology level advances exogenously at a constant exponential rate λ:

$$T(t) = T_0 e^{\lambda t}, \qquad \lambda > 0 \tag{2}$$

[3] For an interesting essay on science policy, in which it is argued that Britain's growth has suffered from a shortage of scientists in management, that too small a fraction of scientists are engaged in using (rather than adding to) the existing stock of knowledge, see C. F. Carter and B. R. Williams, "Government Scientific Policy and the Growth of the British Economy," *The Manchester School*, Sept., 1964.

First model. Our first model is as simple a one as we can invent. It states that the time lag between the creation of a new technique and its adoption is a decreasing function of some index of average educational attainment, h, of those in a position to innovate. (We may think of h as denoting the degree of human capital intensity.) Letting w denote the lag, we can represent this notion as follows:

$$A(t) = T(t - w(h)), \qquad w'(h) < 0. \tag{3}$$

The level of technology in practice equals the theoretical level of technology w years ago, w a decreasing function of h.

Substitution of (2) in (3) yields

$$A(t) = T_0 e^{\lambda[t - w(h)]} \tag{4}$$

If h is constant, two results follow from (4). First, the index of technology in practice grows at the same rate, λ, as the index of theoretical technology. Second, the "level" or path of the technology in practice is an increasing function of h, since an increase of h shortens the lag between $T(t)$ and $A(t)$.

An important feature of this model is that, *ceteris paribus*, the return to education is greater the faster the theoretical level of technology has been advancing. As equation (5) shows, the effect upon $A(t)$ of a marginal increase of h is an increasing function of λ, given $A(t)$, and is positive only if $\lambda > 0$.

$$\begin{aligned} \frac{\partial A(t)}{\partial h} &= -\lambda w'(h) T_0 e^{\lambda[t - w(h)]} \\ &= -\lambda w'(h) A(t). \end{aligned} \tag{5}$$

The same property is displayed by the "marginal productivity of educational attainment." Using (1) and (4) we have

$$Q(t) = F[K(t), T_0 e^{\lambda[t - w(h)]} L(t)] \tag{6}$$

Hence,

$$\begin{aligned} \frac{\partial Q(t)}{\partial h} &= \lambda T_0 e^{\lambda[t - w(h)]} L(t)[-w'(h)] F_2 \\ &= -\lambda w'(h) \times \text{Wage Bill}. \end{aligned} \tag{7}$$

Thus the marginal productivity of education is an increasing function of λ, given the current wage bill, and is positive only if $\lambda > 0$. This feature is not found in the conventional treatment of education described at the beginning of this paper.

This first model is not altogether satisfactory. It is unreasonable to suppose that the lag of the best-practice level behind the theoretical level of technology is independent of the profitability of the new techniques not yet introduced. Further, it is somewhat unrealistic to suppose that an increase of educational attainments instantaneously reduces the lag. In these respects, our second model is somewhat more realistic.

Second model. Our second model states that the rate at which the latest, theoretical technology is realized in improved technological practice depends upon educational attainment and upon the gap between the theoretical level of technology and the level of technology in practice. Specifically,

$$\dot{A}(t) = \Phi(h)[T(t) - A(t)] \tag{8}$$

or equivalently

$$\frac{\dot{A}(t)}{A(t)} = \Phi(h) \left[\frac{T(t) - A(t)}{A(t)}\right], \qquad \Phi(0) = 0, \qquad \Phi'(h) > 0. \tag{8'}$$

According to this hypothesis, the rate of increase of the technology in practice (not the level) is an increasing function of education attainment and proportional to the "gap," $(T(t)-A(t))/A(t)$.

Some results parallel to those in the first model can be obtained if we again postulate exponential growth of $T(t)$, as in (2), and constancy of h. First in the long run, if h is positive, the rate of increase of the level of technology in practice, $\dot{A}(t)/A(t)$, settles down to the value λ, independently of the index of education attainment. The reason is this: if, say, the level of h is sufficiently large that $\dot{A}(t)/A(t) > \lambda$ initially, then the gap narrowed; but the narrowing of the gap reduces $\dot{A}(t)/A(t)$; the gap continues to narrow until, in the limit, $\dot{A}(t)/A(t)$ has fallen to the value λ at which point the system is in equilibrium with a constant gap.

Another result is that the asymptotic or equilibrium gap is a decreasing function of educational attainment. Thus increased educational attainment increases the path of the technology in practice in the long run.

Both these results are shown by Figure 1 and by (9), which is the solution to our differential equation (8), given (2):

$$A(t) = \left(A_o - \frac{\Phi}{\Phi + \lambda} T_o\right) e^{-\Phi t} + \frac{\Phi}{\Phi + \lambda} T_o e^{\lambda t}. \tag{9}$$

As both (9) and Figure 1 imply, the equilibrium path of the technology in practice is given by

$$A^*(t) = \frac{\Phi(h)}{\Phi(h) + \lambda} T_o e^{\lambda t}; \tag{10}$$

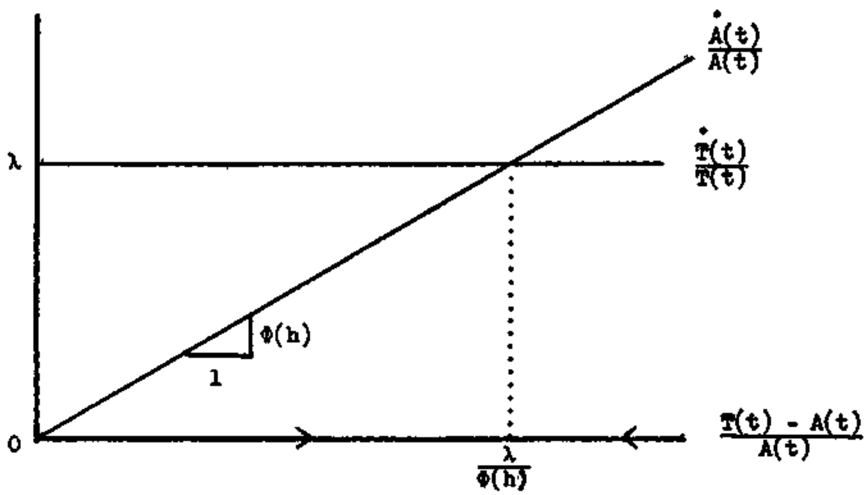

FIGURE 1

the equilibrium gap is given by

$$\frac{T(t) - A^*(t)}{A^*(t)} = \frac{\lambda}{\Phi(h)}\,. \tag{11}$$

In a technologically stangnant economy ($\lambda = 0$), the gap approaches zero for every $h > 0$. In a technologically progressive economy ($\lambda > 0$), there is a positive equilibrium gap for every h and λ. The equilibrium gap is increasing in λ and decreasing in h.

In the first model it was seen that the marginal productivity of educational attainment is an increasing function of λ and positive only if $\lambda > 0$. That is also true of the second model in the long run (once the effect of an increase of h has had time to influence the level of $A(t)$ as well as its rate of change). Equation (12) shows that the elasticity of the long-run equilibrium level of technology in practice, $A^*(t)$, with respect to h is increasing in λ:

$$\frac{\partial A^*(t)}{\partial h}\,\frac{h}{A^*(t)} = \left[\frac{h\Phi'(h)}{\Phi(h)}\right]\left[\frac{\lambda}{\Phi(h) + \lambda}\right] \tag{12}$$

This indicates that the payoff to increased educational attainment is greater the more technologically progressive is the economy.

These are only partial models and excessively simple ones. No machinery has been given for determining educational attainment.[4] The

[4] This is done in a paper by Phelps which develops a Golden Rule of Education. It is shown that Golden Rule growth requires more education the more technologically progressive is the economy.

theoretical level of technology has been treated as exogenous. Finally, it might be useful to build a model which combines elements of both the first and second model: the rate of technical progress in practice may depend both upon the length of time during which a new technique has been in existence and upon its profitability. But we hope that these two models may be a useful starting point.

IV. *Concluding Remarks*

The general subject at this session is the relationship between capital structure and technological progress. Recalling that the process of education can be viewed as an act of investment in people that educated people are bearers of human capital, we see that this paper has relevance to that subject. For, according to the models presented here, the rate of return to education is greater the more technologically progressive is the economy. This suggests that the progressiveness of the technology has implications for the optimal capital structure in the broad sense. In particular, it may be that society should build more human capital relative to tangible capital the more dynamic is the technology.

Another point of relevance for social investment policy may be mentioned. If innovations produce externalities, because they show the way to imitators, then education—by its stimulation of innovation—also yields externalities. Hence, the way of viewing the role of education in economic growth set forth here seems to indicate another possible source of a divergence between the private and social rate of return to education.

Finally, the connection between education and growth which we have discussed has a significant implication for the proper analysis of economic growth. Our view suggests that the usual, straightforward insertion of some index of educational attainment in the production function may constitute a gross misspecification of the relation between education and the dynamics of production.

POPULATION INCREASE

Since Malthus, economists with a classical outlook have always looked askance at population growth. The anti-classical viewpoint of Keynes and Alvin Hansen in the late 1930s produced only a temporary reversal of opinion. Now, more than ever, economists are taking fright at what they see to be the consequences of rapid population increase—not only in the "developing" economies but in the economically mature ones as well. In the former, the objections to present population growth rates run in terms of garden-variety variables like output per head. In the advanced countries, the discussion is more frequently in terms of "amenities" like privacy. Oddly, for an American economist, this increase of concern comes just when the "negative income tax"—with its implicit family allowances (already established in Canada and some other countries)—is getting up steam in the United States.

This paper attempts to sort out some of the influences upon "welfare" of population increase—more precisely, of the birth rate, taking mortality schedules as given. Because the analysis here is merely qualitative and there seem to be pluses as well as minuses attaching to an increase of the birth rate that sheer reason cannot weigh in the balance, no conclusion can be drawn as to whether or not population grows too fast.[1] Indeed I deal mainly with a some-

[1]This open-mindedness does not logically compel one to condone the withholding of birth-control information; there must be better ways to promote births (like family allowances) if a high birth-rate is desired.

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what hypothetical and *avant-garde* question: in a *laissez-faire* economy with universal birth-control knowledge, is the number of children that present parents would choose to have, in view of their knowledge of the private costs and benefits, equal to the number they would choose to have under accurate information on the full "social" costs and benefits?

I / Consumption per head and lifetime family utility

One of the best-known concepts in population analysis is the postulated *stationary state* relationship between output per head (equals consumption per head) and stationary population size. The former increases with the latter in the increasing-returns-to-scale stage; in this range, a larger community can spread its fixed-overhead-capital over more persons. But as population size is further increased so that the fixity of natural resources must eventually outweigh the overhead capital effect, a decreasing-returns-to-scale stage is reached where output per head is declining. The "optimum population size" is conventionally identified as the output-per-head-maximizing population level. (But the calculus of variations tells us that if the community would get enjoyment from enlarging itself and there is some positive discounting of future pleasures, then the long-run optimum exceeds that level.)

The parameters of this relationship are "capital per head" and the technology. Technical progress shifts the relation upwards so that any constant population can enjoy increasing output per head over time. (Technical progress, if it is resource-augmenting to any degree, will also move the maximum-point steadily to the right.) Population can then grow without any decline in output per head. But once continuing population growth is considered, we are faced with a new dimension of the problem: not only does population size have consequences but so does population growth. Indeed the importance assigned by some to the fixity of natural resources is sufficiently small that the time-dimension is the main source of their population worries.[2]

Let us, for the rest of this section, assume constant returns to scale in capital and labour. In those places where I discuss the utility of the representative family, I shall make the family's utility depend only on the time-profile of its own consumption and the number of its children; there will be no externalities, so the consumption and children of other families have no direct effect on the representative family's utility.

Consider first a once-for-all increase of population due to a temporary increase of the birth rate. The bulge of new babies immediately increases the population-capital ratio and so reduces output per head. Later it must reduce output per worker (though output per head may rise due to the labour-force bulge) as the labour-capital ratio increases. In the long run, we can be sure that the capital-labour ratio (like the labour-population ratio) will return to its equilibrium value so that ultimately output per worker and per head will return to their original values.

When the increase of population is steady and continuing, due to a permanent increase of the birth rate, the equilibration that takes place if the system

[2]See, for example, Goran Ohlin, *Population Control and Economic Development* (Paris, 1967).

is stable will prevent output per head and per worker from falling without limit. Correspondingly, we should not expect to find any negative correlation between the rate of growth of output (or of consumption) per head (or per worker) and the rate of growth of population.[3] But neither should we expect, in general, to find that output per head (or output per worker) tends to be restored to its no-growth level.

CONSUMPTION PER WORKER IN GROWTH MODELS

A model illustrating this equilibration is Robert Solow's neoclassical one-commodity growth model.[4] I shall take the investment-output ratio to be not only independent of income per head (all that Solow intended) but independent of the population growth rate as well; later that assumption will be removed. In that model, starting from a balanced-growth or golden-age state, an increase of the population growth rate reduces the capital-labour ratio (once the new people start reaching the labour market) and hence increases the output-capital ratio. The latter increase raises the rate of growth of capital, however, by virtue of the constant investment-output ratio. As the capital-output ratio rises towards the point at which the rate of growth of capital matches the new and higher population growth rate, the capital-labour ratio and hence the output-capital ratio begin to level off and we once again approach a state of balanced growth. In the new golden age, however, output per worker is smaller because the capital-labour ratio is lower. Because the consumption-output ratio is constant (by hypothesis), consumption per worker is also smaller.

In a *laissez-faire* economy, to which some interest attaches, it is not reasonable to suppose that the aggregate saving-income ratio will remain unchanged in the face of the higher birth rate. One piece of support for this proposition is the "life-cycle" theory of saving. Gustav Cassel taught a generation of continental economists that more rapid population growth makes young savers more numerous relative to old dissavers.[5] Hence we should expect the aggregate saving-income ratio to be greater the faster the rate of population growth. The analysis of the utility-maximizing time profile of the individual's wealth by Irving Fisher[6] and Frank Ramsey[7] is fundamental of the theory. Sir Roy Harrod coined the term "hump saving" to describe the individual's accumulation of capital in his early years in preparation for drawing it down in later life.[8]

In what may be the most famous unpublished paper in post-war economics,

[3]For evidence of a lack of correlation, see Richard A. Easterlin, "Effects of Population Growth on the Economic Development of Developing Countries," *Annals of the American Academy of Political and Social Science*, 369 (Jan. 1967), 98–108. I believe that a negative correlation would have emerged only if the rate of increase of population were historically correlated, across countries, with the rate of *acceleration* of population. The question is whether countries with fast-growing populations have been more likely to experience a *rising* population growth rate.

[4]Robert M. Solow, "A Contribution to the Theory of Economic Growth," *Quarterly Journal of Economics*, 70 (Feb. 1956), 65–94.

[5]Gustav Cassel, *The Theory of Social Economy*, trans. from 5th ed. (New York, 1932), 232–56.

[6]Irving Fisher, *Theory of Interest* (New York, 1930), chap. 5.

[7]Frank P. Ramsey, "A Mathematical Theory of Saving," *Economic Journal*, 38 (Dec. 1928), 542–59.

[8]Roy F. Harrod, *Toward a Dynamic Economics* (London, 1948), chap. 1.

Franco Modigliani and Richard Brumberg[9] carried out computations, using a simplified version of their life-cycle model of saving, that showed that the aggregate saving-income ratio is *almost* proportional to the steady-state or golden-age growth rate (equals the population growth rate plus the growth rate of income per head), there being only a little strict concavity. If the saving ratio were strictly proportional to the growth rate, a rise of the population growth rate would eventually (in the limit) be offset by enough additional saving so as to leave the capital-output, capital-labour, and output-labour ratios the same in the new resulting balanced-growth, golden-age state as in the original one. But note that consumption per worker must be smaller in the new golden age; for while output per worker is the same, a smaller proportion of output is being consumed (because of the higher aggregate saving ratio); the higher saving ratio is needed because a larger amount of output per worker must be devoted to capital formation in order to keep capital in unchanged proportion to the now faster growing supply of labour.[10]

It should be pointed out, as a brief digression, that the Modigliani-Brumberg analysis is intentionally a "partial" one: they do not reconcile the wealth-income ratio implied by the saving behaviour of their model with the capital-output ratio implied by their assumed rate of interest. In a paper that undertakes a theoretical and numerical analysis of a "complete" aggregative model,[11] James Tobin finds that the capital-labour ratio and capital-output ratio are smaller in the golden-age state to which the system moves the faster the population growth rate; the induced rise of the saving ratio is insufficient to prevent a fall of capital intensity and of output per worker. That point is clear in the two-period work-and-retire model employed by Peter Diamond.[12] Assume no change of real wage and real rate of interest. Then the saving by young workers (per worker) will not change. True, faster population growth will make their saving greater relative to the numbers and hence to the total consumption of retired persons, so the aggregate saving ratio must rise. But next period's labour force, who are the people who will use the capital saved by this period's young, will also be larger relative to this period's savers. So next period's capital-labour ratio must fall and the real rate of interest must rise. Hence invariance of the golden-age rate of interest (and of capital intensity) is impossible, at least in this simple *laissez-faire* model.

If we leave the world of *laissez-faire*, the response of the investment-output ratio and hence ultimately the response of capital intensity to a change of the population growth rate depends upon the behaviour of the fiscal and monetary authorities. There are several hypotheses about government behaviour that are worth considering. I shall mention here a recent paper of mine in which I

[9]Franco Modigliani and Richard Brumberg, "Utility Analysis and Aggregate Consumption Functions: An Attempt at Integration" (unpublished ms., 1956).

[10]Cassel's *Theory* contains a golden-age model and he makes the point that a higher share of output must be devoted to investment to maintain capital intensity the faster the rate of population growth, 32–41.

[11]James Tobin, "Life Cycle Saving and Balanced Growth," in W. J. Fellner *et al.*, *Ten Economic Studies in the Tradition of Irving Fisher* (New York, 1967).

[12]Peter A. Diamond, "National Debt in a Neoclassical Growth Model," *American Economic Review*, 55 (Dec. 1965), 1126–50.

postulated that the government, whatever the population growth rate it confronts, adjusts the investment-output ratio (by fiscal measures, say) so as to achieve golden-rule capital intensity in the golden-age state.[13] Hence the rate of interest (social rate of return to investment) is equated to the population growth rate (in the absence of technological change) and the consumption time-path is as high as it can be, compared to alternative golden-age paths, given the prevailing population growth rate; consumption per worker, consumption per head, and consumption-per-anything-exogeneous is maximized as of the prevailing growth rate. It is then shown that maximum consumption per worker is smaller the greater the population growth rate. In other words, the golden-rule state offers lower consumption standards in some sense the faster population grows. The same conclusion about consumption per worker would be reached if it were assumed that profits were invariably saved and wages invariably consumed.

LIFETIME FAMILY UTILITY IN A SIMPLE GROWTH MODEL

I have discussed the simple one-commodity growth model on three different assumptions about the investment-output ratio. In all of these cases we reached the conclusion that golden-age consumption per worker is smaller the faster is the rate of population growth (though the rate of growth of consumption per worker in balanced growth is equal to the rate of "labour augmentation" which we are holding constant for the time being). But what about "welfare"? My own tentative notions of the appropriate concept of welfare I leave for the next section. Here I shall pursue the question along what appear to be more conventional lines.

One notion of welfare is consumption per head. Then a difficulty arises: even if golden-age consumption per unit *labour* is smaller the faster population grows, it is not generally impossible that consumption per *head* be greater. The reason is that unless people enter the labour force only rather late in life, faster population growth (due, remember, to a higher birth rate) leads ultimately to a tilting of the age distribution in favour of working-age people relative to pre- plus post-working age people. So, conceivably, a decline of consumption per worker can be swamped by a rise of the labour-population ratio so as to cause consumption per head to rise. Simple calculations show that a small increase of the growth rate produces a *smaller* proportionate decline of the consumption-labour ratio the smaller the capital-consumption ratio in the golden-rule case; the same is true in the case of an investment-output ratio that is adjusted to keep capital intensity constant. A small increase of the growth rate produces a larger proportionate increase of the labour-population ratio the smaller the population growth rate. So we are more likely to obtain the paradox of a rise of consumption per head the smaller the capital-consumption ratio and the smaller the population growth rate. But as the population growth rate is further increased, we must eventually leave the paradoxical range. A simple model supporting these conclusions is presented in Appendix I.

Per capita consumption is certainly a relevant criterion of welfare when, as

[13]Edmund S. Phelps, *Golden Rules of Economic Growth* (New York, 1966), chap. 12.

advocated by some, the consumption of each individual is equalized (in the adult years perhaps). In a golden age, this equalization implies that the consumption of every individual is equalized over his life. Individuals could be led to make such an allocation by confronting them with an after-tax rate of interest equal to individual time preference. A case can be made for government intervention to achieve that allocation; different allocations can also be defended. However, it is also worth studying the welfare effect of faster population growth in a *laissez-faire* model.

In this *laissez-faire* model I am going to identify "welfare" with the lifetime utility of the parents of the representative family. This assumes that "all families are alike," (to misquote Tolstoy). The family is conceived as being born at marriage; at first it consists of two parents plus children, later of only the parents as children marry to form new families.

The number of children in the representative family have a "direct effect" on the lifetime utility of the representative parents; they "cost" the parents during their pre-adult years and, up to a point anyway, they may be assumed to give pleasure to the parents. (The reader is reminded that externalities are excluded here.) The number of children in the representative family also has an "indirect effect" arising from their effect upon capital intensity and hence upon the sequence of real consumption expenditures of the representative family over its life. A rise in the number of children per family causes an increase in the rate of population growth; this affects capital intensity and thus the sequence of real consumption expenditures of the representative family over its life. In this analysis, I shall hold over-all parental tastes constant; an increase of children per family is "allowed" or "coerced" (in an otherwise undistorting way) by the government despite the absence of change in parental preferences for children *versus* other goods. (This does not imply that time preference is invariant to the family's child intensity.) Second, I shall focus on the golden age; it will be pretty clear how things go in the short and medium run once we understand the consequences of faster population growth in the long run, that is, in the golden-age state that is approached.

What then is the indirect effect upon representative parental utility of a virtual movement from a low population-growth golden age to a high population-growth golden age? If the representative parents save so as to maximize lifetime utility, the question can be analysed in terms of the indirect utility function. This function makes *lifetime utility a derived function of the number of children in the family, the real wage, and the rate of interest.* Given the number of children per family, lifetime utility is greater the higher the real wage (clearly) and the higher the rate of interest—the latter being due to the fact that families must save in their early years for later retirement. The indirect effect upon utility, therefore, can be viewed purely in terms of the effect of the faster population growth upon these two factor prices.

In light of this, it is interesting to recall the hypothesis contemplated earlier (loosely inspired by the Modigliani-Brumberg model) that the saving-income ratio is proportional to the golden-age growth rate in golden-age states. Then capital intensity would be independent of the population growth rate in golden-age states and our two factor prices would be the same in these golden-age states. Consequently there would be absolutely no indirect effect upon lifetime parental utility resulting from an increase of the number of children per family, only the direct effect.

As suggested earlier, though, we should expect to find a lower golden-age capital intensity the greater is the population growth rate. This means that the real wage will be smaller in a golden age the faster the rate of growth of population. Taken by itself, that must reduce lifetime utility. But the rate of interest will be higher the faster is population growth. By itself, the rise of the interest rate operates to increase lifetime utility. The problem, therefore, is to weigh these two opposing effects, the sum of which constitute the indirect effect of faster population growth on the lifetime utility of the representative parents.

Of course, one cannot expect to find universally that the indirect effect must be negative (or must be positive). But one can say a bit more than the statement that "it depends upon preferences." One can try to characterise those cases in which the indirect effect is negative (or positive) in terms of relations between observable variables that depend upon those same preferences. To do this we need to work with a complete, internally consistent model. Here I work with the simplest possible model, Diamond's two-period work-and-retire model with no bequests and, for present purposes, no debt and taxation. The decision-making unit here is the parents of the family, who work, save, and bring up children in the first peroid of their parental life, and retire and consume their savings in the second period.

Appendix II contains an analysis of the "indirect effect" upon utility of an increase of the steady population growth rate. The result obtained is the following: a small increase of the population growth rate, by raising (as it must) the golden-age rate of interest and reducing the golden-age wage rate, has a positive, zero, or negative indirect effect on representative parental utility according to whether the rate of interest is smaller than, equal to, or larger than the population growth rate. To state the result another way, the factor-price effect of faster population growth taken alone reduces lifetime parental utility if and only if the capital intensity of the economy is smaller than the golden-rule capital intensity.

A loose and inadequate explanation of why the indirect effect on utility can in principle go either way is that the rise of the interest rate can increase second-period consumption (enjoyed during retirement) enough to offset the decline of first-period consumption entailed by the concomitant fall of the real wage. Undoubtedly the importance of the interest-rate–growth-rate comparison has to do with the fact that, *given* the population growth rate, lifetime utility is maximized when the social rate of return to investment is equated to the growth rate (the golden-rule rate) and families are confronted with a rate of interest equal to the population growth rate (Samuelson's biological rate of interest).[14] Perhaps a rise of the population growth rate has a positive indirect effect on utility when the rate of interest is smaller than the growth rate because such a rise increases the rate of interest. But it should be noted that a

[14]Paul A. Samuelson, "An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money," *Journal of Political Economy*, 66 (Dec. 1958).

rise of the population growth rate may actually increase the excess of the growth rate over the interest rate! And it does not follow from the analysis that the "best" population growth rate—from the point of view of the indirect effect upon utility—is that growth rate which causes the golden-age interest rate to equal the population growth rate. Both of these latter two statements together with the basic result are illustrated in Figures 1 and 2. For diagrammatic purposes only, I have postulated that "time preference" for first-period

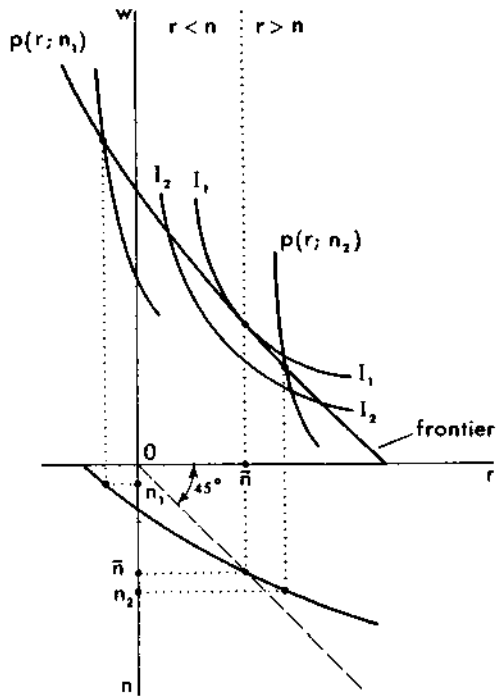

FIGURE 1

versus second-period consumption of the family is invariant to the population growth rate so that the "indifference curves" or indirect-utility contours in the wage-interest rate plane do not shift with a change of the population growth rate (though the total utility that any contour signifies may change with the growth rate due to the "direct effect" on lifetime parental utility). The two diagrams do not cover all possible cases.

The curve ρ $(r; n_1)$ shows the level of the real wage w that is required at any interest rate r and a population growth rate n_1 for the amount of saving done to be just enough to maintain the capital-labour ratio at the level implied by the interest rate. The intersection of that curve with the factor-price frontier indicates the golden-age interest rate and wage rate that correspond to the population growth rate n_1. At the higher population growth rate n_2 we have

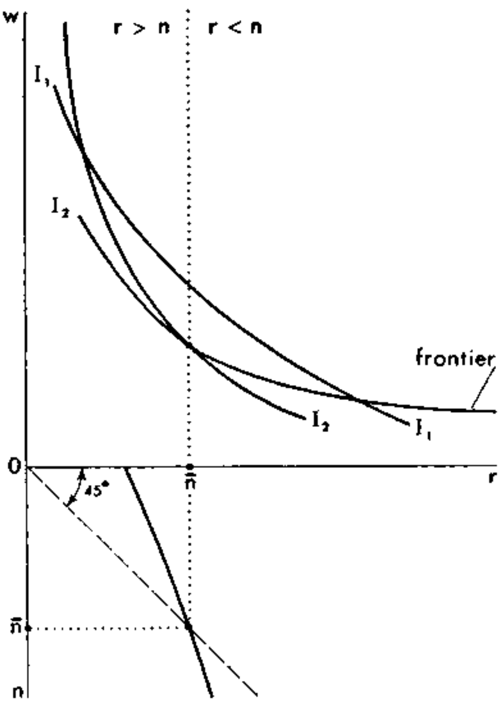

FIGURE 2

a higher curve; its intersection with the factor-price frontier yields a higher rate of interest and a lower wage.

In Figure 1 preferences (and hence these curves) are postulated to be such that the golden-age interest rate is negative when the population growth rate n is zero, as shown by the curve in the lower quadrants that gives the interest rate as a function of growth rate. (This is conceivable and indeed quite Keynesian.) As the population growth rate is increased the corresponding golden-age interest rate increases and does so faster than the population growth rate. At $\bar{n}$ the interest rate has caught up with the growth rate (a golden-rule state) and at greater population growth rates the rate of interest exceeds the growth rate.

These postulates and our result tell us that as the population growth rate is increased (from the zero level) and we thus move down the factor-price frontier, the indirect effect on utility is at first positive and we move to higher indirect-utility contours, as from I_2I_2 to I_1I_1, since at first growth rate exceeds interest rate. Once the population growth rate reaches the level $\bar{n}$, the indirect effect is zero. When the population growth rate is further increased, the interest rate then exceeds the growth rate and we move down the frontier to ever lower utility contours. Since our diagram omits the direct effect on utility of the number of children per family (that is, the population growth rate) it cannot be assumed that $\bar{n}$ is in any sense the "optimum."

Figure 2 illustrates a case in which, paradoxically, the growth rate that makes the interest rate equal to the growth rate is, from the point of view of the indirect effect on utility only, a "pessimum" rather than an optimum. Here, at zero or small enough population growth rates, the golden-age interest rate exceeds the population growth rate. Again, we get a golden-rule state at some $\bar{n}$. At larger population growth rates the interest rate is smaller than the population growth rate. As the population growth rate is increased (from the zero level), therefore, the indirect effect is negative until $\bar{n}$ is reached; thereafter, faster population growth has a positive indirect effect upon lifetime utility since, for large population growth rates, the rate of interest is below the growth rate.

This is enormously paradoxical. It means that sufficiently rapid population growth is better than zero growth from the point of view of factor price effects (neglecting the direct effect). Figure 2 *is* a mathematical possibility; indeed it represents the linear-logarithmic example presented at the end of Appendix II. But it is not realistic. The situation depicted assumes that there is always some increase in second-period consumption that will compensate the parents for a given decline of first-period consumption, no matter how austere first-period living standards to begin with. Moreover, an increase of the population growth rate is likely to increase time preference and thus prevent the interest rate from falling below the growth rate as the latter is increased. Still, even if we exclude Figure 2, Figure 1 and the in-between cases (involving multiple golden-rule states) are somewhat surprising.

As the population growth rate is increased successively to higher levels, what I have called the "direct effect" upon lifetime parental utility must turn negative. This is because the family's first-period consumption (which itself falls as the interest rate rises and the wage falls) must be shared with an increasing number of children. So there seems to be no possibility that "more is better" without limit, quite apart from reproductive capacity. It is said, I believe, that in some societies the children will repay the parents for their first-period sacrifices at a rate of interest sufficiently above the market rate that the parents do not lose from this squeeze of family resources. My guess is that such intertemporal redistributions, like social security, do improve lifetime parental utility in a golden age if the rate of interest is smaller than the population growth rate. I cannot undertake an analysis of these questions here.

POLICY GROUNDS AND WELFARE CRITERIA

I have analysed some of the consequences of one-shot and steady population growth in simple models exhibiting full employment, constant returns to scale, and constant technology. What implications have such an analysis for the question of whether modern societies produce too many (or too few) children? The implications are simple enough to work out if one's welfare criterion is simply consumption per head or lifetime parental utility—or, more precisely, some integral of the possibly discounted values of either of these variables over time. But I doubt that such welfare criteria are satisfactory. I am attracted myself to welfare criteria that give weight only to the preferences of those now living, be they altruistic or selfish. Such criteria do not, of course, approve whatever people do on the ground that they must prefer that or they would not do it. An individual may fail to act in his own interests out of imperfect knowledge concerning private costs and private benefits. And people may sometimes produce spill-over effects on the welfare of others of a type that prevent a Pareto-optimal allocation. Lastly, even with perfect information and no externalities, we know now (or think we know) that an economy can pursue a "dynamically inefficient" path (in the so-called Phelps-Koopmans sense) if it is infinitely long-lived and hence has an infinite number of decision-makers.

I shall ignore here the last consideration; that is, I assume for present purposes that the economy will not go on forever. The subject of externalities I defer to the next section. Here, as a gesture of completeness, I offer a few comments on the matter of imperfect information.

It is a commonplace that children are largely (but not entirely) to be regarded as a consumer durable. As with any durable, the services they yield may be disappointingly small or unexpectedly large; whether their services are typically overestimated is a matter for conjecture. As with most durables, perhaps, the costs of maintenance of children may have risen in recent years faster than anticipated by parents, as acceptable standards of child treatment have improved. On the other hand, parental incomes have undoubtedly risen faster than expected, so that, if children are a non-inferior good, parents may not feel retrospectively that they have had too few children.

But children are unlike other consumer durables in that they work and save. The entry of a family's children into the labour force may have significant effects on other people's incomes. This brings us to the subject of externalities.

II / Cherchez les externalités!

This section does not attempt a comprehensive answer to the question of whether modern societies usually produce too many or too few children (from the point of view of those now living). It does try to present some points bearing on that question.

I shall consider first the question: Do the factor-price effects that occur in a *laissez-faire* economy when parents have more children constitute an externality that implies, other considerations being neutral, that people have more (or fewer) children than is in their self-interest?

The concept of externality that is relevant to such questions is necessarily "marginal" in character; the externality must be significant (relative to other magnitudes) for infinitesimal changes, just as for large changes, of the variables under decision. The failure consistently to remember this point has led to occasional errors in the literature on various problems of economic policy.

PSEUDO-EXTERNALITIES AND INTERDEPENDENCE

The point can be illustrated as follows. Consider a *laissez-faire* economy in competitive equilibrium. Consider some individual who, like everyone else, works each week up to the point where the marginal disutility of effort equals the wage multiplied by the marginal utility of income. Suppose that, with no change in his tastes, he experimentally decides to work substantially longer some week. Given a smooth technology (or in any case if the increase of his working time is great enough) the wage will fall. The increase in total product will exceed the increase in the individual's wage income (plus the increase in his non-wage income which, for simplicity, I shall disregard), the difference being the little "triangle" under the marginal product of labour curve. This difference goes to the rest of the people in the economy; it equals the excess of the increase in their non-wage income over the decrease in their wage income. Thus there is a pseudo-externality here that makes the rest of the people better off. But the gain to the rest of the people is not enough to pay the bribe to the individual that would be needed to induce him to work this extra time; for even if the individual received the whole triangle in addition to his extra wages, it would not pay him to do the extra work since the marginal productivity of work is falling while the marginal disutility of work for the individual is rising (or falling less rapidly). Hence the presence of this non-marginal, pseudo-externality does not disprove that the competitive allocation is Pareto-optimal (though it may be non-optimal for other reasons of course). The phenomenon does not justify a subsidy on effort or a tax on leisure. It is true, though, that other people should give a cheer whenever any individual decides to work more—provided he does not also decide to save more to such an extent that he leaves factor prices unchanged, in which case no more than an indifferent shrug is called for.

The principle here has a well-known application to the theory of optimal immigration. The original residents of the host country gain from immigration provided the immigrants bring with them a configuration of skills, labour, and capital that differs from that already existing in the host country[15] (waving aside various complications outside the model). Yet this does not argue for the subsidization of immigration by the home country.

This analysis compels the following conjecture: the living members of a society (as a whole) ought to welcome any change of tastes which causes parents to produce more children; for these children; when they enter the labour force, will bring with them an assortment of factor supplies that differs from what would exist among the rest of the population. In particular, they will bring a smaller ratio of tangible capital to labour than exists among the older population; they will bring a host of traits and fresh ideas which will complement the wisdom and talents of older people. But, again by analogy with the previous examples, it does not follow from this proposition that the living members of society ought to subsidize or otherwise encourage an increase of births.

Of course the validity of such a conjecture must assume that, as in the examples preceding it, every factor of production receives its marginal social product; we have not yet introduced genuine externalities and distributional considerations. But even on those terms the reader is bound to doubt the conjecture. Surely, he might argue, the conjecture is wrong if the utility of parents

[15]An elegant analysis is contained in a paper by Charles Berry and Ronald Soligo forthcoming in the *Journal of Political Economy*.

depends in part upon the welfare of their children. After all, the preceding section suggests that the fall of wage rates that the children produce when they enter the labour force will not normally be compensated fully by the rise of interest rates (even if those children produce proportionally more children of their own).

While naturally diffident on a matter of such complexity, I am sceptical about the validity of such an objection. Certainly one can specify "externalities in consumption" whereby people have preferences that extend to the choices and decisions of others, which (as do many other factors) deprive the competitive allocation of any presumption of Pareto-optimality. But the objection outlined merely entails that parents care about the welfare of their own children, possibly their children's children, and so on. To take an extreme case, suppose that an extra dollar of discounted income received by their children adds the same amount to the utility of parents as an extra dollar of discounted income of their own. In terms of the two-period model, an increase in the number of children in some family must reduce the wage income of other children when the children born in this period reach the labour force. But there will be a greater increase in interest income going to other parents as their capital (owned in retirement) co-operates with more labour. So even on the extreme supposition of equal marginal utilities, other parents' total utility is increased when some family increases the number of its children. (Nor does the no-subsidy corollary seem to be impaired: even if the entire gain to others is paid to the family having an additional child, this is not sufficient compensation to the family if it has children up to the point where the marginal net disutility of children equals the discounted marginal earnings of its children multiplied by the marginal utility of income.)

However, if parents do not care about the prospective earnings of their children the living members of society may want to subsidize the production of children. On this "don't-care" assumption, parents will have children up to the point where the marginal net disutility of children—the excess of the marginal disutility arising from foregone parental consumption over the marginal utility of the children—equals zero. It is now certainly conceivable, though unlikely I should think, that an extra child in some family will generate enough additional income to others—in particular, enough prospective additional interest income to parents—that the other parents can bribe the family to have more children and still be left better off. No ordinary externality accounts for this result. The result is due to the asymmetrical treatment of wage and interest income.

None of these results depends strictly upon constant returns to scale in capital and labour. For example, the analysis can be applied to an economy in which there is only labour and fixed land, provided land and labour receive their marginal social products.

GENUINE EXTERNALITIES

Let us now consider a number of true externalities which have a bearing on the question of whether a *laissez-faire* society would produce too many or too few children. When one thinks of kinds of externalities which impair the *laissez-faire* determination of investment, saving, and the growth rate, the "isolation paradox" comes to mind.[16] This paradox arises from a consumption externality. If every family gives a weight to a unit of consumption of other people's heirs equal to the weight given to a unit of consumption of other people themselves, while its bequests to its own heirs reflect a smaller weight to the consumption of its own heirs relative to its own consumption, then families ought (through governmental action) to subsidize one another's saving for their heirs. By analogy it can be argued that if each family's utility function gives a weight to the number of children of other families, relative to the living standards of the parents of those families, that is smaller (greater) than the relative weights assigned by the other families themselves then the *laissez-faire* allocation entails too many (few) children and families ought to arrange the taxation (subsidization) of the production of children. The question that arises is whether each family is not in fact content to accept the valuations by other families of children *versus* parental living standards—just as it has been argued that people may in fact adopt their neighbours' evaluations of their neighbours' heirs in the growth context; in that event the consumption externalities do not call for any social action to decrease (increase) the number of children.[17]

The argument that most economies have recently been producing too many children is most commonly based on external diseconomies in consumption. As the population of an economy becomes increasingly crowded on a fixed quantity of land, the consumption and production activities of each individual increasingly impinge on the enjoyment of others. Traffic congestion and air pollution are examples.

Most of these externalities are due to the large size of the population rather than to its rate of growth. But this observation does not remove the implication that population grows too fast on account of these externalities (other considerations being neutral); the "shadow price" to be charged for the production of a new child exceeds the private marginal cost. Nevertheless the sheer rate of change of population may contribute to the external diseconomies generated. Faster population increase tends to produce greater capital formation and the production of capital goods probably generates more externalities than the production of consumption goods. The tearing up of city streets to expand the capacity of public utilities is an example. I have argued elsewhere that faster labour force growth entails a higher unemployment rate for any given rate of inflation due to the imperfect ability of the labour market to allocate heterogeneous unemployed persons to heterogeneous unfilled jobs.[18]

In an ideal world, the government would levy taxes on the production of these diseconomies so that those who suffer them would be appropriately compensated. "Disamenities" would continue to be produced, of course, but from a formal point of view the economy would be like a decreasing-returns-to-scale economy in which all individuals received their marginal social products. In

[16] A. K. Sen, "On Optimising the Rate of Saving," *Economic Journal*, 71 (Sept. 1961), 479–97.

[17] Robert C. Lind, "Comment," *Quarterly Journal of Economics*, 78 (May 1964).

[18] Edmund S. Phelps, "Money-Wage Dynamics and Labor-Market Equilibrium," *Journal of Political Economy*, 76 (Aug. 1968), part II.

a sense, therefore, the problem arises from the failure of societies to introduce the appropriate compensations, not from the population size and rate of increase. But, in practice, the necessary fiscal arrangements and government decisions would be costly to make so that we cannot expect the government to fix up entirely these consequences of population growth.

The extent to which these diseconomies are due to population is open to question. Certainly much of the urbanization since the war in Western economies is due to economic progress and the attendant decline in the importance of agriculture. And economic progress, in giving us the automobile and airplane, has provided the instruments by which the people of a city of given size produce disamenities for one another. Still, whatever the causes of the urban character of mature economies, further population increase seems likely to exacerbate the disamenities now produced in congested areas; further population increase is not likely in most economies simply to multiply the number of cities without increasing the size and the disamenities of existing ones.

It is surprising that, in all the talk about external diseconomies, no one has raised the importance of external economies. One important instance of external economies involves research and technological progress. When an economy increases by, say, 10 per cent its quantities of labour and capital goods, it can expect under constant returns to scale that its material national product will increase by 10 per cent. If that economy maintains the same proportion of that larger labour supply doing technological research, it can expect faster technological progress on top of the proportional increase of material national product. To put the matter another way, if true national income includes some imputed value to the increase of technical knowledge because that technological improvement will promote productivity in the future, then there must be increasing returns to scale. Then one or more factors of production must be paid less than its marginal social product. And this is true even if scarce natural resources entail decreasing returns to scale of material output in labour and capital. In particular, it seems likely that labour (as a whole) receives a wage below its marginal social product on this account. Apart from the previously discussed external diseconomies, which argue in the opposite direction, it follows, I believe, that living members of society (not to mention future ones) have an interest in subsidizing the production of children and hence the production of new minds.

The discrepancy between private and social marginal product of labour for labour as a whole is frequently depreciated in analyses of optimal public expenditure for education. But if one takes the long-run view, which is in keeping with my suggested welfare criterion if those presently living give weight to the welfare of their children, children's children, and so on, then surely these discrepancies are very important. One can hardly imagine, I think, how poor we would be today were it not for the rapid population growth of the past to which we owe the enormous number of technological advances enjoyed today. Certainly until the present time we have been living, and possibly will live for some time into the future, in circumstances of increasing returns to scale by virtue of these technological considerations.

Another instance of external economies is parallel. Our artistic heritage is much like our technology; it is a part of our "public capital." If I could re-do the history of the world, halving population size each year from the beginning of time on some random basis, I would not do it for fear of losing Mozart in the process. No improvement of our dirty air and our traffic congestion could compensate me for that! In our own time, middle-aged and older adults surely benefit from the quantity of artistic public goods produced by young people that results in part from their sheer number.

It is really not so clear to me that these artistic and technological external economies fail to compensate for the external diseconomies. But certainly as the earth becomes ever more crowded there must come a day, if it has not already, when the balance of diseconomies and economies makes a reduction of the population growth rate desirable.

III / Other policy aspects

The preceding analysis has omitted two important considerations, the distribution of income and the adequacy of aggregate demand.

I presume that if society should be attracted to the idea of reducing substantially the birth rate (over and above the reduction that might be achievable through the dissemination of birth-control information), it will have to resort to a "tax on children" (or the elimination of existing subsidies). Since the family allowance plans of some countries do not appear to have increased the birth rate greatly, I should guess that a fairly large tax might be required.

The trouble with such a tax is that it may have distributional effects that are undesirable. If all families were alike, income need not fall for any parents since the government can start all married couples with a fixed tax credit (falling gradually over the reproductive years perhaps) that will pay the child tax; as long as each couple assumes its credit is independent of its children, the child tax will have the desired substitution effect; parents and their children will not suffer any fall of living standards.

But if some families are larger than others and the tax credit is to remain independent of family size so as not to nullify the effectiveness of the child tax, then large families will be made worse off and small families better off. This is of concern if, as is frequently the case, the former are typically far worse off to begin with.

The negative income tax can reduce income inequality among families but if the transfers increase with the number of children in the family they may partly offset the child tax. The subsidization of children in poor families would then entail greater taxes on children among other families if a given reduction of the over-all birth rate were to be achieved.

The rationale for larger negative taxes to larger poor families is, in part, that the children will benefit. An alternative means of benefit is direct public assistance in the form of income in kind to the children. But by freeing parental resources for other uses, such public assistance has much the same effects on the marginal cost to the parents of bringing up children as do family allowances.

If society should be attracted to the idea of increasing substantially the birth rate, a system of family allowances (as contained in the negative income tax proposals) would serve that objective while improving the distribution of family incomes. But there would be distributional effects from the rise of the birth rate itself. The poor who own little capital relative to their labour would suffer from the eventual fall of wage rates more than the capital-owning and land-owning rich who might gain from the rise of interest rates and land rents. And the urban poor are more likely to suffer the external diseconomies of a larger population than the more prosperous people in the suburbs.

The other matter is aggregate demand. A fall of the population growth rate by one percentage point (quite a lot, to be sure) would reduce the "natural" or golden-age growth rate by one percentage point even without any resulting effect on the rate of technical progress.

In Solow's one-commodity model, with a fixed investment-output ratio, this would reduce the golden-age profit rate by a number of percentage points equal to the ratio of capital's share to the saving ratio plus (minus) some amount due to the resulting fall (rise) of capital's share. In addition, slower population growth would unquestionably slow the rate of technical progress if the proportion of the labour force doing research was maintained. If the (Harrod–neutral) rate of labour augmentation fell by one percentage point, as it might in a country as dependent on its own research as the United States, the golden-age profit rate would fall by twice the amount just cited. If these estimates are accepted, profit rates might fall in the long-run from an average of, say, 8 per cent to 4 per cent at normal capacity utilization rates. Any fall of the saving ratio induced by life-cycle-saving considerations would cushion the decline somewhat.

Under a fiscal policy of budgetary deficits as a proportion of national product of one or two per cent, to which most countries are accustomed, such a fall of profit rates might require zero or even negative rates of interest to close the deflationary gap. At the other extreme, if interest rates and capital intensity were maintained so that fiscal policy was called upon to increase the consumption–national-income ratio sufficiently to prevent a deflationary gap, then a much larger deficit would be required. The mathematics of the simplest Keynesian model show that, given government spending, every one percentage point decrease of the golden-age growth rate requires a point increase of the deficit as a percentage of output equal to the capital-output ration divided by the propensity to consume.[19] (The first term of this quotient is the fall of the required investment-output ratio when the capital-output ratio is to be maintained, the second term the rise of the consumption-output ratio that results from a unit increase of the deficit-output ratio.) This expression is nearly four so that a fall of the golden-age growth rate by two points, such as we have been discussing, would increase the deficit-output ratio by nearly eight percentage points. A deficit-output ratio of that size, when government outlays are one-fifth of national income, implies that half of government expenditures are debt-financed.

[19]See, for example, James Tobin, "Money and Economic Growth," *Econometrica*, 33 (Oct. 1965), especially equation (2). The deficits implied by my analysis would, in Tobin's portfolio model, require gradually a rise of interest rates to offset the effect on the demand for capital of the gradually resulting rise of the debt-wealth ratio.

So any reduction of the population growth rate of the magnitude I have considered here would, according to these calculations, require for continuous full employment some rather drastic changes in fiscal policy or monetary policy or both—much more drastic than most "growthmen" have ever envisioned. The deficit taboo would be put to the test. One might be justifiably nervous therefore about the consequences for unemployment of any measures to level off the size of the population. Still, I am optimistic enough to think that the fiscal and monetary authorities would not long tolerate high unemployment.

Appendix I

Let C denote total consumption, L the labour force, and P the population. Then

(1) $C/P = (C/L)(L/P)$.

If for every population growth rate n, capital intensity is set to maximize golden-age consumption (the golden rule), then in the one-commodity model

(2) $C/L = f(\hat{k}) - n\hat{k},\ f'(\hat{k}) = n,\ f(0) = 0,$

where $f(\hat{k})$ is output per unit labour and $\hat{k}$ is the golden-rule capital-labour ratio.

In the simplest, most extreme model, there is no childhood and people work for a "year" and live in retirement for a "year." Then the population consists of this year's labour force plus last year's labour force:

(3) $P = L + L_{-1} = L[1 + 1/(1 + n)]$

whence

(4) $L/P = (1 + n)/(2 + n)$

and

(5) $C/P = [f(\hat{k}) - n\hat{k}][(1 + n)/(2 + n)]$.

If there exists a finite $\bar{k}$ such that $f'(\bar{k}) = 0$, then at $n = 0$ we have

(6) $(C/P)_{n=0} = f(\bar{k}) \cdot \frac{1}{2} > 0$.

Even if $f'(k) > 0$ for all k, (C/L) and hence (C/P) is positive for sufficiently small n.

If $f'(0) = \mu < \infty$ then for all $n \geqq \mu$, (C/L) and hence also (C/P) are equal to zero. If $f'(0) = \infty$ then (C/L) approaches zero only in the limit as n approaches infinity; since (L/P) will approach unity in the limit as n approaches infinity, (C/P) must also approach zero in the limit.

Hence, if $f(0) = 0$, meaning that positive capital is required for positive output, golden-rule consumption per capita must eventually fall as the population growth rate is increased. Nevertheless there can be ranges of n for which (C/P) rises when n is increased. The derivative of interest is

(7) $[d \log(C/P)]/dn = -\hat{k}/(C/L) + 1/[(1 + n)(2 + n)]$.

Using (2) and the relation

(8) $\hat{k}/(f(\hat{k}) - n\hat{k}) = 1/[n(1 - a)/a] \qquad a = (f'(k)k)/f'$

we find that the derivative in (7) is positive if and only if

(9) $n[(1 - a)/a]/[(1 + n)(2 + n)] > 1.$

Since $(1 - a)/a$ is the ratio of labour's share to capital's share, plausible values of this ratio make it difficult to imagine this condition to be satisfied. Similarly, the first term on the right-hand side of (7), the golden-rule capital-consumption ratio, might be thought to exceed three at present values of n.

Note that as n approaches infinity, the left-hand side must be less than one since, if $f(0) = 0$, it can be shown that $(1 - a)/a$ cannot go to infinity.

Appendix II

Lifetime parental utility in the representative family is given by

(1) $U = U(c_1, c_2; n)$

where c_1 is first-period total consumption of the family, c_2 is second-period family consumption and the parameter n is the geometric growth rate of population, which is a suitable proxy for the number of children per family. The parents are postulated to choose c_1 and c_2 so as to maximize their utility, subject to the budget constraint

(2) $c_1 + c_2/(1 + r) = w$

where w is the wage of the parents in their working "year" and r is the rate of interest and marginal productivity of capital. Assuming an interior maximum to exist we have

(3) $U_1 - (1 + r)U_2 = 0$

where U_1 and U_2 denote the two first derivations of U with respect to c_1 and c_2 respectively.

It follows from (3) that c_1 depends only on w, r, and n:

(4) $c_1 = c(w, r; n).$

The same is true of c_2 by (2). Hence there exists an indirect utility function

(5) $U = V(w, r; n)$

whose first derivatives by virtue of (3) are the following:

(6) $$\begin{aligned} \partial v/\partial w &= [U_1 - (1 + r)U_2]\partial c_1/\partial w + (1 + r)U_2 = (1 + r)U_2 \\ \partial v/\partial r &= [U_1 - (1 + r)U_2]\partial c_1/\partial r + (w - c_1)U_2 = (w - c_1)U_2. \end{aligned}$$

The total effect of an increase of the population growth rate can be represented by the derivative

(7) $$\begin{aligned} dV/dn &= \partial V/\partial n + dr/dn[\partial V/\partial r + (dw/dr)(\partial V/\partial w)] \\ &= \partial U/\partial n + dr/dn\, U_2[w - c_1 + (1 + r)dw/dr]. \end{aligned}$$

The first term is the "direct effect," the remainder the indirect effect resulting from the impact of the increase in n upon factor prices.

For the moment assume that $dr/dn > 0$. Then the indirect effect on utility has the sign of the bracketed expression. Now dw/dr is given by the slope of the factor-price frontier if we confine the analysis to alternative golden-age states having a constant capital-labour ratio, k:

(8) $\quad w = \Phi(r),\ \Phi'(r) = -k < 0.$

There is also a relation between k and $w - c_1$ in a golden age since in a golden age this year's saving per working pair of parents is next year's capital per working pair of parents after adjusting for the increase in the number of parents:

(9) $\quad w - c_1 = (1 + n)k.$

Use of (8) and (9) yields

(10) $\quad w - c_1 + (1 + r)dw/dr = k(n - r).$

So the indirect utility effect has the sign of $n - r$.

This result does not require that the marginal rate of substitution between c_1 and c_2 be independent of n, provided non-independence does not reverse the sign of dr/dn, in which event the result would be reversed. Formally, the effect of n upon time preference has no additional indirect effect on utility if the family was already "equating at the margin" since, by (3),

(11) $\quad U_n = U_1 . \partial c_1/\partial n - (1 + r)U_2(\partial c_1/\partial n) + U_3 = U_3.$

As for dr/dn, let us postulate first that time preference is in fact independent of n. Equation (9), when written in the form

(9a) $\quad c(w, r; n) = w - (1 + n)k(r),$

where $k(r)$ denotes the technological relationship between k and r that is implicit in the factor price frontier, implies the steady value of w required to maintain r at any constant (golden-age) value. The required relation will be denoted

(12) $\quad w = \rho(r; n).$

Now "stability", meaning convergence of the system to its golden-age path for any given n, requires $\rho_r < \Phi'(r)$, i.e., that the ρ curve be steeper in a diagram like those of the text than the factor-price frontier. This inequality implies

(13) $\quad k(1 - \partial c_1/\partial w) + \partial c_1/\partial r + (1 + n)k'(r) < 0.$

Combining (9a) and (8) we have the golden-age condition

(14) $\quad c(\Phi(r), r) = \Phi(r) - (1 + n)k(r).$

Differentiating, we have

(15) $\quad dr/dn = -k/[-\Phi'(r)(1 - \partial c_1/\partial w) + \partial c_1/\partial r + (1 + n)k'(r)] > 0$

by virtue of the convergence condition in (13).

It is likely that an increase of n increases time preference if it affects time preference at all. In that case the above analysis of dr/dn understates the rise of r that we should expect from an increase of n.

An equivalent method of obtaining the result in (7) and (10) is to insert the golden-age expressions for c_1 and c_2 as a function of k and n into (1):

$$U = U[f(k) - rk - (1 + n)k, (1 + r)(1 + n)k; n] \tag{16}$$

where $f(k)$ denotes output per worker and $f'(k) = r$. Differentiation then yields

$$\begin{aligned} dU/dn = {} & U_3 + (1 + r)U_2k[(n - r)/(1 + r)]dr/dn \\ & + [-U_1 + (1 + r)U_2][k(1 + dr/dn) + (1 + n)(dk/dr)(dr/dn)]. \end{aligned} \tag{17}$$

This can be seen to be identical to (7) and (10) by virtue of (3).

EXAMPLE

The following linear-logarithmic example is taken from Diamond, "National Debt in a Neoclassical Growth Model," 1134–35:

$$U = c_1^{\beta}c_2^{1-\beta}, 0 < \beta < 1 \tag{18}$$

$$f(k) = Ak^{\alpha}, 0 < \alpha < 1. \tag{19}$$

Then

$$c_1 = \beta w \tag{20}$$

$$c_2 = (1 + r)(1 - \beta)w. \tag{21}$$

The Cobb-Douglas production function in (19) implies the golden-age factor-price frontier relation

$$w = (1 - \alpha)\alpha^{\alpha/1-\alpha}A^{1/1-\alpha}r^{\alpha/\alpha-1}. \tag{22}$$

Use of (20) and (22) then gives the golden-age relation between r and n:

$$r = \alpha(1 + n)/[(1 - \alpha)(1 - \beta)]. \tag{23}$$

At $n = 0$, $r > 0$. If

$$\alpha/(1 - \alpha) < (1 - \beta) \tag{24}$$

then $dr/dn < 1$ and we have a case like Figure 2 of the text in which $r = n$ at some positive n, namely

$$n = \alpha/[(1 - \alpha)(1 - \beta) - \alpha] > 0 \tag{25}$$

and $r < n$ for all greater n. If (24) does not hold, then $dr/dn \geqq 1$ and $r > n$ for all non-negative n.

By (18), (20), and (21) we have

$$U = w\beta^{\beta}(1 - \beta)^{1-\beta}(1 + r)^{1-\beta} \tag{26}$$

and, by (22)

$$U = [(1 + r)^{1-\beta}/r^{\alpha/1-\alpha}][(1 - \alpha)\alpha^{\alpha/1-\alpha}A^{1/1-\alpha}\beta^{\beta}(1 - \beta)^{1-\beta}]. \tag{27}$$

Since r is increasing in n, the indirect effect on lifetime utility has the sign of

$$d \log U/dr = (1 - \beta)/(1 + r) - [\alpha/(1 - \alpha)][1/r]. \tag{28}$$

This is positive, zero or negative according as

$$r \gtreqless [\alpha(1 + r)]/[1 - \alpha)(1 - \beta)] \tag{29}$$

or, substituting (23) for r on the lefthand side, according as

$$(1 + n) \gtreqless (1 + r). \tag{30}$$

INTRODUCTION

My fascination with the question of optimal economic growth must have begun with a chance reading of Ramsey's 1928 masterpiece. To do any creative work in this area requires ridding oneself of the apparition of Ramsey looking very smart and pleased with himself. But while it is assuredly a great paper, I had the growing feeling that it took an unnatural view of optimal fiscal policy: Were we all intuitive Ramseyan utilitarians under the skin, our governments would have been running large enough budgetary surpluses to drive down the real rate of return to investment. If my Golden Rule fable satirized anything it was not the positive or descriptive economics of Solow and Swan as embodied in their seminal growth model but rather the utilitarians' enthusiasm for marching to the Golden Rule state—an enthusiasm correctly foreseen as it turned out. (In the end the rate of return sank anyway, without benefit of budgetary surpluses, as the postwar recovery and modernization of the world's capital stock reached its completion.)

My earliest paper in this part, on the utilitarian–optimal accumulation of risky capital, was actually begun in 1958 and virtually completed in late 1959, when my rebellion against Ramsey amounted only to occasional bouts of moodiness. Actually the paper is a case of "twins," one macroeconomic and the other microeconomic. The macroeconomic paper sprung from a curiosity to see whether Ramsey's requirement of a bliss-level of current-period utility could be lifted upon the introduction of uncertainty in the returns to capital investment; maybe Ramsey's difficulty, that the infinite-horizon nation cannot consume too little (as long as it consumes something) if its intertemporal preferences do not display Ramsey's bliss-level of utility, would be dispelled once we recognize the unattractiveness of saving when it is risky. The microeconomic paper grew out of a curiosity to see how the Markowitz-Tobin model of portfolio choice might be intertemporalized into a multi-period model of risky investment and saving by the multi-period family; however the paper never reached the point of introducing a second asset and the synthesis of the two problems, risky saving and risky investment, was not taken up until much later by Hakkanson, Samuelson, Merton, and others.

While the paper is now hardly known for its contribution to the macroeconomic concern, it did nevertheless show, correctly so far as I can see, that uncertainty makes Ramsey's bliss requirement overly strong. More precisely, the elasticity of substitution in Ramsey's implicit C.E.S. social-welfare functional need not be less than one—but it cannot be too much larger than one. The other contribution is the first demonstration that greater riskiness of saving need not lead to less saving; it will lead to less saving in those cases where a lesser expected rate of return to saving would do the same, namely when the substitution elasticity is greater than one, the case excluded by Ramsey.

The next chapter in my odyssey toward the just conception of optimum growth is my 1965 book, *Fiscal Neutrality toward Economic Growth*. Today that volume is an endangered species, a few copies remaining only in the most distinguished research libraries; and so for many scholars the best access to it, although an incomplete one, is Amartya Sen's 1971 reader, *Economic Growth*, which reprints in its entirety the concluding chapter of the book. The selection reproduced here was prepared for my own 1969 reader for undergraduates, *The Goal of Economic Growth*. Although it is a rather severe condensation and devoid of scholarly references, the essay conveys well enough the notion of fiscal neutrality being examined and some of its defects.

During the 1950s it was increasingly said of the American economy and some others that a tendency toward under-saving existed because of certain imperfections in the market mechanism. Whatever the truth of the underlying intuition and the justice of the conception of optimal economic growth implicit in it, the stated case for under-saving seemed to me to suffer from a certain incoherence. If markets were instead perfect would we then be automatically assured of the right rate of saving no matter what the accompanying fiscal policy, particularly the algebraic budgetary surplus? In a Barro and Bailey world, yes, but otherwise no—or so I argued. The first problem to work out was the characterization of the right fiscal policy, the right budgetary conditions regarding present and expected future taxes, when fiscal policy is not intrinsically neutral because bonds are net wealth and taxes due are net liabilities. Then we might ask in which direction this fiscal policy, being neutral by discretion, must be altered in response to actual market imperfections. This exercise might give us some feel for whether actual fiscal policy was overcorrecting or undercorrecting for the market imperfections prevailing in this or that country.

From the standpoint of economic theory the main limitation of this edifice is its postulate that there exist only dynastic families whose generations are connected like links in a chain. In such a Barro and Bailey world all that is required is a kind of "truth in fiscal policy" plus any one-time redistribution among dynastic families necessary to achieve some sort of static or atemporal justice or social-welfare maximization; there is no room for the concept of intergeneration justice, save for the sort of consumption–externalities that may exist between persons having different birthdates as first raised in Sen's isolation paradox. From the standpoint of moral philosophy, the doctrine of neutralism finds itself in the uneasy camp of intuitionism: Whatever parents feel is right to bequeath as adjusted by the extra taxes they vote out of a more general concern for the future must be right!

An interesting point arose in the course of the analysis that is of surviving importance, although it seems to have been lost in the condensation reprinted here. The formal analysis proceeds as though it were given that the fathers are in the drivers' seats; the children will take the bequests they get and through their government redeem at par their fathers' holdings of public debt. But in fact the young may repudiate the debt, or depreciate it through inflation, and will no doubt choose to do so if the fathers' government deficit exceeds some limit. What fiscal neutrality amounts to in the perfect-foresight case is that the fathers balance their part of the government budget over their lifetime; there may be something stable or self-reinforcing about such a fiscal policy in view of the threat power of the young.

The next two papers, the first of them written with Robert Pollak, propose that there may arise a problem of Strotzian time-inconsistency in the formulation of growth policy by succeeding generations. The problem may be met by the realistic decision of each generation to take as given the policies of future generations and to optimize accordingly; the policies taken as given may be calculated by assuming that future generations will each be doing the analogous thing. This is the Phelps-Pollak game-equilibrium solution to the time-inconsistency problem in optimal growth. It is an equilibrium solution because the regrettable decisions taken in the future are clearly foreseen and discounted in the present. It is a game solution because, in the maximizing tradition of that field of inquiry, no generation acting unilaterally (and what other way can it act?) can improve upon that solution as it sees the matter.

Perhaps the most general of the conclusions reached in the earlier of these papers is that the game-equilibrium solution is not Pareto-optimal among generations. If only their governments could sit down to reason together. The principal point of the second paper, which introduces diminishing returns into the picture, is that there may be two or more (or even as it happened there a continuum of) solutions of the game-equilibrium type.That paper takes the optimistic position that a society is innately capable of rejecting solutions that are Pareto-inferior to another. In the problem modeled there, that axiom leaves only the game-equilibrium solution that leads asymptotically to the Golden Rule state. Others would instead pin down the solution by considerations around the "origin" of the economy. Neither candidate for a unique solution, however, is Pareto-optimal.

These latter three papers must leave one as they left me with the feeling that intuitionist conceptions of optimal growth are deficient solutions to the problem. But what formal ethic might successive generations rally around? I put the question aside in 1968, after drafting the sequel paper on game-equilibrium growth, and did not come back to it until 1974 when, with John Riley, I began to work with the 'maximin' criterion of optimal growth.

The subject of the paper is billed as Rawlsian growth without much license. As readers of Rawls know, he confines the 'maximin' criterion to members of society who, so to speak, mix their sweat in acts of contemporaneous production cum trade—in the division of labor. But surely there is a nearly similar economic cooperation between the young and the old. And what if young workers producing alongside old workers today will tomorrow work alongside younger workers? In any case it is the message of this paper that Rawls can be extended to intergenerational matters without mishap. Only minor difficulties arise in the extension, and these are attributable to the convenient use of the infinite horizon.

Someone said that the role in which I had been cast was to find a satisfactory notion of optimal economic growth. At least I have completed my liberation from Ramsey, fortified at the end by Rawls. But I would not claim that the essay on Rawlsian growth provides a wholly satisfactory solution to my assignment. Guillermo Calvo has gone ahead with testing the adequacy of the maximin criterion in coping with problems of uncertainty over the return to investment and demographic planning. It remains to be seen how the criterion fares in application to a small country in an aggressive world. Is a country adhering to that criterion doomed to perish if other countries are militantly pro-growth? Perhaps simply to vanish? Or will a proper application of the criterion safeguard it from such implications?

With the decline of utilitarianism one might have expected a rush to the Rawlsian position like passengers' running to the opposite rail on a listing ship's deck. This has not happened, at any rate not yet, and it may be that no consensus regarding intergeneration justice or any other domain of justice is destined ever to occur.

THE ACCUMULATION OF RISKY CAPITAL: A SEQUENTIAL UTILITY ANALYSIS

THIS PAPER investigates the optimal lifetime consumption strategy of an individual whose wealth holding possibilities expose him to the risk of loss. The vehicle of analysis is a stochastic, discrete-time dynamic programming model that postulates an expected lifetime utility function to be maximized. All wealth consists of a single asset, called capital.

The problem described belongs mainly to the theory of personal saving. Models of saving behavior thus far have been entirely deterministic [**4, 7, 8, 11, 12, 13**],[2] whereas, in fact, the saver is typically faced with the prospect of capital gain or loss. So it seems appropriate to determine whether the results of the conventional theory carry over or have to be qualified upon admitting capital risk into the theory.[3] The question also arises as to the effect of capital risk itself upon the level of consumption. This neglected factor may play a role in the explanation of certain inter-group differences in saving behavior.

These questions are easier to raise than to answer, and this paper is frankly an exploratory effort. No generality or definitiveness is claimed for the results obtained. A brief outline of the paper and sketch of some of these results follow.

In the first two sections, a utility function and a stochastic capital growth process are postulated and discussed. Subsequently, the "structure" of the optimal consumption policy, that is, the way in which consumption depends upon the individual's age and capital, is established. One's expectations, based on existing "deterministic" theory, are confirmed: Optimal consumption is an increasing function of both age and capital. Little else appears deducible without further restrictions upon the utility function.

[1] For helpful discussions on this subject I am grateful to T. N. Srinivasan and S. G. Winter.

[2] An exception is a Cowles Foundation Discussion Paper by Martin Beckmann [**2**]. That paper (which deals with wage rather than capital uncertainty) uses a technique similar to the one here.

[3] The model below resembles Ramsey's more than contemporary models [**7, 11**] so that it is largely his results that are modified.

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Thereafter attention is confined to certain monomial utility functions. These special cases cannot yield general theorems but they do have the function of providing counter-examples to conjectures and of serving to suggest other hypotheses for empirical test.

For example, it is shown that the classical phenomenon of "hump saving" [8, 12] need not occur, quite apart from reasons of time preference, if capital is risky. Instead a low-capital "trap" region is possible in which it is optimal to maintain or decumulate capital, no matter how distant the planning horizon.

These utility functions all make consumption linear homogeneous in capital and permanent nonwealth income, and linear in each of these variables. But the straight-line classroom consumption function is not really upheld: Consumption cannot be expressed as a function of aggregate expected income because expected wage income (treated as certain) and expected capital income have different variances, whence different impacts upon the level of consumption. The marginal propensity to consume out of risky income is smaller than out of sure income. This result may help to explain why households which depend primarily upon (risky) capital income (e.g., farmers, wealthy heirs) are comparatively thrifty.

Finally, we consider the effect upon the consumption level of variations in the riskiness and in the expected rate of return of capital (given capital and nonwage income). Not surprisingly, the direction of effect of both are unpredictable without knowledge of the type of utility function; the familiar conflict between substitution and income effects applies as much to risk as to the rate of return. Two closely related utility functions give opposite results. But it is interesting that risk always "opposes" return. Where increase of the rate of return raises (reduces) the propensity to consume, an increase in risk reduces (raises) it; and where return has no effect, neither does risk.

1. THE BEHAVIOR OF CAPITAL

Capital is treated as homogeneous in the sense that each unit of the asset experiences the same rate of return.[4]

The individual's consumption opportunities occur at discrete, equally spaced points in time. These points divide the lifetime of the consumer into N periods. The state of the system at the beginning of each period, $n = 1, 2, \ldots, N$, is described by the variable x_n, the amount of capital then on hand. At this time the individual chooses to consume some amount c_n of this capital.

[4] Alternatively, capital might have been envisioned more like identical female rabbits. In any short time period, some units of the asset would multiply while others not. This might be termed subjective or *ex ante* homogeneity.

The unconsumed capital is left to grow at a rate which is not then known. In addition to the capital growth, the individual receives an amount, y, of nonwealth income at the end of the period. This income is the same each period. Consequently the amount of capital available for consumption in the next period is given by the difference equation

$$x_{n+1} = \beta_n(x_n - c_n) + y\,, \qquad x_1 = k, \tag{1.1}$$

where $\beta_n - 1$ is the rate of return earned on capital in the nth period.

We shall assume that the random variables β_n are independent and drawn from the same probability distribution. There are m possible rates of return, $0 \leqslant \beta_i$, $i = 1, 2, \ldots, m$. The probability of the ith rate of return will be denoted by p_i (the same from period to period). In addition we shall assume that $\bar{\beta} = \sum_1^m p_i\beta_i > 1$ so that the consumer expects capital to be productive. However, $\sum_1^m p_i(\beta_i - \bar{\beta})^2 > 0$, and so the realized return may differ from the expected one.

2. THE UTILITY FUCTION

This model postulates a consumer who obeys the axioms of the von Neumann-Morgenstern utility theory. His consumption strategy (or policy) can therefore be viewed as maximizing the expected value of utility, which is determined up to an increasing linear transformation.

Second, we suppose that the lifetime utility associated with any consumption history is a continuously differentiable function of the amount consumed at the beginning of each period.

The lifetime utility function is assumed to be of the independent and additive form

$$U = \sum_{i=1}^{N} \alpha^{n-1}u(c_n)\,, \qquad 0 < \alpha \leqslant 1\,. \tag{2.1}$$

The implications of this functional form are several. Preferences for the consumption "chances" or distributions of any period are invariant to the consumption levels befalling the individual in other periods (separability). Preferences among consumption subhistories in the future are independent of the age of the individual (stationarity). Preference for a consumption strategy is independent of or unaffected by any serial correlation in the random consumption sequence associated with that strategy (independence).[5]

[5] However the necessary and sufficient conditions for independence of utilities when choice takes place under uncertainty have yet to be investigated. The independence of utilities when choice takes place in an environment of certainty has been axiomatized by Debreu [6]. The meaning of additivity with a variable utility discount factor and an infinite number of periods has also been investigated by Koopmans [9].

The same axioms which yield the von Neumann-Morgenstern utility indicators also imply that $U(c_1, \ldots, c_N)$ is bounded from above and below.[6] Consequently $u(c_n)$ is also a bounded function. Let $\bar{u}$ and $\underline{u}$ denote the upper and lower bounds of $u(c_n)$, respectively.

Finally, we postulate that the individual strictly prefers more consumption to less (monotonicity) and that he is strictly averse to risk (concavity). The latter means that for every pair of consumption histories $(c_1, \ldots, c_N)$ and $(c_1^o, \ldots, c_N^o)$ to which he is not indifferent, he will strictly prefer the certainty of the compromise history $\theta c + (1 - \theta)c^o$ to the mixed prospect offering him the history c with probability θ and the history c^o with probability $1 - \theta$, $0 < \theta < 1$. It follows trivially that $u(c_n)$ is a strictly increasing and strictly concave function.

3. DERIVATION OF THE FUNCTIONAL EQUATIONS

We seek the consumption strategy (or, equivalently, policy)—denoted by the sequence of functions $\{c_n(x)\}$ for $x \geqslant 0$, $n = 1, 2, \ldots, N$—which maximizes expected lifetime utility:

$$J_N(c) = \operatorname*{exp}_{\beta} U \tag{3.1}$$

subject to the relation (1.1). Notice that the optimal c_n, $n = 1, \ldots, N$, will be a stochastic rather than a predetermined function of n.

To treat this variational problem we turn to the technique of dynamic programming [3]. Observing that the maximum expected value of lifetime utility depends only upon the number of stages in the process and the initial capital, k, we define the function

$$w_N(k) = \max J_N(c) \tag{3.2}$$

where the maximum is taken over all admissible policies. The function defined may be interpreted as the utility-of-wealth function of the optimizing consumer having N periods of life remaining.

Next one reduces the problem with N decision variables to a sequence of N problems, each involving only one policy variable, the decision which must be taken at the current moment. This approach leads to the following functional equations:[7]

[6] A proof of boundedness may be found in [1] and [5]. The proof uses the "continuity axiom" and a generalization of the St. Petersburg game, the idea for which Arrow [1] credits to K. Menger.

[7] The argument starts with the observation that with the elapse of each period the individual is confronted with another multistage decision problem which differs only in having one less stage and, in general, a different initial capital. By the "principle of

$$(3.3)\quad w_N(x) = \max_{0 \le c \le x} \left[u(c) + \alpha \sum_{i=1}^{m} p_i w_{N-1}(\beta_i(x-c)+y)\right], \qquad N \geqslant 2,$$

and

$$(3.4)\qquad w_1(x) = \max_{0 \le c \le x} u(c)$$

which defines the utility of wealth in the single stage process. Without a subscript, the symbol c shall always denote the value of consumption in the first period of the (not necessarily original) multistage process. Similarly x shall denote capital at the start of whatever process is being considered.

4. PROPERTIES OF THE OPTIMAL CONSUMPTION POLICY

A number of standard results follow from this model: First, the optimal consumption strategy is unique; the optimum value of c_n is a unique function of x_n for every n.

The proof consists of showing that the utility of wealth function is strictly concave if the utility of consumption function is strictly concave; therefore the maximand in each period is a strictly concave function of current consumption, whence the maximizing consumption level is unique.[8]

Second, consumption is an increasing function of capital and age. The latter result depends upon the further assumption made now that $\alpha\bar{\beta} > 1$. It will become clear in the next section that this inequality is also a necessary condition for positive accumulation of capital.

The proof is rather involved and is omitted here. It can be shown that if $\alpha\bar{\beta} > u'(0)/u'(y)$ then, with $N \geqslant 2$ periods remaining, consumption is the following function of capital:

$$(4.1)\qquad c = \begin{cases} 0, & 0 \leqslant x \leqslant \bar{x}_N\,, \\ c_N(x), & x \geqslant \bar{x}_N\,, \end{cases}$$

where $c_N(x) = 0$ at $x = \bar{x}_N$, $c_N'(x) > 0$, and $c_N(x) < x$. The function $c_N(x$

optimality" [3], if the individual's consumption strategy is optimal for the origina N-stage process then that part of the strategy relating to the last N-1 stages must also constitute a complete optimal strategy with respect to the new N-1 stage process. This principle, equation (1.1), the additive utility function (3.1) and the definition (3.2) combine to yield the sequence of equations in the unknown utility of wealth functions in (3.3) and (3.4).

[8] Readers who are unfamiliar with this type of proof may wish to consult [3]. Proofs of the result above and of the other results stated but not proved in this section can be found in an earlier version of this paper (same title) by the author, published as Cowles Foundation Discussion Paper No. 109, which is available on request to the Cowles Foundation.

represents the interior portion of the solution where consumption is not constrained by the nonnegativity requirement.

It can be further shown that the marginal utility of wealth declines with age and capital and that the "consumption function" in (4.1) shifts leftward and upward as age increases:

$$
\begin{aligned}
&w_1'(x) < w_2'(x) < \ldots < w_N'(x) < \ldots, \\
(4.2) \qquad &c_2(x) > \ldots > c_N(x) > \ldots, \\
&0 < \bar{x}_2 < \ldots < \bar{x}_N < \ldots .
\end{aligned}
$$

Of course, when $N = 1$, $c = x$.

In the other case, where $\alpha\bar{\beta} \leqslant u'(0)/u'(y)$, the constraint that consumption cannot exceed capital becomes binding for $N = 2$ and possibly for larger N—when capital is sufficiently small. If there is a value of $x \geqslant 0$ for which $c_N(x) = x$ then, denoting this value by $\hat{x}_N$, we obtain

$$
c = \begin{cases} x, & 0 \leqslant x \leqslant \hat{x}_N\,, \\ c_N(x), & x \geqslant \hat{x}_N\,. \end{cases}
$$

Again, as age increases, N decreases, the marginal utility of wealth function decreases and the consumption function shifts upward. Consequently the intersection where $c = x$ shifts rightward:

$$\hat{x}_2 > \ldots > \hat{x}_N \geqslant 0\,.$$

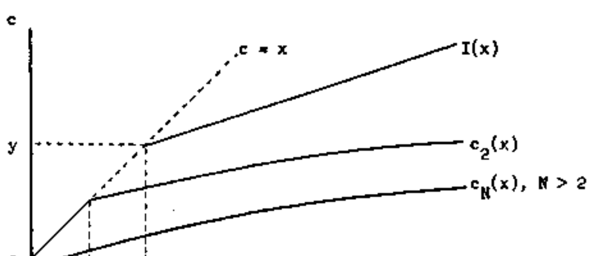

FIGURE 1

A typical possibility is graphed in Figure 1. This consumption function is of the second type. As N becomes small, the consumption schedule shifts upward. When $N = 2$, the function intersects the $c = x$ line. When $N = 1$, $c = x$ at all x.

The $I(x)$ function is defined in the next section.

5. CONDITIONS FOR EXPECTED ACCUMULATION

The preceding theorems confirm our expectations about the qualitative behavior of optimal consumption. They do not go far enough to permit inferences about the behavior of capital as a function of age and initial capital. One might ask if the model generates "hump saving" [8, 12], so important in the theory of aggregate capital formation. The "hump saver" saves when he is young and dissaves as he grows older. Therefore we ask: Can one find a value of N sufficiently large to induce the individual to save —more precisely, to cause the expected value of his subsequent capital to exceed the value of his present capital?[9]

Let us define "expected income," $I(x)$, to be the amount of consumption such that the expected value of capital in the next period equals present capital. Now $\exp x_{n+1} = y + \bar{\beta}(x_n - c_n)$. Expected stationarity, $\exp x_{n+1} = x_n$, implies $c_n = (y/\bar{\beta}) + [(\bar{\beta} - 1)/\bar{\beta}]\, x_n = I(x)$. Expected income is displayed as a function of capital in Figure 1. Our question is then whether, in the limit, as N approaches infinity, $c_N(x) < I(x)$ for all $x \geqslant y$.

The answer is clear cut when capital is riskless. Then $\beta_i = \beta$ for all i and we obtain the following recurrence relation in the limiting utility of wealth function:

$$w(x) = \max_{c} \{u(c) + \alpha w(\beta(x - c) + y)\}\,. \tag{5.1}$$

The maximum is an interior one for $x \geqslant y$ so that $c(x)$ defined by

$$u'(c) - \alpha\beta w'(\beta(x - c) + y) = 0 \tag{5.2}$$

determines c as a function of x.

Differentiating totally with respect to x gives

$$\begin{aligned} w'(x) &= \alpha\beta w'\,(\beta(x - c) + y) + c'(x)[u'(c) - \alpha\beta w'(\beta(x - c) + y)] \\ &= \alpha\beta w'(\beta(x - c) + y) \qquad \text{[by (5.2)]}. \end{aligned} \tag{5.3}$$

Since $w'(x)$ is monotone decreasing, (5.3) implies that $x_{n+1} > x_n$ if and only if $\alpha\beta > 1$. Therefore, denoting the limiting consumption function by $c(x)$, $c(x) < I(x)$ for all $x \geqslant y$.

This simple result fails to extend to risky capital. When $\beta_i \neq \bar{\beta}$ for some i, (5.3) becomes

$$w'(x) = \alpha \sum p_i \beta_i w'(\beta_i(x - c) + y)\,. \tag{5.4}$$

From (5.4) no general conclusions concerning the conditions for expected capital growth can be drawn. Of course capital cannot be expected to grow

[9] Of course, an affimative answer would not be very interesting if the necessary value of N exceeds human life expectancy!

very long unless $\bar{\beta} > 1$. But $\alpha\bar{\beta} > 1$ is insufficient to guarantee "expected" capital growth.[10]

It is clear that the critical value which $\alpha\bar{\beta}$ must exceed if capital growth is to be expected will depend upon the distribution of β_i and the shape of the marginal utility function $w'(x)$. The only practical procedure here is to investigate the implications for capital growth of particular classes of utility functions.

6. IMPLICATIONS OF SELECTED MONOMIAL UTILITY FUNCTIONS

In this section we investigate the implications of certain types of monomia utility functions for the consumption function and for the expected path of capital.

We consider first the utility function[11]

$$u(c_n) = \bar{u} - \lambda c_n^{-\gamma}, \qquad \bar{u}, \gamma > 0, \lambda > 1\,. \tag{6.1}$$

Solving successively for the sequence of unknown functions $\{w_n(x)\}$, $N = 1, 2, \ldots$, yields

$$\begin{aligned} w_N(x) = {} & \bar{u}(1 + \alpha + \ldots + \alpha^{N-1}) - \lambda(\alpha b^{-\gamma})^{N-1}[1 + (\alpha b^{-\gamma})^{\frac{-1}{\gamma+1}} + \\ & \ldots + (\alpha b^{-\gamma})^{\frac{-(N-1)}{\gamma+1}}]^{\gamma+1}[x + (b^{-1} + \ldots + b^{-(N-1)})y]^{-\gamma} \end{aligned} \tag{6.2}$$

and

$$c_N(x) = \frac{(\alpha b^{-\gamma})^{\frac{-(N-1)}{\gamma+1}}}{1 + (\alpha b^{-\gamma})^{\frac{-1}{\gamma+1}} + \ldots + (\alpha b^{-\gamma})^{\frac{-(N-1)}{\gamma+1}}}[x + (1 + b + \ldots + b^{N-2})y] \tag{6.3}$$

where

$$b = (\Sigma p_i \beta_i^{-\gamma})^{\frac{-1}{\gamma}}\,.$$

[10] Several plausible cases are the following. First, there may be no capital level at which the expected returns to saving repays the risks. Or it may be that the individual can "afford" the risks of net expected saving only when capital exceeds a critical value at which $c(x)$ intersects $I(x)$ from above. In the opposite case, additional wealth is worth the risks only as long as capital falls short of the level where $c(x)$ intersects $I(x)$ from below.

[11] The function (6.1) fails to have the boundedness property assumed up to this point and thus it contradicts the "continuity axiom" mentioned in Section 2. Whatever the merits of that axiom, the function has received sufficient study in the context of deterministic models [4, 12, 13] to deserve our attention here.

If the reader applies (6.3) to $c_{N+1}(x)$ and uses (6.2) he will obtain an expression for $w_{N+1}(x)$ having the same form as (6.2). Note also that if $\alpha = \beta_i = 1$ for all i, formula (6.3) calls for consuming a fraction $1/N$ of the individual's net worth, $x + (N-1)y$.

Provided that $\alpha b^{-\gamma} < 1$ (for which $\alpha < 1$, $\beta > 1$, $\gamma > 0$ is sufficient in the certainty case), the expressions in (6.2) and (6.3) converge as N approaches infinity, giving the solutions to the "infinite stage" process:

$$(6.4) \qquad w(x) = \frac{\bar{u}}{1-\alpha} - \lambda \left[\frac{(\alpha b^{-\gamma})^{\frac{-1}{\gamma+1}}}{(\alpha b^{-\gamma})^{\frac{-1}{\gamma+1}} - 1} \right]^{\gamma+1} \left(x + \frac{y}{b-1} \right)^{-\gamma}$$

and

$$(6.5) \qquad c(x) = \left(1 - (\alpha b^{-\gamma})^{\frac{1}{\gamma+1}}\right) \left(x + \frac{y}{b-1} \right).$$

This limiting consumption function is useful as an approximation to $c_N(x)$ for large N.

(i) *Properties of the consumption function.*

A number of properties of the consumption functions (6.3) and (6.5) can be observed immediately. First, the consumption function is linear homogeneous in capital and nonwealth income. Of two households, both having identical utility functions like (6.1), if one household enjoys twice the capital and nonwealth income of the other, it will also consume twice as much.

Second, consumption is linear in capital and nonwealth income. The coefficient of wealth, $\partial c/\partial x$, may be called the marginal propensity to consume (MPC) out of wealth.

The convergence condition $\alpha b^{-\gamma} < 1$ insures that $\partial c/\partial x > 0$. And $\partial c/\partial x < 1$ for all finite α, $b > 0$.

The coefficient $\partial c/\partial y$ may be called the MPC out of "permanent," sure, (nonwealth) income. Clearly $\partial c/\partial y > 0$ if and only $b > 1$ (given the convergence condition). What can be said concerning this condition?

When capital is risky (that is, when $\beta_i \neq \bar{\beta}$ for some i), then $b < \bar{\beta}$.[12] Therefore the postulate $\bar{\beta} > 1$ does not imply $b > 1$. We see thus that Keynes' "psychological law" stating that MPC > 0 applies only if capital has a positive net expected productivity and only if capital is sufficiently productive at that. However, we do observe positive MPC and if we were to

[12] To see this, draw a diagram showing $\beta_i^{-\gamma}$ as a function of β_i. Since $\beta^{-\gamma}$ is a convex function of β, $\sum p_i \beta_i^{-\gamma} > \bar{\beta}^{-\gamma}$ whence $b = (\sum p_i \beta_i^{-\gamma})^{-1/\gamma} < \bar{\beta}$.

fit this model to data we should presumably find that $b > 1$. At any rate, we shall assume $b > 1$ unless we indicate the contrary.

Is the MPC also less than one, as Keynes had it? Of course, with $b > 1$, the MPC out of an income stream beginning sufficiently far in the future is bound to be less than one. Usually one considers the effect on (immediate) consumption of immediate income. To do that in the present model—where the paycheck is received at the end of the period—suppose capital increases by the same amount as y, as if last period's paycheck were increased too. Is this MPC out of "immediate," nonwealth income smaller than one?

This MPC is

$$\left[1 - (\alpha b^{-\gamma})^{\frac{1}{\gamma+1}}\right] \frac{b}{b-1}$$

and is smaller than one if and only if $\alpha b > 1$.

This is an interesting condition. This same condition, we show now, is necessary and sufficient for positive capital accumulation at all possible values of income and capital.

Note first that $c(x) < I(x)$ for all $x \geqslant y$—causing the expected growth of capital—if and only if $c(y) < y$ and $c'(x) \leqslant I'(x)$. Now $c(y)/y$ equals the MPC just analyzed so that $\alpha b > 1$ means $c(y) < y$. The condition that $c'(x) < I'(x)$ is

$$1 - (\alpha b^{-\gamma})^{\frac{1}{\gamma+1}} < \frac{\bar{\beta} - 1}{\bar{\beta}}$$

for which $\alpha b > 1$ is sufficient (although unnecessary).[18]

The significance of this exercise lies in the possibility that $1 < b \leqslant 1/\alpha$, in which case capital will be expected to grow only if it exceeds a certain threshold. Suppose $\alpha b = 1$. Then all nonwealth income is consumed and there is "net expected saving"—that is, $c(x) < I(x)$—only if $x > y$, i.e., only if the individual starts the period with some capital over and above his just-received wage of the previous period. Otherwise there will be no "hump saving" (in this case), even though $\bar{\beta} > 1/\alpha$.

A comparison of the MPC's leads to an interesting finding: The greater nonwealth income, y, as a proportion of total expected income, $I(x)$, the larger is the ratio of consumption to expected income. This is because the MPC out of (sure, immediate) nonwealth income, $c'(x)b/(b-1)$, is greater than the consumption effect of that increase in current capital which is required to raise expected income by one dollar. Writing

$$x = \frac{\bar{\beta}}{\bar{\beta} - 1}\left[I(x) - \frac{y}{\bar{\beta}}\right],$$

[18] Note that all these conditions reduce to $b > 1$ if $\alpha = 1$.

we see that the latter consumption effect is $c'(x)\,\bar{\beta}/(\bar{\beta}-1)$. Recalling that $b < \bar{\beta}$, we find that "sure" income has the stronger effect. This implies that, among households who have like utility functions and who face the same capital growth process, those whose expected income depends relatively heavily on risky capital will be observed to be relatively thrifty. This may help to explain why wealthy heirs, farmers, and certain other groups save a comparatively large proportion of their incomes. Further, the result suggests that capital income and labor income ought not to be aggregated in econometric analyses of consumption.

(ii) *Variations of risk and return.*

The last question taken up here relates to the effect upon consumption of variations in the riskiness and expected return from capital. Since the consumption function is linear homogeneous we can write

$$c = \frac{\partial c}{\partial x}\,x + \frac{\partial c}{\partial y}\,y\,,$$

whence these variations influence consumption through the marginal propensities, which are a function of b (and independent of x and y).

Let us consider first the effect of variations in risk and return on the value of b.

An increase in the expected return on capital is defined here as a uniform shift in the probability distribution of β_t which leaves all its moments the same except the mean, $\bar{\beta}$. Such a shift *increases* $\bar{\beta}$ and b.

What effect has risk on the value of b? When capital is risky, $b < \bar{\beta}$. Thus the presence of risk (as distinct from marginal increases therein) *decreases* b.

Hence, capital's (net) productivity and its riskiness affect consumption in the opposite direction.

A second kind of risk effect results from a change in the degree of risk, somehow measured.

A probability distribution which offers a simple measure of risk is the uniform or rectangular distribution. This is a two-parameter distribution with mean $\bar{\beta}$ and range $2h$. The variance is $h^2/3$ so that h is the measure of risk.

We show now that increases in h reduce b so that the "structural" and "marginal" effect of risk on b are in the same direction. Noting that $db/dh < 0$ means $db^{-\gamma}/dh > 0$, we examine $b^{-\gamma}$.

By definition of b,

$$b^{-\gamma} = \int_{\bar{\beta}-h}^{\bar{\beta}+h} \beta^{-\gamma}\left(\frac{1}{2h}\right) d\beta\,.$$

Evaluating the integral we find

$$b^{-\gamma} = \frac{1}{(1-\gamma)2h}\left[(\bar{\beta}+h)^{1-\gamma} - (\bar{\beta}-h)^{1-\gamma}\right].$$

Differentiating with respect to h yields

$$\frac{db^{-\gamma}}{dh} = \frac{1}{2(1-\gamma)h^2}\left[(\bar{\beta}-h)^{-\gamma}(\bar{\beta}-\gamma h) - (\bar{\beta}+h)^{-\gamma}(\bar{\beta}+\gamma h)\right].$$

Assuming $\gamma > 1$, $db^{-\gamma}/dh > 0$ if and only if

$$\frac{\bar{\beta}-\gamma h}{\bar{\beta}+\gamma h} < \left(\frac{\bar{\beta}-h}{\bar{\beta}+h}\right)^{\gamma}.$$

β equal to zero is excluded, for otherwise b is not defined. Consequently $h < \bar{\beta}$ and the right hand side of the inequality must be positive. But so may be the left hand side (if $\gamma < \bar{\beta}/h$). The following shows the inequality is satisfied for all $\gamma > 1$.

Dividing both sides of the inequality by $\bar{\beta}$, and defining $z = h/\bar{\beta}$, we obtain

$$\frac{1-\gamma z}{1+\gamma z} < \left(\frac{1-z}{1+z}\right)^{\gamma}$$

which, taking the logarithm of both sides, we find to be satisfied if and only if

$$\log(1-\gamma z) - \log(1+\gamma z) < \gamma\left[\log(1-z) - \log(1+z)\right].$$

Expansion of the logarithmic functions into Taylor's series yields

$$\left(-\gamma z - \frac{(\gamma z)^2}{2} - \frac{(\gamma z)^3}{3} - \ldots\right) - \left(\gamma z - \frac{(\gamma z)^2}{2} + \frac{(\gamma z)^3}{3} - \ldots\right)$$
$$< \gamma\left[\left(-z - \frac{z^2}{2} - \frac{z^3}{3} - \ldots\right) - \left(z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots\right)\right]$$

whence

$$\left(\gamma z + \frac{(\gamma z)^3}{3} + \frac{(\gamma z)^5}{5} + \ldots\right) > \left(\gamma z + \frac{\gamma z^3}{3} + \frac{\gamma z^5}{5} + \ldots\right).$$

This inequality can be seen to hold for all $\gamma > 1$. Therefore a margina increase in risk reduces the value of b. Recalling that an increase in the expected return increases b, we note that changes in risk and return have opposite effects on consumption.

We consider now the effect of a change in b upon consumption. Does the substitution effect dominate here—so that a rise in b encourages saving and reduces consumption? Or does the income effect dominate?

Turning first to $\partial c/\partial x$, we see from (6.5) that an increase in b raises $\partial c/\partial x$.

Turning next to $\partial c/\partial y$, we note from (6.5) that $\partial c/\partial y = 1/(b-1) \cdot \partial c/\partial x$. It would appear that a rise in b might reduce $\partial c/\partial y$, because of the downward recapitalization (using $1/(b-1)$) of the y stream, if b were sufficiently small ($b > 1$). It can be shown that $d(\partial c/\partial y)/db \geqslant 0$ if and only if $(\alpha b^{-\gamma})^{-1/(\gamma+1)} \leqslant (1 + b\gamma)/(1 + \gamma)$. If $\alpha = 1$ this is satisfied for all $b > 1$; otherwise it is satisfied only for values of b above some value $\bar{\bar{b}} > 1$.

Thus, if there is no utility discount, the income effect dominates here; then a rise in the expected return on capital weakens the incentive to save and an increase in risk compels more saving in order to reduce the insecurity of the future. But if the future is discounted, the individual feels "poorer"; then a rise in the expected return may encourage saving up to a point, after which the income effect dominates; this point comes sooner the smaller is γ. In either case, risk and return variations have opposing qualitative effects upon consumption.

(iii) *Other utility functions.*

To see that the implications of the utility function (6.1) for the effects of variations in risk and return are not general, one has only to modify the utility function thus:

$$\text{(6.6)} \qquad u(c_N) = \lambda c^{\gamma}, \qquad \lambda > 0,\ 0 < \gamma < 1\,.$$

All the equations (6.2)–(6.5) continue to hold with the difference that λ and γ are then replaced by $-\lambda$ and $-\gamma$, respectively. Hence the limiting consumption function is

$$\text{(6.7)} \qquad c(x) = \left[1 - (\alpha b^{\gamma})^{\frac{1}{1-\gamma}}\right]\left(x + \frac{y}{b-1}\right)$$

where $b^{\gamma} = \sum p_i \beta_i^{\gamma}$.

An increase in $\bar{\beta}$, other moments of the distribution unchanged, will increase b.

Once again the effect of risk is easy to ascertain. Since β^{γ} is a concave function of β, $\sum p_i \beta_i^{\gamma} < \bar{\beta}^{\gamma}$ whence $b = (\sum p_i \beta_i^{\gamma}) < \bar{\beta}$.

Turning finally to the effect of a marginal increase in risk upon b, we find that the "natural" result $db^{\gamma}/dh < 0$ (meaning that global and marginal risk effects have like signs) depends upon the condition $(\bar{\beta} - \gamma h)/(\bar{\beta} + \gamma h) > [(\bar{\beta} - h)/(\bar{\beta} + h)]^{\gamma}$, which is satisfied for all $\gamma < 1$.

Once again, risk and return work in opposite directions.

Consider now the effect of an increase in b upon consumption. Unlike the previous example, $\partial c/\partial x$ decreases with increasing b, as can be seen from (6.7); the substitution effect dominates the income effect. And, as (6.7) clearly shows, $\partial c/\partial y$ is also a decreasing function of b for all values of $b > 1$; the downward recapitalization of future income merely reinforces the substitution effect against the weaker income effect.

Thus an increase in expected return encourages saving while an increase of the riskiness of capital discourages saving. The implications of the utility function (6.6) are essentially opposite to those of the utility function (6.1).

To what can this contrast of results be attributed? The utility function is determined only up to a linear transformation, meaning that we can set $\bar{u} = 0$ in (6.1) without effect. Doing this reveals that both (6.1) and (6.6) are constant-elasticity utility functions with elasticity parameter y. The income effect dominates (unless b is small and y large) in the elastic case and the substitution effect dominates in the inelastic case.

Finally we examine a utility function that can produce some odd results, the logarithmic function in (6.8):

$$u(c_N) = \log c_N \,. \tag{6.8}$$

It appears to be impossible to solve for $c_N(x)$ explicitly in terms of x and y except in the case $y = 0$. Then we easily find

$$w_N(x) = (1 + \alpha + \ldots + \alpha^{N-1}) \log x + v(\theta, \alpha, N) \tag{6.9}$$

where $v(\theta, \alpha, N)$ depends only upon the parameters, denoted by θ, of the probability distribution of β_t, α and N, and not upon x.

Also

$$c_N(x) = \frac{x}{1 + \alpha + \ldots + \alpha^{N-1}} \,. \tag{6.10}$$

When the utility function is logarithmic, the optimum consumption rate is independent both of the expected return and riskiness of capital. Consumption is linear homogeneous in capital. As N is increased, the consumption function flattens asymptotically until, in the limit,

$$c(x) = (1 - \alpha)x \,. \tag{6.11}$$

A limiting function exists only if $\alpha < 1$.[14]

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[14] For certain utility functions the existence of a limiting solution does not require $\alpha < 1$. Ramsey [12] argued that boundedness was sufficient but a condition on the elasticity or rate of approach to the upper bound is also necessary, at least in models not containing risk. Samuelson and Solow [14] assume that the upper utility bound is attained at a finite consumption rate, which is not a necessary condition.

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FISCAL NEUTRALISM AND ACTIVISM TOWARD ECONOMIC GROWTH

One of the folkloric propositions about the predominantly capitalistic economy is that its consumers are sovereign over its rate of growth; at least they are sovereign over the volume of private investment, notably the rate of tangible private capital accumulation, on which the growth rate heavily depends. The subscribers to this proposition include many political conservatives who are pleased to think that consumers acting individually in the free marketplace, not the government, choose the investment rate.[1] The subscribers include many socialists who believe that capitalism is inferior to socialism precisely because capitalist governments cannot control the aggregate investment rate. And they include some liberals whose arguments for new public policies to boost the growth rate implicitly assume that the market or the government cannot already err on the side of excessive investment.

The fundamental error of this proposition, viewed from the standpoint of contemporary fiscal and monetary theory, arises from its neglect of the role played by government taxation. If consumers happen to be sovereign when the central government

1. Some conservatives would qualify the proposition. They would concede that the marketplace is an imperfect instrument by which consumers can realize the growth—more precisely, the time paths of family consumption—that they really want. No system after all can solve perfectly the awesomely complex problems of intertemporal choice in this uncertain world. But they hold that the free choice which the capitalist system gives to consumers comes tolerably close to giving households the time-profile of consumption goods that they wish among all feasible profiles. I shall return soon to questions of the efficiency of the markets in making intertemporal allocations for they lie at the heart of the controversy over growth policy.

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runs a 10 billion dollar deficit, say, they surely cannot also be sovereign if tax rates should place the government budget in deficit by 20 billion dollars. If a sufficiently tighter money policy is coupled with an easier tax policy, aggregate employment will be left unchanged; but we shall have a greater share of aggregate production devoted to consumption goods and a smaller share left for capital formation.

To save the proposition of consumer sovereignty, its supporters must fall back on the contention that, fortuitously or not, the "mix" of taxes and monetary policy over the business cycle has happened to be about right to make consumers in fact sovereign. It must be contended that the treasury of the central government, together with the other taxing authorities, has in fact been "neutral" toward consumer spending and saving. What does this mean? In what sense—by what standard—can we say that the treasury is or is not "neutral"?

CLASSICAL "NEUTRALITY" TWICE SPURNED

The effect of tax policy on the rate of private investment, accepted now by neo-Keynesian economists, was orthodox doctrine in classical economics. David Ricardo and others recognized that an already existing government debt, internally owned, does not subtract from existing productive resources.[2] After denying that taxation to service an already existing public debt is a real burden —on the ground that the interest on the debt is a transfer from one citizen to another—Ricardo added: "From what I have said it must not be inferred that I consider the system of borrownig as the best calculated to defray the . . . expenses of the state. It is a system which tends to make us less thrifty—to blind us to our real situation."[3] The suggestion here is that had the government expenditure been financed instead by taxes, consumers would have felt appropriately poorer in terms of their command over resources for *private* use and would have reduced their (private)

2. It should be added that governmental indebtedness to the private sector will have some effect upon the uses of existing resources, particularly upon consumption and hence upon the volume of resources left available for capital formation, unless taxes are increased enough to offset this effect.

3. David Ricardo, *Principles of Political Economy and Taxation* (E. P. Dutton, 1911), p. 162.

consumption accordingly; the reduction of consumption would have released resources for the government's use and thus reduced or eliminated any diversion of resources from the investment sector.[4]

Ricardo's objection to deficit finance makes the earliest statement of what I call fiscal *neutralism,* a doctrine that government taxes ought to be at a level which conveys to consumers the value or "opportunity costs" of the resources being diverted from private to public use. He presumed that the neutral level of taxation is that which puts the government budget in "balance," neither in surplus nor deficit.

The notion that there is some special virtue in a balanced budget from the point of view of investment and growth was first spurned by the early post-Keynesians who wished to employ tax-rate variations to keep the economy operating at the desired level of employment. The "budget" was placed exclusively in the service of economic stability and any yearning for budget balance was put down as prescientific. If the private components of aggregate demand were buoyant, taxes must be set high, high enough perchance to put the government budget in surplus; if private demand were depressed, taxes would have to be low and the budget might thereby be placed in deficit. Monetary policy was given little part to play.

Monetary policy came back to intellectual life in the 1950s, not so much as an anticyclical weapon but as a long-run force. Neo-Keynesians like Paul Samuelson wrote of the choice between having, on the one hand, easy money to promote high investment and growth with tax policy to restrain consumption demand and, on the other, tight money to restrain growth with low taxes to promote consumption.[5] Taxes were once again assessed for their

4. In a "lifetime saving" model of consumer behavior, only a fraction of the extra tax bill would be financed by a reduction of consumption in the early months, the remainder of the reduction occurring later on.

5. See P. A. Samuelson, "Public Responsibility for Growth and Stability," in this volume, pp. 70–74.

In the wake of Russia's Sputnik in 1957 a number of economists desiring faster growth urged a new "mix" between monetary and fiscal policy in favor of resolutely easier money and a postponement or cessation of tax-rate cuts. The inhibiting effect of the balance-of-payments problem on the monetary authorities and the large growth of "potential" or capacity effects on growth, as in classical doctrine. Taxes might still be called upon to a limited extent for stabilization purposes, especially if monetary policy was slow to have effect or was harmful in quick and large doses. But monetary instruments would be employed to make the tax level expected to be appropriate in the future consonant with long-run government fiscal objectives. A more serious qualification arises if changes in interest rates and rates of return on capital have an important permanent effect on the balance of payments, given aggregate demand. Then the possibility of exchange-rate adjustments must be assumed to keep international reserves at desired levels.

Several writers drew the conclusion that if fiscal and monetary tools give rise to political control over investment, then the volume of private investment will be determined by a political process. Some writers evidently envisioned that the governing political party (and perhaps the rival party) would formulate an investment policy or a growth policy for popular approval. Samuelson wrote: "This [governmental] power over the community's rate of capital formation should constitute a sobering responsibility for the voters in any modern democracy."

There was no intent here to deny any individual the free choice to consume whatever portion of his disposable after-tax income he likes. Households need not be equally thrifty, nor some thriftier than others. The instruments of the politically determined growth policy were to be just the everyday tools of fiscal and monetary policy. Nor was there any intent by and large to engineer a rate of investment at variance with popular desires. Consumers might yet be sovereign over growth. But through a political process. The market would have little or no role. Indeed, changes in market behavior as such might not have even a marginal effect upon the politically determined growth rate. In the absence of political expressions for greater investment and growth, an overall increase in the thriftiness of consumers would presumably be offset by a reduction of tax rates lest consumption decline.

The marketplace was apparently not seen as even potentially useful in promoting consumer sovereignty over investment. It was

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output that mounted over the years, however, ultimately won economists and legislators over to another tax cut (legislated in 1964) as the one means available to bring the economy back towards high employment.

not conceived that a suitable fiscal environment might bestow sovereignty on the consumer in his market role. It was tacitly assumed that there exists no fiscal principle, no rule of taxation, the application of which would cause private markets to find the sovereign rate of investment through the dollar votes of consumers and savers. In particular the rule of taxing so as to balance the budget was not presumed to induce private markets to choose the "right" rate of investment. The concept of neutrality was again rejected, or else neglected.

A MODERN CONCEPT OF FISCAL NEUTRALITY

Contrast this novel position on the role of politics in determining resource allocations to economic growth with the position of most Western economists on resource allocations to individual consumer goods. These economists recognize that when there are externalities and decreasing-cost phenomena the government will usually need to intervene with public expenditures to achieve a more "efficient" mix of consumption goods. Without (perfect) intervention the allocation of resources will not generally be "Pareto optimal": that is, it will be possible by some reallocation of resources to make everyone better off. Consider, however, the residue of private consumption goods for which there are perfectly competitive markets and which give off no externalities. It is agreed that the government does not, in order to achieve a Pareto-optimal mix of consumption goods, require a "consumption policy" to determine the mix of outputs of these goods. There is no need to submit alternative consumption mixes to a popular vote. From the fact that the various levels of governments, through their excise taxes and subsidies, can *control* the mix of these goods it does not follow that the government must in any natural sense of the term *decide* the consumption mix. Although society can alter the relative production of pots and pans by political means, few suggest that we must therefore have a pot-or-pan policy. The government can exercise its power in a "neutral" way, hoping that the market will allocate resources among these consumption goods at least as well as can be done by a political process.

With respect to economic growth, how might a neutral policy be defined? I shall say that tax policy is neutral if it produces the same allocation of resources—between aggregate consumption and investment especially, but also among different consumption goods and among individuals—as would occur if there were no government treasury at all (hence no government debt and no government taxes of the ordinary kind) but only a government agency to conscript resources for use in the production of public goods supplied by the government and an agency to redistribute wealth so as to achieve the desired distribution of lifetime income. The conscription or draft of resources here is taken to be efficient: the government is imagined to conscript that set of resources, among all sets sufficient to produce the programmed public goods, which entails the least reduction of the output of private goods valued at their relative prices. And it is imagined that future conscriptions on each person are known insofar as they interest any living person. It does not matter that such a conscription system would be impractical compared to the potentialities of a system of taxation using money or credit. The purpose is only to use conscription as a standard for judging whether tax policy causes the economy to duplicate the way an economy would grow under this idealized system of conscription. Note finally that a neutral tax policy causes the economy to imitate the resource allocations of an economy with the public sector the present economy actually has, not to imitate some otherwise similar economy without any public sector at all.

The virtue of the hypothetical conscription system, though not necessarily an adequate virtue, is that it would not, to use Ricardo's term, "blind" the members of the economy to their true command over private consumption goods now and in the future, as might an arbitrary level of taxes. Each household would know the real costs to itself, in terms of the consumption it expects to be able to make, of the planned program of government-supplied goods. (That the government must form expectations of its future provisions of public goods is not only a prerequisite of a neutral tax policy but, presumably, a prerequisite of any rational use of fiscal instruments.)

Though a neutral tax policy does not distort people's vision of their lifetime consumption possibilities, it should quickly be admitted that people's eyesight in this respect may be astigmatic to begin with. If people see themselves as poorer than they really are, for example, a non-neutral policy that makes them feel a little richer might be better. The shortcomings of fiscal neutrality will shortly be discussed. But let us first examine the workings of fiscal neutrality to understand its strengths. Consider an increase in the government debt without any accompanying change in other economic conditions. We know that such a change in "paper wealth" does not increase the community's true power to consume goods now and in the future, given its present assortment of capital goods. A neutral tax policy will therefore respond with additional taxes to make people feel poorer by just the amount that the increase of the public debt makes them feel richer. The "neutralizing" tax addition will normally exceed the interest on the added national debt, for if taxes were raised just enough to service the additional debt, finite-lived taxpayers, who can sell the debt and thus consume the principal, would still feel that they were richer on balance. A balanced budget therefore is not generally neutral. Or consider a planned increase in future free government services. The Rockefeller Brothers Fund in 1958 saw in the expected enlargement of government expenditures within ten years a reason for larger current private investment. A neutral tax policy would give this result. It would raise taxes by an amount such that people's expectations of their present and future private consumption possibilities would fall by the amount of the increase in future government withdrawals of resources (suitably discounted back to the present according to their distance); present private consumption expenditures would thus fall and resources would be freed for private capital formation.

THE TROUBLE WITH FISCAL NEUTRALITY

Operating in the fiscal environment of a neutral tax policy, the private market will produce a Pareto-optimal growth path of "private" consumption goods if a competitive equilibrium is attained; if there is complete information about future as well as current prices (including wage rates and interest rates) and also perfect information about current and future supplies of public goods; if producers have complete information about the future as well as the current technology; if there are no externalities in production; if consumers know their tastes and their preferences are unchanging over time; if there are no externalities in consumption other than the public goods whose production we take as given.

These are stringent "ifs" and of course they are not satisfied precisely by any economy. In what follows the most frequently cited and perhaps most important ways in which market economies fail to satisfy these conditions will be discussed.

Many of the points listed here as objections to a policy of fiscal neutrality were originally voiced as objections to the growth rate produced by a *laissez faire* economy. Many of these arguments suggest that investment and growth would be too little under *laissez faire*. Since modern economies are not *laissez faire*, and especially since they contain fiscal and monetary instruments that influence the growth rate, these particular arguments do not necessarily show that present-day capitalist economies grow too little. What they may suggest is that growth would be too little under a neutral fiscal policy. But not all the objections to fiscal neutrality have this same upshot.

Pigovian "Myopia" · One of the oldest objections to the *laissez-faire* market solution to growth problems—and by extension to the market solution under a neutral tax policy—is the "myopia" argument of Alfred Pigou. He wrote that "our telescopic faculty is defective," that we "see future pleasures on a diminished scale" and accordingly consume in the present to an extent we later regret. Thus "people distribute their resources between the present, the near future and the remote future on the basis of a wholly irrational preference." Pigou wanted government policies that would steer the economy along a growth path dictated by consumer preferences which were rid of these irrational elements. But the very meaning of consumer sovereignty and "optimality" comes into question.

Absence of Comprehensive Futures Markets · One of the facts of economic life is that our economy lacks futures markets that would bring together buyer and seller of future goods at known prices. The reasons for this lie partly in the awkwardness of allowing for technological and demographic uncertainty. In any case, the absence of these markets compounds the uncertainty about future prices, wage rates and interest rates. No one knows how much other people are going to save in the future, and hence what future interest rates and wage rates will be. Thus people may misjudge their lifetime purchasing powers even under neutral taxation. It would be interesting to know whether today's middle-aged overestimated or underestimated their real earning possibilities today when they were making work and saving decisions ten and twenty years ago.

Social vs. Individual Risk · Another way that capitalism fails to produce the right rate of growth, under a neutral fiscal environment, arises from the tendency of firms and wealthowners on the whole to shy away from risky investments. Robert Solow, James Tobin, and others have argued that to the extent that the risks of various investments are statistically independent, society would ideally "pool" these risks in the manner of an insurance company, making investments pretty much on the basis of their "actuarial" returns, on the mathematical expectations of their returns, without much worry that any individual investment would go sour. But even the largest firms are not large enough to spread the risks of investments on the scale, say, of the supersonic transport. This is one reason why there is frequently a discrepancy between the rates of interest that savers and lenders typically earn and the rates of return that firms require their investments to promise. As a result there is likely to be the wrong amount of saving as well as too little risky high-yield investment relative to safe low-yield investment. Some departure from fiscal neutrality is required to improve the situation. Rather than simply tax above the neutral level, however, which could be the wrong medicine, it would be best that the government arrange to subsidize risky investment or to engage in risk-sharing with firms.[6] Such fiscal actions would cause interest rates to rise, bringing them nearer to actuarial returns, and this rise might increase total saving; at least private saving would (on this account alone) be better attuned to the prospective average returns on the economy's aggregate investment.

6. To some extent, the corporate income tax causes the government to share in the profits and losses from investments and thus reduces risk as well as expected returns.

Monopoly · Another reason that many firms do not accept all investments which have an expected social rate of return in excess of the rate of interest available to savers is that these firms have monopoly power. Where there is a natural monopoly or there are artificial restrictions on entry, other firms are prevented from competing away the resulting monopoly profit. The latter shows up as an excess of the value of market value of the firm over the replacement cost of the firm's assets. (Other factors, like the costs of expansion and the costs of increasing customers can impose this excess valuation though they do not have the same significance.) One effect of this monopoly is likely to be an increase in the quantity of capital allocated to the more competitive sectors of the economy. The inflated asset valuation and the reduction in the rate of interest available to savers is also likely to reduce total saving and investment.

Externalities from Investment · It is often argued nevertheless that without monopoly we would have less growth and ultimately less capital formation on the ground that we depend upon patent protection and other restrictions upon entry in order to encourage technological research. To this Kenneth Arrow, Richard Nelson, and other economists reply that the system of monopoly produces less than the ideal amount of research, especially an inadequate amount of pure research. This is because the benefits of research are potentially "external" to the firm undertaking it. Since information of any kind is technically a "public good" whether offered by the government or not—it can be shared by all with only a negligible transmission cost—there would ideally be no price charged to producers for the use of and access to research performed by another producer. There seems to be a dilemma for economic policy here: no price should be charged for the use of research yet denial of the right to charge a price, to demand royalty payments for its use or to retain monopoly rights in its use, would virtually destroy any incentive for firms to do research. Many economists see a way out of this dilemma through extensive government finance or subsidization of research with little or no patent protection allowed. But clearly the government cannot be expected to make perfect decisions in so complex an area as technological research. So it is probably wise to leave some room for private initiative. Whatever the best practical solution to these problems, the shortcomings of our technological institutions do not seem to signal clearly that the neutral level of overall taxes is definitely too low or too high.[7]

Externalities from Consumption · Just as production, like the production of knowledge, can produce an external effect (neighborhood effect) or at least a potential one if no obstacle to it is placed in the way, so consumption can produce externalities. Amartya Sen and Stephen Marglin have offered the example of the "isolation paradox." Suppose that everyone living today feels a kind of generalized altruism toward his contemporaries as well as people in the next generation. Suppose further that the increased amount of consumption by each person in the next generation which is necessary to compensate the representative individual living today for a one-unit reduction in the consumption of each of his contemporaries is less than the increase of the amount of per capita consumption the next generation can have if each member of the present generation gives up one unit of consumption. If the number of people is large enough, there will then be a net gain for all if every individual is made to give up some consumption. This is because the favorableness of the terms at which society can exchange present consumption for future consumption by investing capital exceeds the favor the representative man has for his contemporaries relative to the next generation—and it does so by more than enough to compensate each individual for the reduction of his own consumption. Note that despite this gain, the individual is not implied to be willing to consume less *on his own*. He will be happy consuming less only if his contemporaries must match him. If these externalities are important, a neutral fiscal policy will produce the wrong quantity of consumption. Some departure from neutral taxation is required to induce people collectively to consume less when there is a gain from so doing (as just described) or to consume more when there is a loss from a collective increase of consumption. But the market behavior of savers cannot tell us by how much to depart or even in what direction to depart from the neutral level of taxation.

7. Note that other kinds of investment, like education or even tangible capital formation, may give off externalities.

Overlapping Generations · Fresh troubles assail the policy of fiscal neutrality when generations overlap, when the old coexist for a time with the young. Samuelson, Peter Diamond and others have studied the possibility of a market malfunction if the procession of generations is going to be infinitely long. In some no-government models of such an economy it can be shown that, while each household acts rationally, the economy may very well oversave in a certain sense, driving the capital stock beyond what is fruitful if it is to be maintained at that level forever. In a stationary economy with no population growth and no technological progress, this phenomenon would be signalled by a continuously negative rate of return on capital. This possibility exists in these models because there is no way that people can save for their retirement years except by accumulating tangible capital. The same oversaving can occur in an economy with a government and neutral taxation. A departure from fiscal neutrality would be called for if this phenomenon arose in order to eliminate the oversaving.

Another difficulty arises when generations overlap from the fact that the true private consumption possibilities of those living today, particularly those who will survive to coexist the succeeding generations, depend not only on the technology and upon future government uses of resources but also upon the way the succeeding government distributes the tax burden among the various generations then living. (The size of any negative taxes paid to the aged is a case in point.) This consideration alters somewhat and makes conjectural the neutral level of taxation. It is the taxes future governments will impose on the present generations, not the level of overall future government expenditures they will make, that will determine (along with current government outlays) the neutral level of taxes. The overlap of generations may also create an opportunity for those living today to improve upon a neutral tax policy. By reducing its present tax level, the present generation may be able to foist some of the cost of its current government programs onto future generations to the extent that future governments do not penalize surviving members of the present generation for so doing. Whether those with an interest in their heirs' well-being would only deceive themselves in thinking themselves to be better off and whether they might anticipate the larger tax burden on their heirs and accordingly bequeathe more are intricate issues in the theory of saving behavior that cannot be pursued here.

Exact Neutrality Unrealizable · The last objection to fiscal neutrality is that precise neutrality is not realizable in practice. Some economists believe that neutrality is a will-of-the-wisp. It is not only the overall budgetary position, the overall level of taxes, that influences consumption, after all, but also the specifics. A precisely neutral tax policy requires exclusive reliance on "lump-sum" taxes, taxes which the taxpayer believes to be independent of the amount of leisure he takes, the amount of saving he does and the amount of investment he undertakes. Such taxes, of course are not feasible, not for any length of time. Thus governments must decide upon a mix of income taxes, profits taxes, wage taxes, excise taxes and so on; each of these has substitution effects upon the incentive to work and to save. Yet the quantitative importance of these substitution effects is not yet established and if they were quantified one could adjust the overall tax level so as approximately to nullify these substitution effects.

THE TRIALS OF ACTIVISM

What should we conclude from this long (though much condensed) critique of fiscal neutrality and of the marketplace it intends to make sovereign over growth? Note that any *non*-neutral policy towards the overall level of taxation would be as afflicted by many of the faults of capitalism as would a neutral policy. The questions of monopoly and technological progress would bedevil any architect of a politically determined growth policy. But some of the objections—especially uniformed judgments about future prices, externalities in consumption and the possibility of gain at the expense of future generations (who will probably be better off than we anyway)—strike at the heart of the case for using markets to realize consumers' wishes regarding growth.

The good neutralist will answer that the estimates of the future by the uncoordinated market, while imperfect, are not worse than would be the imperfectly informed estimates made by the government; that the failure of the market to cope with external effects is not less than the ability of governments to encompass the side-effects of its actions. Some neutralists will contend that the errors from following a neutral policy would be as small as those from any other policy and that it is hard to tell whether fiscal neutrality would produce too little growth or too much. Other neutralists will argue that calculation of the neutral tax level will indicate the right general magnitude of growth and that back-of-the-envelope estimates of the resulting errors may provide a basis for deviating a little in some direction from the neutral tax level.

Others will conclude in favor of activism towards growth decisions. Finding the objections to fiscal neutrality overwhelming, they will want to rethink economic growth from scratch. The presumption of these activists is that the marketplace under a neutral policy would err so badly that a carefully constructed growth program is likely to be superior.

It should be understood that the activists' road is the hard one, not the easy one. The problems which activists see as unsolved by fiscal neutrality the activists must set for themselves to solve. The imperfectly informed voter is as unlikely to meet these problems as is the decision-maker in the marketplace.

In place of the estimates by consumers of what future the market will bring, calculations must be made of the consequence for future consumption possibilities of alternative government tax programs for growth. Production functions and technological-progress functions must be estimated; future technologies and demographic patterns must be studied. Thorniest of all these calculations is the problem of estimating how future governments will respond to the resources made available to them. If the present government departs radically from the tradition of balanced budgets, will future governments also? If future governments are assumed by each government to behave as it would in their shoes, this problem takes on a conjectural game-theoretic aspect, and it can lack a solution. How convenient by contrast to imagine that the consumer is expert in these matters!

Digging into the preferences of the people is the other half of the task. The difficulty is compounded when people are deemed to care about the external effects of growth upon their contemporaries and upon future people. When we widen reasonably further their areas of concern, the mind boggles. A decision, say, to reduce significantly the rate of economic growth could affect human happiness more than consumption. It might bring a mood of tedium and futility. Rapid growth may perform deeper functions than the economist ordinarily supposes. It is said that many a household saves and invests in private securities in order to feel a participant in the development of the economy. Surrogate saving and investing in government bonds used to finance a portion of public expenditure instead of taxes might not give the same sense of satisfaction. On the other hand, the diversion of resources and of national attention away from investment might pave the way for new social and humanistic pursuits.

The groundwork for "optimal economic growth" is just beginning to be laid. Economics is not yet able to describe the welfare consequences of our growth choices with any confidence. Pending the further development of this research, intuitions and perceptions about the rightness of our present fiscal norms and investment rate will continue to be offered. The complexity of optimal economic growth should warn us against acting upon any of them uncritically. But neither should we fear to drift from the arbitrary fiscal policies of the present.

SECOND-BEST NATIONAL SAVING AND GAME-EQUILIBRIUM GROWTH

Nearly thirty years ago Frank Ramsey [13] pioneered in a new field of economic theory, which we now call optimal economic growth. In many respects his analysis was quite general and recently his model has been extended to cases of many capital goods, population growth, technological progress and uncertainty.[2] There has, however, been no modification of Ramsey's treatment of preferences.

Ramsey made the remarkable postulate that each generation possesses what we shall call *perfect altruism*. By this we mean that each generation's preference for their own consumption relative to the next generation's consumption is no different from their preference for any future generation's consumption relative to the succeeding generation. This is a *stationarity postulate*: the present generation's preference ordering of consumption streams is invariant to changes in their timing.[3] Thus Ramsey did not admit the possibility that the current generation would assign its own consumption a place of importance somewhat out of proportion to its proximity. In his analysis he allowed *time preference* only of an extraordinarily selfless kind: The pure time-preference or discount rate used in discounting the rate of utility from consumption t years hence is required to be constant with respect to time. A positive discount rate favours the present generation only because of and to the extent of the proximity of its consumption.

Presumably Ramsey was not so optimistic as to believe that the current population in fact experienced a pleasure from the prospect of any future generation's consumption, relative to pleasure from its own consumption, that is diminished only by its sheer futurity. He must have regarded such " preferences " as really an ethic to which all generations ought to subscribe. Indeed he termed positive utility discounting of any sort " ethically indefensible ", though he admitted a constant, positive discount rate into his analysis.[4]

But what if people do not subscribe to this ethic? Then the rate of national saving that is optimal from the standpoint of the present generation is not the Ramsey solution. If a truly democratic government attempts to cater only to the preferences of the individuals who are presently members of the body politic[5], then it is *their* optimum, rather than the Ramsey solution, in which a democratic government will interest itself. (Whether the government needs to compute this optimum instead of relying on certain fiscal rules or principles together with well-functioning markets is a separate matter.) Accordingly, this paper will investigate the optimal saving policy of an " imperfectly altruistic " present generation under various assumptions about future saving behaviour and its control.

Part I sets forth the assumptions concerning preferences and technological consumption possibilities that run throughout the paper. In Part II we analyze the " first-best " optimiz-

1 This research was supported in part by a grant from the National Science Foundation. The authors, both former students at Amherst College, would like to take this statistically improbable occasion to thank two of their Amherst teachers, Arnold Collery and James R. Nelson, for introducing them to the pleasures of economics.

2 For a survey, some new results and many references to recent work, see Phelps [11] Chapter 5.

3 This is Postulate 4 in Koopmans' study of ordinal utility functions which exhibit stationarity yet allow the utility discount factor to vary with the magnitude of consumption. (Note that the failure of the present generation's utility function to exhibit such stationarity does not prevent the " stationarity " of a different sort that exists if each generation's preferences are alike.) See Koopmans [5].

4 Ramsey, [13], p. 543.

5 This assumption is gaining ground in the theory of optimal economic growth. See, for example, Marglin [7], and Phelps [10].

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ation problem that arises when the present generation can commit future generations to save the amounts which the present generation wish them to save. If, however, the present generation lacks the power to commit future generations' decisions, then the saving policies of future generations constitute additional constraints for the present generation and the optimal saving decision of the present generation becomes a problem of " second best ". In Part III we derive the second-best saving policy of the present generation when all future saving income ratios equal an arbitrary constant. Of particular interest here is the question of whether second-best saving is greater or smaller than first-best saving when given future saving is non-optimal from the standpoint of the present generation. (Clearly future generations might save non-optimally in the present generation's view if they pursued certain arbitrary fiscal rules or if they were themselves " maximizing ", imperfect altruists. In either case, the present generation would face a second-best problem—even if it were itself perfectly altruistic.)

Finally, in Part IV, we suppose that all generations are alike in their preferences: they exhibit the same imperfect altruism, the same time preference and so on. We postulate that all generations expect each succeeding generation to choose the saving ratio that is second-best in its eyes. This somewhat game-theoretic model leads to the concept of an " equilibrium " sequence of saving-income ratios having the property that no generation acting alone can do better and all generations act so as to warrant the expectations of the future saving ratios. This equilibrium is compared to the first-best optimum and its intertemporal non-optimality in the Pareto sense is shown.

I. PREFERENCES AND CONSUMPTION POSSIBILITIES

Each generation is supposed to live, save and consume over just one period. These periods are equally spaced and infinite in number. All generations are taken to be equal in size.

The preferences of the present generation are represented by the utility function

$$U = u(C_0)+\alpha\delta u(C_1)+\alpha^2\delta u(C_2)+\ldots,\quad 0<\delta<1,\quad 0<\alpha<1, \qquad \ldots(1)$$

where C_0 is the consumption of the present generation, C_1 the consumption of the next, and so on. The " period utilities ", $u(C_t)$, are identical functions of current consumption but the " utility " of consumption t periods hence is " discounted " by the factor $\delta\alpha^t$. The constant factor α reflects time preference or myopia while the constant factor δ, applied equally to all future generations regardless of timing, is a measure of the degree to which the present generation values other peoples' consumption relative to their own. " Imperfect altruism " here denotes $0<\delta<1$ while " perfect altruism " means $\delta = 1$. All the equations shown here are valid for any positive δ but for simplicity of exposition we suppose $\delta<1$. In contrast, $\alpha<1$ is often necessary for the existence of the various optima considered here.

In this paper we confine ourselves to period utilities which exhibit a constant elasticity of marginal utility:

$$u'(C_t) = C_t^{-\rho},\quad \rho>0. \qquad \ldots(2)$$

If the present generation's preferences satisfy (1) and can be represented by an indifference map which is homothetic to the origin, then the " period utility functions " must satisfy (2).[1]

[1] Since any increasing monotonic transformation of U yields an equivalent representation of preferences there is no loss of generality in replacing each u by v, where $v = a+bu$, $b>0$. This justifies the omission of additive and multiplicative constants in (2).

It might be objected that the present generation has no interest in the time profile of future consumption; but surely it would save less if it thought that the next generation would run its capital stock to zero, leaving subsequent generations impoverished. The assumption that the intertemporal utility function is additive is equivalent to the assumption that our marginal rate of substitution of consumption in period i for consumption in period j is independent of the level of consumption in period k. An additivity assumption, together with a homogeneity assumption, implies that the intertemporal utility function is either of the form $\Sigma a_t C_t^{1-\rho}$ or $\Sigma a_t \log C_t$. Our assumptions about " discounting " and " imperfect altruism " imply $a_0 = 1$ and $a_t = \alpha^t\delta$, $t \neq 0$.

There are three types of period-utility functions satisfying (2); these correspond to $\rho<1$, $\rho = 1$ and $\rho>1$. For $\rho \neq 1$ we have

$$u(C_t) = \frac{1}{1-\rho} C_t^{1-\rho} + b. \qquad ...(2a)$$

For $\rho = 1$ we have (neglecting the constant of integration)

$$u(C_t) = \log C_t \qquad ...(2b)$$

When $\rho<1$, $u(0) = b$ and $u(\infty) = \infty$. When $\rho = 1$, $u(0) = -\infty$ and $u(\infty) = \infty$. When $\rho>1$, $u(0) = -\infty$ and $u(\infty) = b$.

As for the production side, we postulate a constant marginal productivity of capital, $\lambda - 1 > 0$, and no depreciation. Capital, K, is consumable, like rabbits. The process of capital growth is described by the relation

$$K_{t+1} = \lambda(K_t - C_t), \quad \lambda > 1 \qquad ...(3)$$

with the initial capital stock historically given:

$$K_0 = K_o, \quad K_o > 0. \qquad ...(4)$$

The variable s, the present " saving ratio ", is defined by

$$s = \frac{K_0 - C_0}{K_0}, \quad 0 \leqq C_0 \leqq K_0 \qquad ...(5)$$

and σ_t will denote the " saving ratio " t periods from the present:

$$\sigma_t = \frac{K_t - C_t}{K_t}, \quad 0 \leqq C_t \leqq K_t, \quad t = 1, 2, 3, ... \qquad ...(6)$$

Hence

$$K_1 = \lambda s K_0 \qquad ...(7)$$

$$K_{t+1} = \lambda \sigma_t K_t, \quad t = 1, 2, 3, ... \qquad ...(8)$$

If the σ's are all equal to some constant σ we have geometric growth of capital and consumption beginning in period 1:

$$K_t = \lambda^{t-1}\sigma^{t-1}K_1, \quad t = 1, 2, 3, ... \qquad ...(9)$$

$$C_t = (1-\sigma)\lambda^{t-1}\sigma^{t-1}K_1 \qquad ...(10)$$

or

$$C_t = (1-\sigma)s\lambda^t\sigma^{t-1}K_0. \qquad ...(11)$$

This leads to geometric growth (or decay) of the undiscounted marginal period-utilities by virtue of (2) which, as we shall see, is a property of considerable convenience. (There exists another production model in which consumption cannot exceed " current production " that also has this convenient property; the present model is merely the simplest available.)

Pigovian income, Y_t, defined as the consumption level which keeps the capital stock " intact " ($K_{t+1} = K_t$) is defined by

$$K_t = \lambda(K_t - Y_t) \qquad ...(12)$$

whence

$$Y_t = \frac{\lambda - 1}{\lambda} K_t. \qquad ...(13)$$

If capital is not to decrease we must have $C_t \leqq Y_t$. It is readily shown that

$$\sigma_t \geqq \frac{1}{\lambda} \text{ if and only if } C_t \leqq Y_t, \qquad ...(14)$$

but it would be artificial to impose such a constraint in the present model. Note finally that constancy of our σ_t is equivalent to constancy of the more familiar ratio $(Y_t - C_t)/Y_t$.

II. THE FIRST-BEST OPTIMUM

Consider now the first-best optimum. This would be realized if, for example, the present generation could control not only their own saving ratio but future saving ratios as well.

We observe that whatever the present saving ratio, an optimal programme must have the property that the σ_t's are chosen optimally with reference to the current capital stocks inherited from the past; one can determine each optimal σ_t as a function of the corresponding K_t, finding the optimal s function at the end of the problem once the policy functions governing the σ_t's have been determined. Hence, if we write

$$U = u(C_0) + \delta\alpha V \qquad \text{...(15)}$$

where

$$V = \sum_{t=1}^{\infty} \alpha^{t-1} u(C_t), \qquad \text{...(16)}$$

our problem is first to find the future consumption policies which maximise V (and express these in terms of σ_t). Let $V_*(K_1)$ denote the maximized value of V for given K_1. The problem can then be formulated in the usual manner of dynamic programming[1] by the recursive relation (suppressing time subscripts)

$$V_*(K) = \max_{0 \leq C \leq K} \{u(C) + \alpha V_*(\lambda(K-C))\} \qquad \text{...(17)}$$

in the unknown function $V_*(K)$. Since the V-maximization problem is an infinite-horizon one and V is " stationary " in Koopmans's sense (the discount factor declines geometrically), the optimal consumption policy $C_t = C_*(K_t)$ is independent of time. (The single asterisk denotes first-best optimality.)

By solving finite N-period processes for the current C_* function and the current V_* function and by taking limits as N approaches infinity, one can find the functions $C_*(K)$ and $V_*(K)$. In the case $\rho \neq 1$, using (2a), we have

$$V_*(K) = \frac{b}{1-\alpha} + \frac{1}{1-\rho}\left[\frac{1}{1-(\alpha\lambda^{1-\rho})^{1/\rho}}\right]^{\rho} K^{1-\rho} \qquad \text{...(18)}$$

$$C_*(K) = [1-(\alpha\lambda^{1-\rho})^{1/\rho}]K, \qquad \text{...(19)}$$

whence the first-best future saving ratio, σ_*, is a unique constant, independent of K:

$$\sigma_* = (\alpha\lambda^{1-\rho})^{1/\rho}, \qquad \text{...(20)}$$

There exists such an optimum if and only if $V_*(K)$ is finite (neglecting the b term which can always be set equal to zero). It is easy to show from (18) that this requires

$$(\alpha\lambda^{1-\rho})^{1/\rho} < 1 \text{ or equivalently, } \alpha\lambda^{1-\rho} < 1. \qquad \text{...(21)}$$

If this inequality is not satisfied, every $\sigma < 1$ is inferior to a σ closer to unity, and, since there is no " closest σ ", there is no optimum σ. The existence condition then is that the calculated $\sigma_* < 1$. Note that if $\rho < 1$, condition (21) is stronger than the condition that $\alpha < 1$. If $\rho > 1$ this condition is weaker.

In the logarithmic case, $\rho = 1$, we find, using (2b),

$$V_*(K) = \frac{\log K}{1-\alpha} + \frac{1}{1-\alpha}\log(1-\alpha) + \frac{\alpha}{(1-\alpha)^2}\log(\alpha\lambda) \qquad \text{...(22)}$$

$$C_*(K) = (1-\alpha)K \qquad \text{...(23)}$$

[1] See Bellman [2].

whence σ_* is again independent of K:

$$\sigma_* = \alpha. \qquad \text{...(24)}$$

Note that (24) is a special case of (20) for $\rho = 1$ so that (20) is our general formula for σ_*. In this case an optimum exists if and only if $\alpha < 1$, which is postulated.[1] That these formulae are indeed solutions to the functional equation (17) the reader can easily verify by showing that the formulae satisfy

$$V_*(K) = u(C_*(K)) + \alpha V_*[\lambda(K - C_*(K))]. \qquad \text{...(25)}$$

There is no doubt that the solution is unique so that the above formulae are the only correct ones.

The remaining problem is to optimize with respect to the present saving ratio, given that future saving ratios are to be set at their optimal value, σ_*. Thus we want to maximize U, as given below, with respect to s:

$$U = u[(1-s)K_0] + \delta\alpha V_*[\lambda s K_0]. \qquad \text{...(26)}$$

By virtue of the strict concavity of u and V_*, the stationary value at which $\partial U/\partial s = 0$ is a unique maximum. We shall let s_* denote the maximizing value of s; the single asterisk denotes the fact that it is the *first-best* present saving ratio, the appropriate present saving ratio when the present generation can control the future saving ratios in its own interest.

Using (2), (18), (22) and

$$0 = -K_0 u'[(1-s_*)K_0] + \delta\alpha\lambda K_0 V_*'(\lambda s_* K_0) = \frac{\partial U}{\partial s}, \qquad \text{...(27)}$$

we calculate that

$$\left(\frac{1-s_*}{s_*}\right)^{-\rho} = \delta\left(\frac{(\alpha\lambda^{1-\rho})^{1/\rho}}{1-(\alpha\lambda^{1-\rho})^{1/\rho}}\right)^{\rho} \qquad \text{...(28}a\text{)}$$

or equivalently

$$s_* = \frac{1}{1+\delta^{-1/\rho}\left(\dfrac{(\alpha\lambda^{1-\rho})^{1/\rho}}{1-(\alpha\lambda^{1-\rho})^{1/\rho}}\right)^{-1}}, \qquad \text{...(28}b\text{)}$$

which are valid for all $\rho > 0$. From (28b) we see that $0 < s_* < 1$ so the maximum is an interior one—on the condition, of course, that an optimum exists, hence that (21) is satisfied.

It is immediately apparent from (28a) and (20) that s_* would equal σ_* if δ were equal to one; in that case we would have the standard Ramsey problem, with stationarity, so that present saving would be the same function of current capital as future saving and, since our special utility function makes the saving ratios constants, all present and future saving ratios would be equal.

In our model, with $\delta < 1$, one sees from (28a) and (20) that

$$s_* < \sigma_*. \qquad \text{...(29)}$$

Imperfect altruism causes the present generation to choose a present saving ratio that is smaller than the future saving ratio it would like future generations to select. It can easily be verified that s_* increases monotonically with δ and that $\partial\sigma_*/\partial\lambda$ and $\partial s_*/\partial\lambda$ have the same sign as $1-\rho$.

III. THE SECOND-BEST OPTIMUM FOR ARBITRARY σ

We suppose now that future saving behaviour is beyond the present generation's control. In particular, we postulate that the future σ's are known constants and are equal:

$$\sigma_t = \sigma = \text{constant}, \quad 0 < \sigma < 1, \quad \text{for all } t \geqq 1. \qquad \text{...(30)}$$

[1] The above formulae are presented for the case in which λ is uncertain in Phelps [9].

The inequalities imply that consumption will be positive in all future periods if $K_t > 0$. It is conceivable that the economy might exhibit constancy of the future saving ratio through the interaction of certain government fiscal policies with certain private saving propensities. As we shall show in the subsequent section, such behaviour could also come from certain maximizations by future generations. We note that the formulae of this section are valid for $\delta = 1$ as well as $\delta < 1$ on the part of the present generation.

Using (1) and (11) one sees that the second-best present saving ratio, to be denoted s_{**}, maximizes

$$U = u[(1-s)K_0] + \alpha\delta u[(1-\sigma)\lambda s K_0] + \alpha^2\delta u[(1-\sigma)\lambda^2 \sigma s K_0] + \ldots + \alpha^t \delta u[(1-\sigma)\lambda^t \sigma^{t-1} s K_0] + \ldots \qquad \ldots(31)$$

with respect to s subject to (30) and the constraint $0 \leqq s \leqq 1$ in (5). Upon calculating the partial derivative $\partial U/\partial s$ and substituting the marginal utility formula in (2) we obtain

$$\frac{\partial U}{\partial s} = [-(1-s)^{-\rho} + \delta(1-\sigma)^{1-\rho} s^{-\rho} \sigma^{\rho-1} M] K_0^{-\rho} \qquad \ldots(32)$$

where

$$M = \alpha\lambda^{1-\rho}\sigma^{1-\rho} + (\alpha\lambda^{1-\rho}\sigma^{1-\rho})^2 + \ldots + (\alpha\lambda^{1-\rho}\sigma^{1-\rho})^t + \ldots.$$

This infinite series converges if and only if

$$\alpha\lambda^{1-\rho}\sigma^{1-\rho} < 1. \qquad \ldots(33)$$

On that condition we have

$$M = \frac{\alpha\lambda^{1-\rho}\sigma^{1-\rho}}{1-\alpha\lambda^{1-\rho}\sigma^{1-\rho}}. \qquad \ldots(34)$$

Equating the derivative in (32) to zero and using (34) yields our basic equation

$$\left(\frac{1-s_{**}}{s_{**}}\right)^{-\rho} = \delta\left(\frac{1-\sigma}{\sigma}\right)^{1-\rho}\left(\frac{\alpha\lambda^{1-\rho}\sigma^{1-\rho}}{1-\alpha\lambda^{1-\rho}\sigma^{1-\rho}}\right) \qquad \ldots(35a)$$

or equivalently

$$s_{**} = \frac{1}{1+\delta^{-1/\rho}\left(\dfrac{1-\sigma}{\sigma}\right)^{(1-\rho)/-\rho}\left[\dfrac{\alpha\lambda^{1-\rho}\sigma^{1-\rho}}{1-\alpha\lambda^{1-\rho}\sigma^{1-\rho}}\right]^{-1/\rho}}. \qquad \ldots(35b)$$

Our use of the double asterisk in (35) indicates that the value of s which satisfies this equation is the second-best value of s. That s_{**} is utility-maximizing rather than minimizing follows from the fact that $\partial^2 U/\partial s^2 < 0$. The maximum is clearly unique and an interior one. The common-sense explanation of the latter is that when $s = 1$ the marginal utility of present consumption is infinite while future marginal utilities are finite (and their sum converges) so $s = 1$ cannot be optimal; similarly, when $s = 0$ future marginal utilities are infinite.

The convergence condition in (33) is thus sufficient for the existence of this (second-best) optimum. But it is not always a necessary condition. If (33) does not hold—it must hold when $\rho = 1$, given $\alpha < 1$—then total utility diverges either to plus infinity (when $\rho < 1$) or to minus infinity (when $\rho > 1$) for all s. Let us however adopt the over-taking criterion according to which one policy is preferred to another if it produces greater cumulative utility over T periods for every T greater than some $T^\circ \geqq 1$, and according to which a feasible decision is optimal if it is preferred or indifferent to all others.[1] On that criterion, when (33) does not hold, every increase of s, $s < 1$, is an improvement. If $\rho < 1$, $s = 1$ is best of all since such a policy will "overtake" any policy of $s < 1$; thus the second-best optimum exists in this case and gives $s_{**} = 1$. If $\rho > 1$, the policy $s = 1$ gives a present-period utility of minus infinity by (2) and hence cannot overtake policies making $s < 1$; in

[1] See Weizsäcker [14] and Atsumi [1]. An exposition is contained in Phelps [11].

this case there exists no optimum since there is no value of s which is nearest to one yet not equal to it.

As a matter of notation, let $F(\sigma, \delta, \alpha, \rho, \lambda)$ denote the right-hand side of (35b). Then our results can be stated as follows:

$$s_{**} = \begin{cases} F(\sigma, \delta, \alpha, \rho, \lambda) \text{ if } \alpha\lambda^{1-\rho}\sigma^{1-\rho} < 1, \\ 1 \text{ if } \alpha\lambda^{1-\rho}\sigma^{1-\rho} \geqq 1 \text{ and } \rho < 1 \\ \text{Does not exist otherwise.} \end{cases} \qquad \ldots(36)$$

In what range must σ lie in order to satisfy the convergence condition in (33)? If $\rho = 1$, the condition reduces to $\alpha < 1$ so that all values of σ satisfy the condition. If $\rho \neq 1$ we solve for the value of σ, denoted by $\bar{\sigma}$, which gives equality in (33), i.e., the σ value which just fails to satisfy the convergence condition:

$$\bar{\sigma} = (\alpha\lambda^{1-\rho})^{-1/(1-\rho)}. \qquad \ldots(37)$$

Then, if $\rho < 1$, convergence will occur if and only if $\sigma < \bar{\sigma}$. We note in this case that

$$\bar{\sigma} > 1 \text{ if and only if } \alpha\lambda^{1-\rho} > 1 \text{ when } \rho < 1. \qquad \ldots(38)$$

Hence, if this latter inequality holds we have convergence for all σ, $0 < \sigma < 1$. Note that this condition is identical to the condition for the existence of the first-best optimum.

Figure 1 illustrates the case with $\bar{\sigma} > 1$ so that convergence occurs and the F function exists for all admissible σ. In Figure 2 we illustrate the $\bar{\sigma} < 1$ case in which F exists only for $0 < \sigma < \bar{\sigma}$ and $s_{**} = 1$ for σ such that $\bar{\sigma} \leqq \sigma < 1$. (The case $\bar{\sigma} = 1$ requires separate treatment which we omit.)

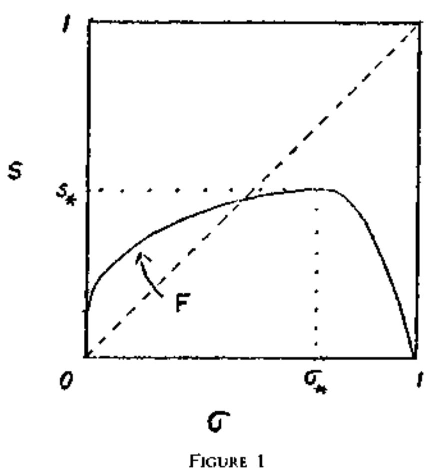

FIGURE 1

$\rho < 1$ $\bar{\sigma} > 1$

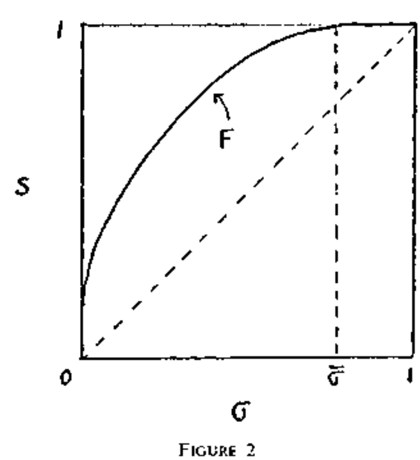

FIGURE 2

$\rho < 1, \bar{\sigma} < 1$

If $\rho > 1$ convergence will occur if and only if $\sigma > \bar{\sigma}$ where

$$0 < \bar{\sigma} < 1 \text{ when } \rho > 1. \qquad \ldots(39)$$

This is illustrated in Figure 3 where s_{**} exists only in the interval $\bar{\sigma} < \sigma < 1$.

Note that, when $\rho > 1$, $\bar{\sigma} < \lambda^{-1}$ so that, by (14), capital will be shrinking toward zero for all $\sigma \leqq \bar{\sigma}$; this makes the region of Figure 3 in which no optimum exists one of little interest. In contrast, when $\rho<1$, $\bar{\sigma}>\lambda^{-1}$ so the region in Figure 2 where $s_{**}=1$ is one of future capital growth (as is the adjacent interval between λ^{-1} and $\bar{\sigma}$).

Finally, Figure 4 illustrates the logarithmic case with $\rho=1$ where convergence occurs, so that the F function is defined, for all σ.

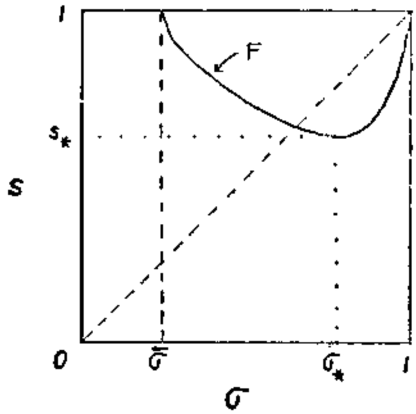

FIGURE 3

$\rho>1$

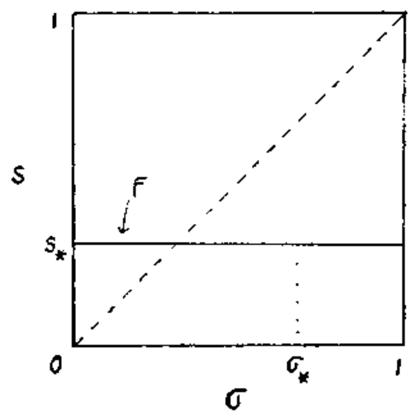

FIGURE 4

$\rho=1$

We shall now discuss the interesting, essential properties of these diagrams and compare second-best to first-best saving behaviour.

The logarithmic case is simple. Letting $\rho=1$ we obtain from (35b) the result

$$s_{**}=\frac{\delta\alpha}{1-\alpha(1-\delta)} \text{ for all } \sigma. \qquad \ldots(40)$$

Thus s_{**} is independent of σ in this special case. It is therefore clear, as a comparison of (28b) and (35b) confirms, (using $\rho=1$), that $s_{**}=s_*$. Hence a departure of σ from its optimal value, σ_*,—optimal from the viewpoint of the present generation—while reducing the present generation's utility, should not cause any adjustment of present saving. This logarithmic case must be added to the curious list of examples in which first-best and second-best decisions do not differ.[1]

Consider now the other cases, where $\rho\neq 1$. It is straightforward to prove that the F function approaches the boundary values shown in Figures 1-3 as σ approaches the values zero, $\bar{\sigma}$, and one.

As for the slope, we calculate

$$\frac{\partial F}{\partial\sigma}=\left(\frac{1-\rho}{\rho}\right)\frac{s_{**}(1-s_{**})}{\sigma}\left[\frac{1}{1-\alpha\lambda^{1-\rho}\sigma^{1-\rho}}-\frac{1}{1-\sigma}\right]. \qquad \ldots(41)$$

Of course, this derivative is meaningful only for σ such that the convergence condition, $\alpha\lambda^{1-\rho}\sigma^{1-\rho}<1$, is satisfied. (41) shows that $\partial F/\partial\sigma=0$ if and only if

$$\sigma=\alpha\lambda^{1-\rho}\sigma^{1-\rho} \text{ or equivalently, } \sigma=(\alpha\lambda^{1-\rho})^{1/\rho}. \qquad \ldots(42)$$

Recalling (20), we see therefore that a stationary value occurs at $\sigma=\sigma_*$, the " optimal " value of σ, and only there. If σ_* exists, the convergence condition must be satisfied at

[1] These cases have been characterized in terms of " separability ". See Davis and Whinston [4], and Pollak [12].

least for σ in the neighbourhood of σ_*, so that there must be a stationary value of F at $\sigma = \sigma_*$. Recalling (21), σ_* must exist if $\rho > 1$ (Figure 3) and, when $\rho < 1$, if $\bar{\sigma} > 1$ (Figure 1). Since σ_* is unique, the stationary value is unique if it exists, which is to say, if σ_* exists.

The insensitivity of s_{**} to σ in the neighbourhood of σ_* may come as a surprise to those who would have expected F always to be monotone in σ. Analogous results arise, however, in short-run, least-cost production theory.

We can now deduce that the stationary value, if it exists, must occur below the 45-degree line. For it is intuitively obvious and easy to show that

$$F(\sigma_*, \delta, \alpha, \lambda, \rho) = s_*; \qquad \text{...(43)}$$

i.e., when $\sigma = \sigma_*$, the second-best saving ratio is equal to the first-best ratio that would be chosen were the present generation able to choose σ. Since $s_{**} = s_*$ at $\sigma = \sigma_*$ and $s_* < \sigma_*$ [(29)] we have

$$F(\sigma_*, \delta, \alpha, \rho, \lambda) < \sigma_*. \qquad \text{...(44)}$$

The stationary value is a maximum or a minimum according as $1-\rho$ is positive or negative. If $\rho < 1$ then (41) shows that $\partial F/\partial\sigma$ is positive if $\sigma < \alpha\lambda^{1-\rho}\sigma^{1-\rho}$—which is to say, if $\sigma < \sigma_*$—and negative if $\sigma > \alpha\lambda^{1-\rho}\sigma^{1-\rho}$, i.e., $\sigma > \sigma_*$; thus the stationary value is a maximum in this case. Similarly, if $\rho > 1$, the stationary value must be a minimum. (Actually these results follow simply from the uniqueness of the stationary value and the values of the end-points of the F function.) These results explain the shape of the F functions in Figures 1 and 3.

We now consider the case of $\rho < 1$ when σ_* does not exist. This means that $\alpha\lambda^{1-\rho} > 1$. (This inequality can occur only if $\rho < 1$.) In this case $\partial F/\partial\sigma$ is clearly positive for all σ such that $\sigma < \alpha\lambda^{1-\rho}\sigma^{1-\rho}$ or $\sigma < (\alpha\lambda^{1-\rho})^{1/\rho}$, hence for all admissible σ for which F is defined. But when $\alpha\lambda^{1-\rho} > 1$ then $\bar{\sigma} < 1$ by (37) so that F is defined only for $\sigma < \bar{\sigma}$. This case is illustrated in Figure 2.

The significance of the inverted-U-shaped curve in Figure 1, with its maximum at $\sigma = \sigma_*$, is obvious. It means that $s_{**} < s_*$ for all $\sigma \neq \sigma_*$. When $\rho < 1$, the non-optimality of future σ in our eyes should cause us to save less than we would if we could impose our desires on future generations. When $\rho > 1$, as in Figure 3, $s_{**} > s_*$ for all $\sigma \neq 1$ so the non-optimality of σ is a reason for saving more. (Of course, in Figure 2 no first-best optimum exists so no comparison of first- and second-best present saving can be made.)

It can be shown that $\partial F/\partial\lambda$ has the same sign as $1-\rho$. Hence, where both first- and second-best optima exist, so that comparisons are possible, a divergence of σ from its optimal value will decrease (increase) optimal present saving if and only if an increase of λ would increase (decrease) optimal present saving. The ubiquitous conflict between income and substitution-effects will produce one pair of results or the other according as the marginal utility in (2) is elastic or inelastic.

It can also be shown that $\partial F/\partial\delta$ is everywhere positive, meaning that an increase of altruism will increase s_{**} (where an interior maximum exists) for any given σ.

IV. " EQUILIBRIUM " SAVING IN THE COURNOT-NASH SENSE

The concept and calculation of the second-best optimum is of interest even if that analysis does not explain actual national saving because society as a whole has no notion of such an optimum. Let us suppose now that the present society (or eventually some generation) acquires the notion of the optimum and that it becomes a conscious, calculating maximizer. Then the present generation will want to know what the future saving ratios are going to be, for its optimal s is not generally independent of σ.

A theory of future saving ratios is suggested by the observation that if the present generation has eaten from the fruit of knowledge, it is reasonable for this generation to expect that subsequent generations will likewise seek to optimize (in *their* eyes), and similarly for each future generation. Hence our problem is to look for a sequence (or sequences) of saving ratios each one of which is second-best from the point of view of the generation that chooses it. Such a sequence will be called an *equilibrium.*

We shall suppose that the consumption-possibility relation implicit in (3) is known and common to all generations. In addition, we postulate that the preferences of each generation for its own consumption and consumptions one, two, three . . . periods subsequent to it are identical to such preferences for every other generation; i.e., each generation has the same imperfect altruism, the same time preference and the same u function. Thus the next generation maximizes

$$u(C_1)+\alpha\delta u(C_2)+\alpha^2\delta u(C_3)+\dots$$

(subject to the subsequent saving ratios that it takes as given), the following generation similarly maximizes

$$u(C_2)+\alpha\delta u(C_3)+\alpha^2\delta u(C_4)+\dots$$

and so on, where the subscripts denote the dates of the consumptions. There is still an infinite number of periods.

Let us note that this problem possesses " stationarity " in a relevant sense (even though every generation's preferences are non-stationary in Koopmans's sense): If the present generation thinks it faces future saving ratios ($\sigma_1 = x_1$, $\sigma_2 = x_2$, ...) and, say, the next generation thinks (not always compatibly) that it faces the identical sequence of saving ratios ($\sigma_2 = x_1$, $\sigma_3 = x_2$, ...) then they will adopt the same second-best policy, independently of the fact that their dates in history differ. Since a second-best policy makes the saving ratio independent of current capital stocks they will adopt the same saving ratio—even if, unlike our earlier assumption, the future σ's are unequal.

It will now be clear that there may very well exist at least one equilibrium having the simple form that all the saving ratios are equal. Such an equilibrium exists if there is a number, say $\hat{\sigma}$, such that, if every generation expects all subsequent generations to choose a saving ratio equal to $\hat{\sigma}$, every generation will find that its own second-best saving ratio is equal to $\hat{\sigma}$. Each generation's assumption that subsequent generations will save the fraction $\hat{\sigma}$ of their respective capital stocks is self-warranting in that if the generations make this assumption they will act so as to validate it. The resulting sequence of saving ratios is an " equilibrium " in the sense (customary in other contexts) that expectations are fulfilled—albeit posthumously. It is also an equilibrium in the game-theoretic sense, used previously, that *ex post facto* no generation acting alone could have increased its total utility, given the saving policies of the other generations.

Thus we say that a sequence of equal saving ratios, $s = \sigma_1 = \sigma_2 = \dots = \sigma_t = \dots = \hat{\sigma}$, is an equilibrium one if and only if

$$\hat{\sigma} = F(\hat{\sigma}, \delta, \alpha, \rho, \lambda) \qquad \dots(45)$$

Such a " fixed point " occurs at the intersection of the F function with the 45-degree line in Figs. 1-4. In one case, as we shall show, there may be two such fixed points or none.

This concept of equilibrium was, of course, discussed (with reference to duopoly) by Cournot [3] in terms of the intersection of " reaction curves " such as our F function. Nash [8] in the past decade proved the existence of at least one " equilibrium point " in n-person, non-cooperative games in which each player has available to him a finite set of pure strategies—where an equilibrium point is a collection of strategies (possibly mixed strategies), one for each player, such that no player is able to increase his payoff when the others hold their strategies fixed.[1] The type of game here clearly differs somewhat from that studied by Nash. Nevertheless the equilibrium concept here does appear to be essentially that used by Cournot, Nash and other game theorists; hence the term " Cournot-Nash equilibrium " in this part title. Such an equilibrium is not necessarily of the sort customarily meant by many growth theorists.

[1] Nash [8]. For a survey of non-cooperative games, see especially Chapters 5 and 7 in Luce and Raiffa [6].

Using (45) and thus replacing s_{**} and σ by $\hat{\sigma}$ in (35) yields the following equation determining $\hat{\sigma}$

$$\frac{\hat{\sigma}}{1-\hat{\sigma}} = \delta\left(\frac{\alpha\lambda^{1-\rho}\hat{\sigma}^{1-\rho}}{1-\alpha\lambda^{1-\rho}\hat{\sigma}^{1-\rho}}\right) \qquad \text{...(46}a\text{)}$$

or equivalently

$$\hat{\sigma}^{\rho} = \alpha\lambda^{1-\rho}[\delta+(1-\delta)\hat{\sigma}]. \qquad \text{...(46}b\text{)}$$

We shall now briefly discuss the existence and uniqueness of such fixed points. We then compare the fixed point(s) with the first-best optimum and test for Pareto-optimality.

If $\rho>1$ and $\sigma>1$ it is clear that the F function (Fig. 1) must intersect the 45-degree line at least once. In fact, we shall show that in this case the F function intersects the 45-degree line only once, so that the fixed point is unique. If $\rho<1$ and $\bar{\sigma}>1$, the F function need not intersect the 45-degree line at all (Fig. 2); but we shall show that it is also possible for the F function to intersect the 45-degree line twice or to be tangent to the 45-degree line for some σ. If $\rho>1$, the F function (Fig. 3) clearly intersects the 45-degree line at least once. We shall show that in this case there is always exactly one fixed point. In the logarithmic case, $\rho = 1$, there clearly exists a unique fixed point (Fig. 4).

To examine the existence and uniqueness of the fixed point of the F function, we define two new functions. We let $L(\sigma)$ denote the left-hand side of (46b) and $R(\sigma)$ the right hand side:

$$L(\sigma) = \sigma^{\rho}, \qquad \text{...(47)}$$

$$R(\sigma) = \delta\alpha\lambda^{1-\rho}+\alpha\lambda^{1-\rho}(1-\delta)\sigma. \qquad \text{...(48)}$$

A value of σ is a fixed point if and only if it is admissible (between the zero and one) and $L(\sigma) = R(\sigma)$. We begin by calculating the first and second derivatives of L and R:

$$L'(\sigma) = \rho\sigma^{\rho-1}, \quad L''(\sigma) = \rho(\rho-1)\sigma^{\rho-2}, \qquad \text{...(49)}$$

$$R'(\sigma) = \alpha\lambda^{1-\rho}(1-\delta), \quad R''(\sigma) = 0. \qquad \text{...(50)}$$

If $\rho<1$ and $\bar{\sigma}>1$, $R(\sigma)$ is an increasing linear function of σ, and lies above $L(\sigma)$ at $\sigma = 0$ and below $L(\sigma)$ at $\sigma = 1$. Because its second derivative does not change sign, the monotonically increasing L function can intersect the R function only once. Hence, for $\rho<1$ and $\bar{\sigma}>1$, the F function has a unique fixed point, $\hat{\sigma}$.

As the geometry of Fig. 1 leads us to expect,

$$\hat{\sigma}<s_*<\sigma_*, \qquad \text{...(51)}$$

since s_* is the maximum value assumed by the F function and lies below the 45-degree line.

If $\rho>1$, the fixed point must occur for a value of σ greater than $\bar{\sigma}$, so we examine the behaviour of L and R for $\bar{\sigma} \leqq \sigma \leqq 1$. Again, $R(\sigma)$ is an increasing linear function of σ whose initial value, $R(\bar{\sigma})$, lies above the initial value of L, $(\bar{\sigma})$, and whose terminal value, $R(1)$, lies below the terminal value of L, $L(1)$. The L function is monotonically increasing, and since its second derivative does not change sign, there is clearly one and only one value of σ for which $L(\sigma) = R(\sigma)$. Thus, for $\rho>1$ the function has exactly one fixed point.

As the geometry of Fig. 3 suggests,

$$s_*<\hat{\sigma}<\sigma_*. \qquad \text{...(52)}$$

The first inequality follows from the fact that s_* is the minimum value assumed by the F function and is not on the 45-degree line. The second is a consequence of the fact that for $\rho>1$ the F function must have a negative slope at the fixed point and that the F function has a negative slope for and only for values of σ between $\bar{\sigma}$ and σ_*.

If $\rho<1$and $\bar{\sigma}<1$, a fixed point is a value of σ, $0<\sigma<\bar{\sigma}$, such that $L(\sigma) = R(\sigma)$. It is easily shown that $R(\sigma)$ is an increasing linear function of σ and that it both begins and ends above the L function; that is, $R(0)>L(0)$ and $R(\bar{\sigma})>L(\bar{\sigma})$. L is a monotonically increasing function with a negative second derivative. From this we may conclude that there are three possible cases: (i) there may be no fixed point, (ii) there may be one fixed point, if the L function is tangent to the R function for some σ, and (iii) there may be two fixed points.

By returning to the F function itself, it is possible to say considerably more about the existence or non-existence of fixed points in this case. From (41), it can be shown that the slope of the F function at a fixed point is less than one if (but not only if) $\rho+\delta>1$. But there can be two fixed points only if the slope of the F function at the second fixed point is greater than one, and one fixed point only if the slope of the F function at the fixed point is equal to one. Hence, if $\rho+\delta>1$ there can be no fixed point. If $\rho+\delta<1$ it is possible to have two fixed points, as the reader can verify by taking $\rho = \frac{1}{2}$, $\alpha = \frac{4}{5}$, $\lambda = \frac{16}{9}$, $\delta = \frac{11}{60}$, and computing the values of the F functions at $\sigma = \frac{1}{16}$ and $\sigma = (\frac{44}{49})^2$.

Note that in this case ($\rho<1$, $\bar{\sigma}<1$) there is no first-best optimum to be compared to the fixed point(s).

In the logarithmic case ($\rho = 1$), as already remarked, it is immediately clear that a unique fixed point exists and that

$$s_* = \hat{\sigma} < \sigma_*, \qquad \ldots(53)$$

In this analysis we have confined ourselves to fixed-points described by a constant saving ratio over time. We are unsure whether or not there may exist fixed-point sequences with non-constant saving ratios. It is possible therefore that the discussion which follows is somewhat incomplete.

We shall next show that the Cournot-Nash-fixed point equilibrium is not Pareto-optimal and further that there is a sense in which the equilibrium point displays "under-saving". Non-Pareto-optimality is not surprising for the basic situation has much in common with the "prisoners' dilemma" of game theory in which the equilibrium strategy of every partner-in-crime is to "confess". The question of under-saving is much subtler but if we consider only alternative *constant* saving ratios then, within this class of paths, there is under-saving at the game-equilibrium point. For we show now that there exists at least one point on the 45-degree line above the fixed point (or above both fixed points if there are two) which dominates the fixed point and dominates every point on the 45-degree line below the fixed point. This, of course, implies that the equilibrium point is non-Pareto-optimal.[1]

Total utility of the present generation in (31) depends upon s and σ so that we may write $U = U(s, \sigma)$, given initial capital and the four parameters. We wish to calculate $dU(s, \sigma)/ds$, subject to the side relation $\sigma = s$; this is given by

$$\frac{dU(s,\sigma)}{ds} = \frac{\partial U(s,\sigma)}{\partial s} + \frac{\partial U(s,\sigma)}{\partial \sigma} \qquad \ldots(54)$$

when evaluated at $\sigma = s$.

Consider first Figs. 1 and 4 where there is a unique fixed point. At $s = \hat{\sigma}$, i.e., at the fixed point $(\hat{\sigma}, \hat{\sigma})$, $\partial U/\partial s = 0$ since the fixed point lies on the F function. For all $s<\hat{\sigma}$, i.e., for all points (s, s) below $(\hat{\sigma}, \hat{\sigma})$, $\partial U/\partial s>0$ since such points must be below the F function and $\partial^2 U/\partial s^2>0$ for all σ. Hence

$$\frac{\partial U(s,\sigma)}{\partial s} \geqq 0 \text{ for all } \sigma = s \leqq \hat{\sigma}. \qquad \ldots(55)$$

Thus $\partial U/\partial\sigma>0$ for all $s \leqq \hat{\sigma}$ suffices to show that $dU(s, s)/ds$ in (54) is positive for all $s \leqq \hat{\sigma}$. Differentiation of (31) with respect to σ yields

$$\frac{\partial U(s,\sigma)}{\partial \sigma} = -\frac{\delta s^{1-\rho}\alpha\lambda^{1-\rho}(1-\sigma)^{-\rho}(1-\alpha\lambda^{1-\rho}\sigma^{-\rho})K_0^{-\rho}}{(1-\alpha\lambda^{1-\rho}\sigma^{1-\rho})^2}. \qquad \ldots(56)$$

[1] In the preliminary version it was shown that no point on the F function other than the first-best point (s_*, σ_*) is Pareto-optimal.

It is easily verified that this expression is positive for all $\sigma < \sigma_*$. Since $\hat{\sigma} < \sigma_*$ in all cases, $\partial U/\partial\sigma > 0$ at the fixed point and below it on the 45-degree line. Hence (54) is positive at the fixed point and below it. Thus the present generation and any future generation would be willing to increase its own saving ratio beyond $\hat{\sigma}$ by some amount if every succeeding generation were bound to imitate it. Further, such an increase of the saving ratio makes succeeding generations better off for an additional reason for each succeeding generation will inherit more capital if past generations have saved more. Similarly, a reduction of the common saving ratio below $\hat{\sigma}$ will make all generations worse off. So there may be said to be under-saving at the fixed point.

The Fig. 3 case requires only slight modification of the above argument. There $\partial U/\partial s$ and $\partial U/\partial\sigma$ are not defined for $\sigma \leqq \bar{\sigma}(<\hat{\sigma})$. But all points in this region yield infinite, negative utility so none of them can be preferred to the fixed points or to any point (s, s) above the fixed point; hence the undersaving argument carries over to this case.

In the Fig. 2 case, the above argument goes through if $\hat{\sigma}$ is unique, upon replacing σ_* in the argument by the number one (so that $\partial U/\partial\sigma$ is everywhere positive). If there exist two fixed points the above argument is invalid but the conclusion remains. For as we choose 45-degree points closer to $(\bar{\sigma}, \bar{\sigma})$, total utility goes to infinity so there is always a point (s, s) sufficiently close to $(\bar{\sigma}, \bar{\sigma})$ that dominates the fixed point and all points on the 45-degree line below it.

Hence, if we confine ourselves to constant saving ratio sequences, thus sticking to our 45-degree line, there can be said to be under-saving at any fixed point equilibrium. But we have not and shall not attempt to rule out the existence of some non-constant saving-ratio sequence which is both Pareto-optimal and which causes $\sigma_t < \hat{\sigma}$ for some t. So the under-saving hypotheses has not been completely sustained and possibly cannot be.

Our final topic is the consequence for the equilibrium saving ratio of an increase of altruism. If we write (46) in the form

$$\hat{\sigma} = G(\delta, \alpha, \rho, \lambda), \qquad \ldots(57)$$

where we consider only unique fixed points then, by definition of $\hat{\sigma}$,

$$G(\delta, \alpha, \rho, \lambda) = F[G(\delta, \alpha, \rho, \lambda), \delta, \alpha, \rho, \lambda] \qquad \ldots(58)$$

whence

$$\frac{\partial G}{\partial \delta} = \frac{\partial F/\partial\delta}{1-(\partial F/\partial\sigma)}, \qquad \ldots(59)$$

the right-hand side of which is to be evaluated at $\sigma = \hat{\sigma}$.

Since $\partial F/\partial\delta > 0$ everywhere, $\partial G/\partial\delta > 0$ if and only if $\partial F/\partial\sigma < 1$ at $\sigma = \hat{\sigma}$. The latter inequality clearly holds in Figs. 1, 3 and 4 where $\hat{\sigma}$ is unique. In the case $\rho < 1$, $\sigma < 1$ (Fig. 2), if two fixed points exist, an increase of δ, since it shifts up the F function, will increase the lower of the two fixed points while decreasing the upper fixed point. As δ increases, the fixed points come together at a tangency point corresponding to some $\delta < 1$; for larger δ no fixed point exists.

We remark that as $\delta \to 1$, s_* and $\hat{\sigma}$ approach σ_* so that the first-best sequence of saving ratios and the equilibrium sequence merge, both approaching σ_* which is, of course, the Ramsey solution for all σ_t (in our model) on his assumption that $\delta = 1$. In Fig. 2, the fixed points disappear which is as it should be since σ_* does not exist in that case.

V. CONCLUDING REMARKS

In studying the first-best optimization problem under imperfect altruism, we found that the present generation would save less as a proportion of income or capital than it would have future generations save. If the present generation cannot control future generations' saving and it expects future generations to choose a common saving ratio that is non-optimal in its view, the second-best present saving ratio will be smaller or greater than the first-best amount according as marginal utility is consumption-inelastic or consumption elastic. Then, upon imputing to each generation the expectation that succeeding generations would likewise seek a second-best optimum saving policy, we investigated an " equilibrium " sequence of saving ratios in the game-theoretic sense. We showed that such an equilibrium, where it existed, is not Pareto-optimal and that there is at least a natural and limited sense in which any such equilibrium entails " under-saving ".[1] We showed that the game-equilibrium saving ratio, if unique, was greater the greater is each generation's altruism and we remarked that the equilibrium sequence of saving ratios and the first-best sequence merge and become equivalent to the Ramsey-optimal sequence as altruism becomes perfect.

If we are right that the approach here represents a gain over previous approaches, then much more work needs to be done. One would like to suppose diminishing returns to saving, that capital consumption is possible only within limits, that population grows and the technology improves. Uncertainty about future decisions and even future existence need to be introduced. The over-lapping of generations should be treated. And ultimately one wants to know the implications of more general utility functions.

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[1] We saw that the present generation would increase its saving ratio if all future generations were bound to do likewise. Can the present generation so bind future generations? In the context of the present model the answer would appear to be no. Presumably a constitution establishing a saving ratio exceeding the game-equilibrium saving ratio would have no defenders and would be amended; but perhaps sentimental attachment to the constitution would save it. In a different model with over-lapping generations, the anticipated survival of people for two or more generations (" periods ") may lend stability to a constitution requiring national saving in excess of the game-equilibrium amount implied by that model. Further, in a model with people of varying altruism, the more altruistic may be able to block constitutional amendment.

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THE INDETERMINACY OF GAME-EQUILIBRIUM GROWTH

A recent paper by Robert Pollack and myself[1] essays the optimal saving problem when, by reason of (limited) selfishness on the part of each generation, future generations will not consume and save the capital they inherit in the proportions that the current generation would like them to do. This situation poses for each generation a "second best" problem or "sub-optimization" problem—neither term is wholly appropriate—whose solution depends upon the assumptions made by each generation about future saving behavior.

If each generation expects future generations to behave as it would behave in their situations, then there may result a kind of game-equilibrium growth path which is self-warranting: Every generation acts in such a way as to validate earlier assumptions as to how it will act and, given these assumptions, no generation acting alone can increase its estimated overall utility—despite the fact that cooperative action among the generations, were it enforceable, could produce an improvement in every generation's utility.

That paper postulated a discrete-time production process in which the

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*This paper is a revision of a discussion paper dated May 1968 and presented at the Econometric Society Congress in September 1970. This version corrects the admissible range of the asymptotic capital-labor ratio and it elaborates the potential role of an ethic in rescuing the determinacy of the growth path.

In the minds of some, my dynamic-programming analysis makes excessive demands on intuition. I am grateful therefore to Dr. Pauwels for recasting the argument in terms of differential game analysis. See his Mathematical Note which follows this chapter.

[1] E. S. Phelps and R. A. Pollak, "On Second-Best National Saving and Game-Equilibrium Growth," *Review of Economic Studies* 35 (April 1968).

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output-capital ratio is constant and labor is inessential and unproductive. The population is constant in size and is completely replaced each period.

The present contribution postulates an exponentially growing population and continuous-time production under possibly variable proportions and constant returns to scale along the lines of the one-good Solow-Swan neoclassical growth model.[2] Every generation is born directly into the labor force and dies with its boots on. There is no overlap—no births intervene during a generation's tenure. I shall study only the limiting case in which every generation's tenure shrinks to zero. In any finite length of time, then, infinitely many generations hold sway in a continuum, one after the other, each for an infinitely short time. It is hoped that there is heuristic value in such a model.

I. BOUNDED SUSTAINABLE CONSUMPTION PER CAPITA AND UNBOUNDED RATE OF UTILITY

The production-and-growth equations are

$$\dot{L}/L = \gamma = \text{constant} > 0 \tag{1}$$

$$f(k) = F(k, 1) = F(K, L)/L, \qquad k = K/L \tag{2}$$

with the following specifications in the first model:

$$f(0) \geqslant 0,\ f'(k) > 0,\ f''(k) < 0,\ f'(0) > \gamma,\ f'(\infty) = 0 \tag{2a}$$

$$\dot{k} = f(k) - \gamma k - c = g(k) - c, \tag{3}$$

with

$$g(0) \geqslant 0,\ g'(0) > 0,\ g''(k) < 0,\ g'(\hat{k}) = 0,\ 0 < \hat{k} < \infty. \tag{3a}$$

Here K and L denote capital and labor (or population), respectively, k the capital-labor ratio and c the rate of consumption per head (or per unit of labor). The lower quadrant of Figure 1 graphs the g function. Note that g is bounded and, further, a unique global maximum occurs at the Golden Rule capital intensity, $\hat{k}$.[3]

Now to the matter of saving behavior. The objective is to find and characterize some "policy function" or "consumption function," $c(k)$, that is

[2] R. M. Solow, "A Contribution to the Theory of Economic Growth," *Quarterly Journal of Economics* 70 (February 1956), and T. W. Swan, "Economic Growth and Capital Accumulation," *Economic Record* 32 (November 1956).

[3] For an exposition of the Golden Rule concept, see for example E. S. Phelps, *Golden Rules of Economic Growth* (New York: Norton, 1966), Chapter 1.

consistent with utility maximization by the generations and possesses the game-equilibrium property.

Imagine first some generation, born at time b, with lifetime Δ. Let it consider the contemporaneous per capita consumptions of the survivors and newborn during the interval Δ to be equivalent in worth to its own per capita consumption. Consumption standards are equalized among people at any given moment of time. Let the generation have complete control of national saving until $b + \Delta$ at which it bequeaths $k(b + \Delta)$ per capita. The latter will have a utility for the generation of $V_b(k(b + \Delta))$. Let $u(c(t))$ denote the instantaneous "rate of utility," untainted by time preference, produced by consumption at time t, $b \leqslant t \leqslant b + \Delta$. The generation's "lifetime utility," $U_b(\Delta)$, is

$$U_b\ (\Delta) = V_b(k(b + \Delta)) + \int_b^{b+\Delta} u(c(t))dt \tag{4}$$

where it will be specified in this first model that

$$u'(c) > 0,\quad u''(c) < 0,\ u(\infty) = \infty,\quad u(0) = -\infty. \tag{4a}$$

See Figure 1 for a picture of the u function.

For any provisional size of per capita bequest, x, the generation must, for a utility maximum,

$$\text{maximize} \qquad \int_b^{b+\Delta} u(c(t)dt$$

$$\begin{aligned} \text{subject to} \qquad & k(b) = k_b > 0 \\ & k(b + \Delta) = x \geqslant 0 \\ & \dot{k} = g(k) - c \end{aligned} \tag{5}$$

The solution is of the form

$$c(t) = h(t; k_b, x), \qquad b \leqslant t \leqslant b + \Delta \tag{6}$$

The utility-maximizing bequest must thus satisfy

$$\frac{\partial U_b\ (\Delta)}{\partial k(b + \Delta)} = V_b'\ [k(b + \Delta)] + \int_b^{b+\Delta} u'\left\{h[t; k_b, k(b{+}\Delta]\right\} \frac{\partial h}{\partial k(b + \Delta)} \cdot dt \tag{7}$$

Letting $\bar{c}$ denote the average per capita consumption rate over the interval Δ, we can, for small Δ, approximate the latter integral by

$$\Delta u'\ (\bar{c})\ \frac{\partial \bar{c}}{\partial k(b + \Delta)} \tag{8}$$

which, using the further approximation,

$$k(b + \Delta) = k_b + \Delta \, [g(k_b) - \bar{c}]$$

can then be written

$$\frac{\Delta u'(\bar{c})}{\partial [k_b + \Delta (g(k_b) - \bar{c})] \; / \; \partial \bar{c}} = - \, u' \; (\bar{c}) \, \frac{\Delta}{\Delta} = - \; u' \, (\bar{c}) \, . \tag{9}$$

Hence utility maximization equates the marginal utility of average consumption to the marginal utility of the bequest.

In the limit, as $\Delta \to 0$ and overlapping vanishes, utility maximization by any generation requires consuming so as to equate the marginal utility of consumption to the marginal utility of the bequest,

$$u'(c(b)) \; = \; V_b'(k_b), \tag{10}$$

the latter being completely predetermined in the limiting case.

The marginal utility for the present generation of the capital it bequeaths depends upon the value the present generation assigns to future consumptions and the disposition of capital for consumption by future generations.

Let each generation make the assumption that all generations infinitely far into the future will consume according to some unknown, to-be-calculated, stationary and continuous consumption function, $c(k)$; such a function derives from *their* utility maximization and their making the like assumption. Further, let it be assumed by all that $c(k)$ makes $k(t)$ approach some constant, $\bar{k} > 0$, independent of initial k_0. This means that $c(k)$ intersects $g(k)$ from below at one and only one point, as illustrated in the lower quadrant of Figure 1. This convergence of $k(t)$ to $\bar{k}$ might, for example, arise from a constant saving-income ratio, equivalently $c(k) = \alpha f(k)$, $0 < \alpha < 1$.

Formally, then,

$$\begin{aligned} & c(k) \begin{bmatrix} \leqq \\ > \end{bmatrix} g(k) \text{ according as } k \begin{bmatrix} \leqq \\ > \end{bmatrix} \bar{k}, \qquad 0 < \bar{k} = \text{constant}, \\ & \bar{c} = c(\bar{k}) \; = \; g(\bar{k}) \; > \; 0, \\ & c'(\bar{k}) \; \geqslant \; g'\,(\bar{k}). \end{aligned} \tag{11}$$

I postulate that every generation exhibits symmetrical or identical preferences for its own consumption vis-à-vis the consumption of generations subsequent to it. Specifically, each generation's preferences with respect to the *per capita* consumptions of generations *following it* are describable by application of the "overtaking principle" and an identical Ramsey-like utility functional lacking "pure time preference" or "myopia."[4] The overtaking principle states that one path is better than another if its functional at some point in time exceeds the functional corresponding to the other and does so continuously thereafter. This principle can be implemented by use of a functional whose integrand is the excess of the utility rate over that steady instantaneous rate of utility to which the utility rate is asymptotic if $k(t)$ is asymptotic to $\bar{k}$, namely $u(g(\bar{k}))$ or $u(\bar{c})$.

But there is this departure from the Ramsey model with population growth: Though generations are "alike," they disagree about the best growth path—each one tending to like best the path that gives *it* higher consumption. Each generation is selfish to the extent that, for equal per capita consumption rates, it assigns itself a marginal utility that is $1/\delta$ times the marginal utility it assigns to any future generation's per capita consumption, $\delta < 1$. Each generation tilts the marginal rate of substitution in its favor by a constant amount, yet refuses to discriminate among future generations on considerations of their relative proximities *per se*. (Making per capita consumption the desideratum already fails to attach weight to the fact that swapping a given amount of capital from sparsely to densely populated generations will increase the mean living standards in the two generations combined if only $f'(k) > 0$.)

The lifetime utility of our Δ-lived generation born at b can therefore be written

$$U_b(\Delta) = \int_b^{b+\Delta} [u(c(t)) - u(\bar{c})]\,dt + V_b(k(b+\Delta)) \tag{12}$$

where, with $\delta < 1$,

$$\begin{aligned} V_b(k(b+\Delta)) &= \delta \int_{b+\Delta}^{\infty} [u(c(t)) - u(\bar{c})]\,dt \\ &= \delta \int_{b+\Delta}^{\infty} [u(c(k(t))) - u(\bar{c})]\,dt, \end{aligned}$$

The marginal utility of bequeathed capital, $V_b'(k(b+\Delta))$, resides in the implication that additional capital will advance the time schedule with which $k(t)$ approaches the unknown asymptote and hence the time schedule of the advance of $u(c(t))$ toward $u(\bar{c})$. In formal terms, noting that the function $c(k)$ makes k a derived function of time,

$$\frac{dV_b\,(k(b+\Delta))}{dk(b+\Delta)} = \frac{dV_b(k(b+\Delta))}{d(b+\Delta)} \Bigg/ \frac{dk(b+\Delta)}{d(b+\Delta)} \tag{13}$$

[4] For a simple exposition of Ramsey, with and without population growth, and of the overtaking principle, see E.S. Phelps, *op. cit.*, Ch. 5. See also F.P. Ramsey, "A Mathematical Theory of Saving," *Economic Journal* 38 (December 1928).

where the denominator is the rate at which capital is accumulating by virtue of the $c(k)$ policy of the immediately following generations. Upon calculating the derivative in the numerator we obtain

$$V_b'(k(b+\Delta)) = -\frac{\delta[u(c(k(b+\Delta))) - u(\bar{c})]}{\dot{k}(b+\Delta)} \tag{14}$$

The trick here is that used by Keynes to show Ramsey how his no-discount formula for the optimal saving rate could be obtained without the calculus. Suppose "tomorrow's" generation would save one unit if it inherits the amount today's generation contemplates bequeathing to it. Then if today's generation adds one unit to its bequest, tomorrow's generation will be in the position that the next day's would otherwise have been in. Since it will then save precisely as the next day's would otherwise have, the next day's generation will similarly find itself as its successor would have. *Und so weiter.* Consequently, the utility shortfall that would otherwise have occurred tomorrow, $\delta[u(\bar{c}) - u(c)]$, is forever eliminated and the rest of the future is unchanged. If there were positive time preference, the advancement of the consumption schedule would have additional effects upon the utility of bequeathed capital.

In the limit, as $\Delta \to 0$, the marginal utility of consumption is equated to the limiting marginal utility of bequeathed capital:

$$u'(c(b)) = -\frac{\delta[u(c(k(b))) - u(\bar{c})]}{\dot{k}(b)} \tag{15}$$

But note that since $k(t)$ must be continuous and $c(k)$ is assumed by every generation to be continuous, our generation's calculated per capita consumption rate, $c(b)$, must equal $c(k(b))$, the per capita consumption rate immediately in the future or else our generation will realize it has assumed the wrong value of $c(k(b))$. For the next moment's generation, having essentially the same per capita capital and symmetrical preferences, will calculate essentially the same per capita consumption rate. This means that the present generation must assume a value of $c(k(b))$ which causes it to calculate an "optimal" consumption rate equal to it. Of course, there is a corresponding requirement on other generations. The game-equilibrium consumption rate at any time t, $c(t)$, therefore satisfies

$$u'(c(t)) = -\frac{\delta[u(c(t)) - u(\bar{c})]}{g(k(t)) - c(t)} \tag{16}$$

This differs from the Ramsey-Keynes relation with respect to δ, which is unitary in Ramsey's model, and with respect to $\bar{c}$ (about which more will be said).

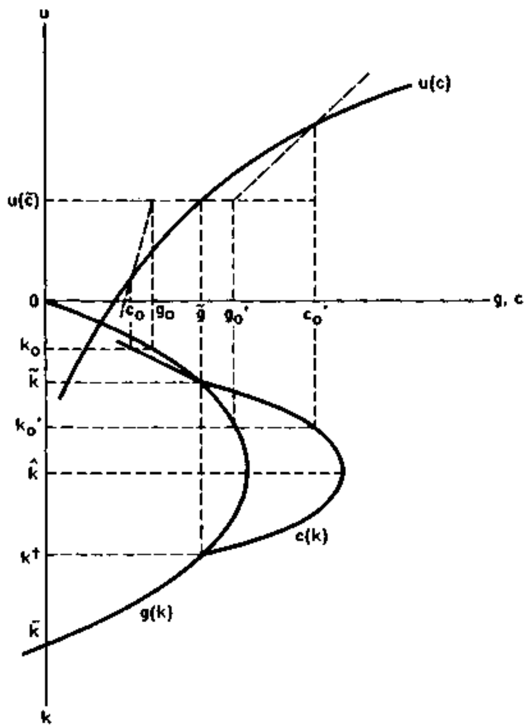

Figure 1. Unbounded Instantaneous Utility Rate and Bounded Sustainable Consumption Per Head.

Figure 1 shows a geometric method, due to Ramsey, of calculating $c(t)$ and displaying its dependence upon $k(t)$. Consider the initial per capita endowment k_0. The corresponding $g_0 = g(k_0)$ is smaller than $g(\tilde{k})$, the asymptotic consumption rate. To find the corresponding game-equilibrium consumption rate, c_0, we construct a straight line—the lefthand sloping dashed line in Figure 1—from the point $(\bar{u}, g_0)$ which intersects $u(c)$ with a slope $1/\delta$ times the slope (which is marginal utility) of the $u(c)$ function at that point. There must exist just one such point if $u'(0)$ is infinite. Its abscissa, c_0, must be less than g_0, whence k will be increasing.

With alternative endowment k_0', the corresponding $g_0' = g(k_0')$ exceeds $g(\tilde{k})$. A symmetrical construction of a line cutting through $u(c)$ with a slope $1/\delta$ times the slope of u determines a corresponding c_0' that is greater than k_0' so that k will be decreasing.

The geometry of Figure 1 assures us that, when $k(0) < \tilde{k}$, game-equilibrium c is monotone increasing in $g(k)$, therefore also in k. But when $k(0) > \tilde{k}$, our geometrical sense warns us that straight lines emanating from the u curve and bearing the right multiple, δ, to the slopes of the curve at their origin could *intersect* one another *below* $u(c)$ on the $u(\bar{c})$ horizontal. Hence, for $g_0 > g(\tilde{k})$, $c(k)$ need not be monotone and there may exist no single-valued function $c(k)$. This is borne out by calculating from (16) the derivative

$$c'(k) = \frac{u'(c)\ g'(k)}{u'(c)\,(1-\delta)\ -\ u''(c)\,[g\ -\ c]} \tag{17}$$

The denominator may vanish and change signs any number of times as k is increased over the range in which $g - c > 0$. So a continuous function $c(k)$ apparently need not exist over the whole domain of k.

As k is increased beyond the Golden Rule value, $\hat{k}$, every g value encountered in the climb to $\hat{k}$ is encountered again. Since c depends only on $g(k)$, not upon k itself, $c(k)$ must, in a sense, "double back." In particular, another intersection of the $c(k)$ curve with the $g(k)$ curve must occur at some $k^{\dagger}$ where $g(k^{\dagger}) = g(\tilde{k})$. For $k > k^{\dagger}$, if not for smaller k, any such $c(k)$ function is nonsense, since present saving sufficient to make $\dot{k} = g - c > 0$ there only *reduces* future per capita consumption, plunging society into a headlong rush toward self-destructive altruism at consumptionless $\bar{k}$. At $k < k^{\dagger}$, however, some "gross" per capita saving in the ordinary sense of $f(k) - c > 0$ is not obviously senseless as it will ultimately retard the slip-back of k and the *average* future per capita consumption rate, even if it depresses or slows the rise of per capita consumption for a while.

Assume however that a well-behaved $c(k)$ function exists over some relevant domain. Is there just one such function or many? The question involves the determinateness of the asymptote $\tilde{k}$.

First of all, it is worth establishing that any admissible asymptote $\tilde{k}$ cannot be larger than the Golden Rule $\hat{k}$. Despite the absence of positive pure time preference, no game-equilibrium growth path will drive capital intensity beyond the Golden Rule level.

We know that any "stable" asymptote $\tilde{k}$ must have the property of equality between the marginal utilities of consumption and bequeathed capital as in (10).

We can calculate $V'(\tilde{k})$ directly from the following linearization argument. An extra bequest Δk_0 received at the beginning of "day zero" will produce a vanishing sequence of capital-bequest deviations around $\tilde{k}$ in subsequent "days." A linear approximation of this sequence is the following:[5]

[5] The linear approximation is inadequate in the singular case where the functions $c(k)$ and $g(k)$ are tangent at $\tilde{k}$, as equations (19) and (20) clearly reveal.

$$\Delta k_1 = \Delta k_0 [1 - c'(\bar{k})] \frac{1 + f'(\bar{k})}{1 + \gamma} \tag{18}$$

$$\Delta k_2 = \Delta k_1 [1 - c'(\bar{k})] \frac{1 + f'(\bar{k})}{1 + \gamma}$$

$$= \Delta k_0 \left\{ [1 - c'(\bar{k})] \left[\frac{1 + f'(\bar{k})}{1 + \gamma} \right] \right\}^2$$

$$\vdots$$

$$\Delta k_n = \Delta k_0 \left\{ [1 - c'(\bar{k})] \left[\frac{1 + f'(\bar{k})}{1 + \gamma} \right] \right\}^n$$

For sufficiently short "days," we make the further approximation

$$[1 - c'(\bar{k})] \frac{1 + f'(\bar{k})}{1 + \gamma} = 1 + f'(\bar{k}) - \gamma - c'(\bar{k}) \tag{19}$$
$$= 1 + g'(\bar{k}) - c'(\bar{k})$$

The sum of the per capita bequest deviations thus exhibits a "multiplier" effect:

$$\sum_{t=0}^{\infty} \Delta k_t = \Delta k_0 \left(\frac{1}{c'(\bar{k}) - g'(\bar{k})} \right) \tag{20}$$

Therefore the sum of the increments in utility from the consumption increments induced by the geometrically declining bequest increments is approximately

$$\delta \sum_{t=0}^{\infty} \Delta u_t = \frac{u'(\bar{c}) c'(\bar{k})}{c'(\bar{k}) - g'(\bar{k})} \Delta k_0 \tag{21}$$

Since the departures from $\bar{c}$ and $\bar{k}$ are small for small Δk_0, in the limit, as Δk_0 shrinks to zero and as we move to continuous time, the following result holds exactly:

$$V'(\bar{k}) = \delta \frac{u'(\bar{c}) c'(\bar{k})}{c'(\bar{k}) - g'(\bar{k})} \tag{22}$$

Equality of our two marginal utilities therefore entails

$$u'(\bar{c}) = \delta \frac{u'(\bar{c}) c'(\bar{k})}{c'(\bar{k}) - g'(\bar{k})} \tag{23}$$

Note that in the Ramsey model, where $\delta = 1$, $g'(\bar{k})$ is implied to be zero, whence $\bar{k} = \hat{k}$.

Now $c'(\bar{k}) - g'(\bar{k}) \geqslant 0$ for stability. And $u'(c) > 0$ for all c. Therefore $c'(\bar{k}) \geqslant 0$. At any *stationary* equilibrium capital intensity, consumption is nondecreasing in capital. From (23) we have

$$c'(\bar{k}) = \frac{g'(\bar{k})}{1 - \delta} \tag{24}$$

It follows that $g'(\bar{k})$ must also be non-negative, like $c'(\bar{k})$. Hence $\bar{k}$, whatever its value or values, must be no larger than the Golden Rule $\hat{k}$ at which $g'(\hat{k}) = 0$.

Note that (24) is consistent with the more general expression for $c'(k)$ in (17) since $g - c = 0$ on any stationary path. One might, one would think, have used (17) directly to obtain (24). However the existence of two independent routes is reassuring; and my derivation of (17) requires that $\dot{k}$ be non-zero.

When $\dot{k} \neq 0$, differentiation of (16) with respect to time yields a kind of Euler equation that is of some interest:

$$\dot{k}\frac{u''(c)c}{u'(c)}\frac{\dot{c}}{c} = -\frac{d\dot{k}}{dt} - \delta\frac{dc}{dt} \tag{25a}$$

$$\dot{k}\left\{\frac{\frac{d}{dt}u'(c)}{u'(c)}\right\} = -[f'(k) - \gamma - \frac{dc}{dk}]\dot{k} - \delta\frac{dc}{dt} \tag{25b}$$

$$-\frac{\frac{d}{dt}u'(c)}{u'(c)} = f'(k) - \gamma - (1 - \delta)\frac{dc}{dk} \tag{25c}$$

so that marginal utility of consumption may "level off" *before* $f'(k)$ is driven down to γ, unlike the Ramsey case where $\delta = 1$.

The puzzle is that $\bar{k}$ cannot apparently be made determinate. I am not raising any question of whether any given "solution," a $c(k)$ function, produces just one "stable" $\bar{k}$.[6] The question is whether there is just one such $c(k)$ function or many.

[6] In fact, the uniqueness of admissible $\bar{k}$ points at which any continuous $c(k)$ function crosses from below the $g(k)$ function appears to be deducible. Suppose that in addition to one "stable" $\bar{k}$ there exists at least one other intersection point. Then at least one of these must be an unstable value, $k\dagger$, at which $c(k^\dagger) = g(k^\dagger)$ and $c'(k^\dagger) < g'(k^\dagger)$. Equation (17) indicates that, for $c(k)$ continuously differentiable, the limit as $k \to k^\dagger$ of $c'(k)$ is $g'(k^\dagger)/(1 - \delta)$ since $g = c$ in the limit. Therefore $g'(k^\dagger) > 0$ is impossible; any unstable intersection point must lie on the falling part of the $g(k)$ function. Consequently, it is impossible that there should exist a pair of stable $\bar{k}$ values—each one necessarily on the rising part of the $g(k)$ curve—for they would have to bracket a nonexistent unstable $k\dagger$ on the rising part of the $g(k)$ curve.

There seems to be no obstacle to finding *another* $c(k)$ function that is anchored to a different $\bar{k}$ in the lower quadrant of Figure 1 such that the upper quadrant's constructions and given u function will generate it and thus validate it. The algebra appears to show that *any* assumed $\bar{k}$, $0 < \bar{k} \leqslant \hat{k}$, will generate a $c(k)$ function that will intersect the g curve at $\bar{k}$ with a slope, by virtue of (17), that equals the required slope given by (24). Choose any value of capital intensity smaller than the Golden Rule level, say k°. If $\bar{c}$ is assigned the value $g(k^{\circ})$, then (17) makes $c(k) = g(k)$ and $dc/dg = 1/(1 - \delta)$ in the limit as $k \to k^{\circ}$. Hence any value of $k < \hat{k}$ can constitute the $\bar{k}$ anchor for a $c(k)$ function calculable from u and g.[7]

To obtain a $c(k)$ function at all, therefore, it appears necessary in the present model that each generation assign to its successors the same expectation of $\bar{k}$ that it expects itself. But such an assignment is apparently arbitrary within limits and there is no reason why any arbitrary expectation should be shared.

"Variable proportions" itself has nothing to do with this conclusion. Even if $g'(k)$ were a positive constant up to $\hat{k}$, more than one $c(k)$ anchored on different $\bar{k}$ cannot apparently be ruled out. In this case we know simply that $c'(\bar{k})$ is the same number for every admissible $\bar{k}$.

If this conclusion of indeterminacy is correct, how does the Phelps-Pollak paper obtain determinateness? In that model, $g(k)$ is unbounded. Utility satiation was postulated in the only case where zero time preference was admissible. This satiation could not be realized in any steady state with finite k; the satiety rate of utility was approached asymptotically. This is a known quantity (in the model) and it played the role of $u(\bar{c})$ in the new model just discussed. Knowledge of the asymptotic $u(t)$ permits $c(k)$ to be uniquely determined. But how is asymptotic utility saturation deduced? I now examine a continuous-time model more like the Phelps-Pollak model in order that this question of the determinacy of game-equilibrium growth paths can be more broadly understood and the results here can be reconciled with the Phelps-Pollak conclusions.

II. UNBOUNDED SUSTAINABLE CONSUMPTION PER CAPITA AND BOUNDED RATE OF UTILITY

We now modify (3a), specifying that $f'(k)$ is bounded above γ, so that g is monotone increasing and unbounded:

$$g(0) \geqslant 0, \quad g'(k) > 0, \quad g''(k) \leqslant 0, \quad g(\infty) = \infty. \tag{3a}$$

See Figure 2.

[7] As for the admissibility of $\hat{k}$, this is possible with $c'(\hat{k}) = g'(\hat{k}) = 0$. Applying L'Hôpital's rule and (17) we have $dc/dg = c''(k)/g''(k) = 1/(1 - \delta)$.

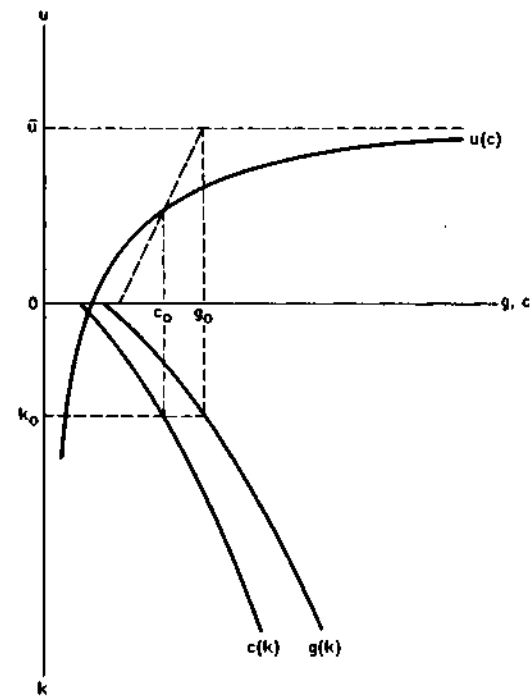

Figure 2. Bounded Instantaneous Utility Rate and Unbounded Sustainable Consumption Per Head.

The instantaneous rate of utility is now bounded, the upper bound, $\bar{u}$, being reached only in the limit as c goes to infinity. This is also illustrated in Figure 2.

$$u(0) = -\infty, \quad u'(c) > 0, \quad u''(c) < 0, \quad u(\infty) = \bar{u} < \infty. \tag{4a'}$$

(This specification entails that the marginal utility of per capita consumption be at least asymptotically elastic with respect to per capita comsumption.)

First, let us postulate that each generation takes it for granted that all future generations will find their optimal game strategy to call for increasing $k(t)$ at a non-vanishing rate. Then the asymptotic rate of utility is the satiety rate, $\bar{u}$. Using the overtaking principle, they will introduce the "subtractor," $\bar{u}$, to compare the respective total utilities of per capita consumption streams that drive c to infinity. A generation living for an instant at time b will calculate the marginal utility of bequeathed capital to be

$$V_b'(k(b)) = \delta \frac{\partial}{\partial k(b)} \int_b^\infty [u(c(k(t))) - \bar{u}] \; dt$$
$$= \frac{-\delta [u(c(k(b))) - \bar{u}]}{\dot{k}(b)} \tag{26}$$

On this postulate, the postulates of utility maximization and of a continuous $c(k)$ function, we find that the relation

$$u'(c(t)) = \frac{-\delta [u(c(t)) - \bar{u}]}{g(k(t)) - c(t)} \tag{27}$$

must hold along the game-equilibrium path. In fact, (27) determines a unique $c(t)$ path and unique $c(k)$ function, with k, $c(k)$ and $g(k)$ all monotonically going to infinity. This solution is illustrated in Figure 2.

The Phelps-Pollak paper utilized the particular class of utility-rate functions of the form

$$u(c) = \bar{u} - c^{\eta+1}, \quad \eta < -1 \tag{28}$$

where

$$\eta = \frac{u''(c)c}{u'(c)}$$

is the (negative) elasticity of the marginal utility function, $u'(c)$. Substitution of this function for $u(c)$ in (27) yields

$$\frac{\dot{k}}{g(k)} = \frac{\delta}{-\eta - (1-\delta)}$$
$$\frac{c(k)}{g(k)} = \frac{-(\eta + 1)}{-(\eta + 1) + \delta} \tag{29}$$

In the Ramsey case where $\delta = 1$ —he actually constructed an example using (28) with the further assumptions that $\gamma = 0$ and $f' = r = \text{const.} > 0$ —there is per capita growth by virtue of the essential restriction $\eta < -1$. With $\delta < 1$, growth of capital per head is slower. In either case, there is constancy of a kind of "saving ratio" though the ordinary saving-income ratio will be constant only if $f(k)$ is proportional to $g(k)$.

One can liken the Phelps-Pollak model to the present one if labor is made inessential and constant over time and if $g(k)$ is made proportional to k. Hence the solution of that model made consumption proportional to capital or to "income." It was postulated there that each generation expects future generations to adopt a *common* "saving ratio." Each generation therefore asked: What common future saving ratio would cause it to wish to choose the same saving ratio? The answering saving ratio, shown to be unique, was the game-equilibrium solution in that paper.

The assumption that future generations will follow a linear-homogeneous comsumption policy has some merit, for if future people were assumed so to behave then every present generation would find its optimal consumption function likewise to be linear homogeneous when the utility rate takes the homogeneous form in (28). This was one way of imagining that the present generation could break the Gordian knot of indeterminacy among game-equilibrium paths. The assumption earlier in this section that each generation expects future generations to accumulate capital per head without bound is another way. While the assumed equality of future saving ratios is a commendably natural assumption—and not demonstrably false with the utility function in (28)—it is not logically necessary. Neither is the weaker and equally natural assumption of unbounded capital growth.

It will be indicated now that the expectation of inhomogeneous future consumption behavior can induce actual inhomogeneous consumption policies along a game-equilibrium path.

Let every generation expect that $c(k) = g(k)$ at just one k with $c'(k) > g'(k)$. Then we are back to the problem of Part I of this paper. If such a $\bar{k}$ is to be utility maximizing for the present generation to maintain when $k_0 = \bar{k}$, it need satisfy only

$$g'(\bar{k}) = (1 - \delta)c'(\bar{k}) \tag{30}$$

The utility-maximization condition governing $c(k)$ states that

$$u'(c) = -\delta \frac{[u(c) - u(\bar{c})]}{g(k) - c}, \qquad \bar{c} = g(\bar{k}) \tag{31}$$

whence

$$c'(k) = \frac{u'(c)g'(k)}{u'(c)(1 - \delta) - u''(c)(g - c)} \tag{32}$$

and

$$\lim_{k \to \bar{k}} c'(k) = \frac{g'(\bar{k})}{1 - \delta} \tag{33}$$

These conditions cannot invalidate any assumed $\bar{k}$. To illustrate this, the reader can simply introduce a construction like that of Figure 1 onto the u and g functions of Figure 2.

III. CONCLUSIONS

If sustainable consumption per head is unbounded, the common assumption that consumption per head will grow without bound offers a natural means to anchor a game-theoretic consumption function. But the common assumption that consumption per head will instead converge to some finite value also admits the calculation of a game-equilibrium path. If steady-state consumption per head is bounded in the manner of the Golden Rule model, one has to know the capital intensity level towards which the economy is commonly assumed to converge. In both types of models, the asymptotic capital intensity remains a parameter, undetermined by the model, at best arbitrary only within limits. It does not appear that economics alone can completely determine the game-equilibrium path. What, then, can (if anything can)?

I suggest that, in otherwise indeterminate situations like this, there may develop an "ethic" that specifies some obligations that each generation is expected to meet. By telling each generation what to expect of other generations, morals may make determinate the altruistic behavior of each generation.

Public morals may be grounded in some underlying ethical axioms that express what the society considers just in relationships between persons and between generations. I shall give an example of that. But it is also possible that the role of morals may be filled more primitively by a myth that recounts the evil consequences to a society that would depart from some traditional pattern of behavior. The myth of the disaster that befalls the society that indulges in deficit spending is an example within the present context of economic growth. The deficit taboo encourages each generation to believe that the capital bequest its balanced budget would produce will not be dissipated to some unknown degree at some future time. In this case, intergenerational capital accumulation could be viewed not as the economist's game equilibrium but rather as the sociologist's ritual equilibrium.

In some intertemporal choice problems, however, ethics might operate instead of a taboo. In the problem of national capital accumulation modelled here, it is conceivable that the society's ethics (together with its technology and utility functions) would permit each generation to deduce the asymptotic capital intensity and thus serve to anchor the game-equilibrium path. Under the Paretian ethic—where any change that is preferred by at least one person is counted a social gain if it is not opposed by the others—each generation would presumably anticipate the economy's approach to the Golden Rule state because each prefers that equilibrium path to the other ones. Then the game-equilibrium growth path, shaped by the partially selfish preferences of each passing generation, would indeed approach the Golden Rule state.

Appendix

These notes are intended to provide a more rigorous basis, or at any rate a fancier one, for the results obtained in the text and also to extend those results to the case of exponential "myopia." The pertinent scripture, freely adapted here, are the imperishable pp. 263–64 of R. E. Bellman's *Dynamic Programming* (Princeton, 1957).

Let us define

$$f(b, m, \Delta) = \max_{c[b,\, b+\Delta]} \left\{ \int_b^{b+\Delta} e^{-\rho t} [u(c(t)) - \delta u(\bar{c})] dt + V(b + \Delta, k(b + \Delta)) \right\} \tag{A.1}$$

subject to

$$k(b) = m \tag{A.2}$$

and

$$\dot{k}(t) = G(k(t),\ c(t)) = g(k(t)) - c(t) \tag{A.3}$$

where

$$V(b + \Delta, k(b + \Delta)) = \delta \int_{b+\Delta}^{\infty} e^{-\rho t} [u(c(k(t))) - u(\bar{c})] dt \tag{A.4}$$

$$= \delta e^{-\rho(b+\Delta)} \int_0^{\infty} e^{-\rho s} [u(c(k(b + \Delta + s))) - u(\bar{c})] ds$$

We are interested in the limit as $\Delta \to 0$. The subtractor, $\delta u\ (\bar{c})$, is gratuitous if $\rho > 0$ but implements the overtaking principle if $\rho = 0$.

Letting c denote the average consumption rate around time b, we have, for small Δ,

$$f(b, m, \Delta) \cong \underset{c}{\text{Max}} \left\{[u(c) - \delta u(\bar{c})]\, e^{-\rho b}\, \Delta + V(b, m) + V_b(b, m)\Delta + V_m(b, m) G(m, c)\Delta\right\} \tag{A.5}$$

or, since $V(b, m)$ is independent of c, given b and m,

$$f(b, m, \Delta) - V(b, m) = \underset{c}{\text{Max}}\ [u(c) - \delta u(\bar{c})]\, e^{-\rho b}\, \Delta + V_b(b, m)\Delta + V_m(b, m) G(m, c)\Delta\Big\} \tag{A.6}$$

From (A.4) it is clear that $f(b, m, \Delta) - V(b, m)$ is not a variable but depends only upon b, m and Δ through the unknown function $c(m)$. We can write, by virtue of (A.1) and (A.4) the relation, for small Δ,

$$f(b, m, \Delta) - V(b, m) \cong u(c(m))(1 - \delta) e^{-\rho b}\, \Delta \tag{A.7}$$

Upon substituting (A.7) into (A.6), dividing both sides by Δ and letting $\Delta \to 0$, we obtain the equation

$$u(c(m)(1 - \delta)e^{-\rho b} = \underset{c}{\text{Max}} \left\{[u(c) - \delta u(\bar{c})]\, e^{-\rho b} + V_b(b, m) + V_m(b, m)\, G\,(m, c)\right\} \tag{A.8}$$

Note that, by (A.4),

$$V(b, m) = e^{-\rho b}\ V(0, m) \tag{A.9}$$

$$V_m(b, m) = e^{-\rho b}\ V_m(0, m) \tag{A.10}$$

$$V_b(b, m) = -\rho V(b, m), \text{ each for every } m. \tag{A.11}$$

Maximization of the righthand side of (A.8) with respect to c thus entails, at the maximum,

$$u'(c) e^{-\rho b} + V_m\ (b, m) G_c(m, c) = u'(c) - V_m(0, k) = 0 \tag{A.12}$$

The second order condition for a maximum, given m and V_m, is

$$u''(c) \leqslant 0 \tag{A.13}$$

At any time b, therefore, the currently game-optimal consumption rate, c, must satisfy

$$u(c)(1-\delta) = u(c) - \delta u(\bar{c}) + V_b(0, m) + V_m(0, m)G(m, c) \quad \text{(A.14)}$$

or, using (A.11) and (A.12),

$$\frac{\rho V(0, m) - \delta[u(c) - u(\bar{c})]}{G(m, c)} = u'(c) \quad \text{(A.15)}$$

Implicit differentiation of (A.15) yields

$$\frac{dc}{dm} = \frac{\rho V_m(0, m) - u'(c)g'(m)}{-(1-\delta)u'(c) + u''(c)[g(m) - c]} \quad \text{(A.16)}$$

$$= \frac{u'(c)[g'(m) - \rho]}{(1-\delta)u'(c) - u''(c)(g - c)} \qquad \text{by (A.12)}$$

This differs from (17) and (32) of the text only in that the excess, $g'(m)$, of the social rate of return over the Golden Rule rate of return is replaced by the excess of the rate of return over the so-called Golden Utility rate of return whose value is $\gamma + \rho$. It follows that, when $\rho > 0$, the Golden Utility capital intensity, rather than the Golden Rule capital intensity, places the upper bound upon the set of admissible values of $\check{k}$.

RAWLSIAN GROWTH: DYNAMIC PROGRAMMING OF CAPITAL AND WEALTH FOR INTERGENERATION 'MAXIMIN' JUSTICE

The volume of national saving seems to have been governed, in most countries if not all, by fiscal myth rather than an understanding over intergenerational justice. It used to be believed that public borrowing would lead to further borrowing in a process terminating in ruin. That dogma worked like a charm, warding off the carnal appetite for larger consumption now. Though attitudes toward public debt are today more permissive, the contemporary myth that large deficits are intrinsically inflationary keeps a lid on the growth of real government indebtedness.

Nevertheless the justice (or injustice) of increasing the public debt has long been a topic of academic discussion. Ricardo held that deficits set back the growth of capital and that was reason enough (for him) to oppose them.

In modern economics the traditional standard of justice is undoubtedly utilitarianism. The Ramsey-Weizsäcker model of utilitarian accumulation closely resembles Fisher's theory of household saving under zero time preference. If initial capital is short of the capital-saturation Golden Rule path, the normal case, the best available path of *equalized* utilities would be a stationary state that leaves positive the social rate of return to investment. But then the sacrifice of a util by the present generation could (be made to) yield an increment of more than a util to any future generation—a utilitarian gain. It follows that, for an optimum, there must be saving, generation after generation, to drive the capital stock toward the Golden Rule level. Under " typical " conditions, it may be added, the necessary consumption restraint will require the austerity of budgetary surpluses—the initial shortfall of capital being due in some part to the " displacement effect " or " burden " of past deficits or insufficient surpluses.[1]

Yet it must be a bit startling, on first encounter with the utilitarian doctrine, to see the saving by a multi-period household become an allegory of the proper accumulation by a multi-generation society. Fisherine households do not sacrifice enjoyment after all, they only postpone enjoyment for the sake of larger lifetime enjoyment. Why should a generation have to sacrifice some (lifetime) enjoyment for the sake of any generation no less fortunate merely if the investment would pay (to its beneficiary) a positive rate of return? The present generation might well complain that it was being asked to suffer for the sheer accident of its place in the chronological ordering:

> The time is out of joint; O cursed spite,
> That ever I was born to set it right!

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In *A Theory of Justice*, John Rawls has presented the first radical[2] alternative to utilitarianism to have appeared in several decades—the principle of " maximin ".[3] While Rawls himself drew back (for a mistaken reason)[4] from such an application, the principle seems especially inviting as a conception of *intergenerational* justice. A just society would so programme its taxes and resulting stocks of capital and national debt as to maximize the lifetime utility of the generation (or generations) having least utility. With this paper we investigate some of the properties of " maximin " growth.

In the prevailing wisdom, of course, " maximin " growth is no growth at all.[5] The reasoning is that, intergeneration externalities absent, the " maximin " path (where it exists) must equalize generation utilities at the largest available level. If each generation's utility depends only upon its own " consumption ", then both income and consumption must also be equal—constant from generation to generation.[6] Our results, however, refute the notion that " maximin " is generally " anti-growth ". Yet they also free society from any forced march to the Golden Rule state.

The key feature of the model studied here is its overlapping generations. Each generation is egoistic, working and accumulating tangible capital for its own life-cycle purposes—subject to its prior allegiance to " maximin " justice. Nevertheless the self-interest of every generation is itself intergenerational because its desire to consume will span its period of retirement when the next generation will provide the economy's labour supply. We show that, for every initial condition inside some domain, there exists a " maximin " solution characterized by a unique sequence of intergeneration " trades ". A generation that adds (say) to the capital stock receives in return a moral claim to additional old-age consumption. A generation receiving added capital accepts an obligation to work more. It is further shown how the " maximin " allocation can be " supported " by institutions of private wealth owning, perfect markets, public grants and public debt. If a generation (optimally) adds to the capital stock it is entitled to issue *more* public debt to itself so that it and future generations will benefit equally.

In this model the " maximin " solution does indeed equalize utilities as the utilitarians contend. But not even that property survives once certain externalities are accorded their place. Rawls, perhaps taking the utilitarian objection too much to heart, suggests that " ties of sentiment " ensure that a generation would like to improve the opportunities of its successors if some satisfactory level of development has not been reached. Yet a principle of justice is still needed to mediate differences in desires.[7] We show in the last section that Rawls's sentiments, when imbedded into our " maximin " framework, can indeed lead to a growth-path of rising utility.

1. FORMULATION AS A DYNAMIC PROGRAMMING PROBLEM

At the beginning of each period a new generation of identical individuals is born into the economy. Each generation can work in its first period and can consume at the end of its first and second period. All generations are alike in size, tastes and technology. Whether they will have identical endowments of capital and obligations to the old, of course, is a matter to be determined.

Consider the situation of the tth generation born under justice, $t = 1, 2, \ldots$. It has available for use in current period production a stock of capital, k_{t-1}, left over by the previous generation (now old). It faces a predetermined claim by the latter for second period consumption x_{t-1}. The two-dimensional description of the state in terms of (k_{t-1}, x_{t-1}) reflects the fact that two generations, young and old, co-exist in period t.

Among the current variables to be determined in each period t are the fraction of the period in which the young are to work, l_t, and that portion of the resulting gross output, $F(k_{t-1}, l_t)$, the young are to consume, c_t.

$$k_t = F(k_{t-1}, l_t) - c_t - x_{t-1}; \quad c_t, k_t \geqq 0 \quad 0 \leqq l_t \leqq 1. \qquad \ldots(1.1)$$

In (1.1) all variables are in *per capita* form.

The production function F is assumed linear homogeneous, concave and twice differentiable, with first derivatives $F_k(k, l)$ and $F_l(k, l)$ positive everywhere. For every l there is some $\bar{k}(l)>0$ beyond which the gross marginal product of capital is less than or equal to 1. Also $F_k(k, l)\to\infty$ as $k\to 0$. Finally we suppose $F(0, l) = F(k, 0) = 0$.

Each generation's preferences are " identical " and " egoistic ". They are represented by an ordinal utility function which is (functionally) independent of t and in which only the generation's own experiences figure:

$$U_t = U(c_t, x_t, l_t). \qquad \ldots(1.2)$$

The utility function U is strictly quasi-concave and twice differentiable with derivatives $U_c(c, x, l)>0$, $U_x(c, x, l)>0$ and $U_l(c, x, l)<0$ everywhere. Whenever it is desired to avoid corner solutions it will be assumed that $U(\cdot)\to -\infty$ as either c or x or " leisure ", $1-l$, goes to zero.

Associated with each allocation $\{c_t, x_t, l_t \mid t = 1, 2, \ldots\}$ is a corresponding sequence of intergenerationally commensurate ordinal utilities $\{U_1, U_2, \ldots\}$. Such an allocation is feasible if the implied $(k_t, x_t) \geqq 0$ for all $t = 1, 2, \ldots$, given the initial state (k_0, x_0). Our problem is, loosely, to find from the feasible allocations one that makes the smallest of the utilities as large as possible.[8]

We adopt the infinite time horizon in order to maintain the time-independence (or stationarity) of the optimization problem from generation to generation. The resulting analytical gain comes, however, at the cost of a difficulty over the existence of a maximin path, at least for some subset of initial states.

Initially, we consider the problem of maximizing the infimum of utility levels over all periods beginning with the tth. This " max-inf " problem is to find the path or paths from some predetermined state (k_{t-1}, x_{t-1}) which yield an infimum $m(k_{t-1}, x_{t-1})$ satisfying:

$$m(k_{t-1}, x_{t-1}) = \max_{\{c_\tau, x_\tau, l_\tau, k_\tau\}} \left[\inf_{\tau \geqq t} U(c_\tau, x_\tau, l_\tau)\right]$$

$$\text{s.t. } k_\tau = F(k_{\tau-1}, l_\tau)-c_\tau-x_{\tau-1} \geqq 0.$$

We begin by establishing the existence of $m(k_{t-1}, x_{t-1})$. Since F is bounded, the utility of the first generation is bounded from above. Then the infimum of any feasible utility stream is certainly bounded and there must exist some least upper bound $s(k_{t-1}, x_{t-1})$. A property of the infimum function is

$$\inf_{\tau \geqq t} U(c_\tau, x_\tau, l_\tau) = \min\left[U(c_t, x_t, l_t), \inf_{\tau \geqq t+1} U(c_\tau, x_\tau, l_\tau)\right].$$

Therefore the least upper bound, $s(k_{t-1}, x_{t-1})$, can be described by the typical functional equation of dynamic programming, that is:

$$s(k_{t-1}, x_{t-1}) = \sup_{\{c_t, x_t, l_t\}} \{\min\left[U(c_t, x_t, l_t), s(k_t, x_t)\right]\}$$
$$\text{s.t. } k_t = F(k_{t-1}, l_t)-c_t-x_{t-1} \geqq 0. \qquad \ldots(1.3)$$

In the appendix it is demonstrated that $s(k, x)$ is a continuous function of k and x. It then follows immediately from (1.3) that the least upper bound is attained, that is, $s(k_{t-1}, x_{t-1})$ is the maximized infimum $m(k_{t-1}, x_{t-1})$.

Before examining the form taken by $m(k, x)$ it is useful to consider the alternatives available to generation t given a prior decision to leave a capital stock of k_t and to consume x_t in old age. Consumption when young and labour supply must be in the set bounded by (1.1). The best the tth generation can then do for itself is to achieve the utility level

$$W(k_t, x_t \mid k_{t-1}, x_{t-1}) = \max_{l_t} U(F(k_{t-1}, l_t)-x_{t-1}-k_t, x_t, l_t). \qquad \ldots(1.4)$$

Since U is strictly increasing in c_t and x_t, the conditional utility function W is strictly increasing in x_t and strictly decreasing in k_t. Furthermore, given the concavity of F and strict quasi-concavity of U, it is a straightforward matter to check that $W(k_t, x_t)$ is a strictly quasi-concave function.

The functional equation (1.3) can then be rewritten as:

$$m(k_{t-1}, x_{t-1}) = \max_{k_t} \{\min_{x_t} [W(k_t, x_t \mid k_{t-1}, x_{t-1}), m(k_t, x_t)]\} \qquad ...(1.5)$$

We next prove that m is semi-strictly quasi-concave. That is, if two initial states yield different returns, m' and m'' with $m'' > m'$, any convex combination of these two states yields a return which is greater than m'.

Corresponding to the initial state (k_0', x_0') is an optimal sequence of vectors

$$\{c_t', x_t', k_t', l_t' \mid t = 1, 2, ...\}$$

such that

$$U(c_t', x_t', l_t') \geqq m'. \qquad ...(1.6)$$

Similarly for (k_0'', x_0'') there is an optimal sequence such that

$$U(c_t'', x_t'', l_t'') \geqq m'' > m'. \qquad ...(1.7)$$

Next consider the initial state

$$(vk_0' + (1-v)k_0'', vx_0' + (1-v)x_0''), \quad 0 < v < 1.$$

Since the production function is assumed to be concave the sequence of vectors

$$\{(vc_t' + (1-v)c_t'', vx_t' + (1-v)x_t'', vl_t' + (1-v)l_t'', vk_t' + (1-v)k_t'') \mid t \geqq 1\}$$

is certainly feasible. Moreover, from the appendix, the feasible vectors $q_t' = (c_t', x_t', l_t')$ and $q_t'' = (c_t'', x_t'', l_t'')$ are bounded from above by some vector $\bar{q}$

$$\text{i.e.} \quad q_t' \in L' = \{q \mid U(q) \geqq m', q \leqq \bar{q}\}$$

$$q_t'' \in L'' = \{q \mid U(q) \geqq m'', q \leqq \bar{q}\}.$$

But L' and L'' are compact, hence for all v the continuous function

$$f(q', q'') = U(vq' + (1-v)q'') \quad \text{defined on} \quad L' \times L''$$

achieves its minimum on $L' \times L''$.

That is, for all v, $0 < v < 1$ and for all $(q', q'') \in L' \times L''$ there exists some vector $(\bar{q}', \bar{q}'') \in L' \times L''$ s.t.

$$U(vq' + (1-v)q'') \geqq U(v\bar{q}' + (1-v)\bar{q}'')$$
$$> m',$$

where the strict inequality follows from the definitions of L' and L'' and the assumption that U is strictly quasi-concave.

In particular, this strict inequality must be true for the feasible vectors q_t', q_t'' defined above. Then,

$$m(vk_0' + (1-v)k_0'', vx_0' + (1-v)x_0'') \geqq \inf_t \{U(vq_t' + (1-v)q_t'')\}$$
$$> m'. \quad \|$$

Summarizing, we have:

Lemma 1.1. *The return function $m(k, x)$ is semi-strictly quasi-concave.*

This Lemma and the continuity of $m(k, x)$ imply that the latter is also quasi-concave.

Given the infinite time horizon, a high initial capital stock can only benefit all generations by a finite amount if it can be used productively. Then the assumption of a bounded production set implies that no matter how favourable the initial conditions, there exists some greatest value[9] of the return function m^G. We denote the smallest initial capital stock associated with an initial x_0 that achieves m^G by $k^G(x_0)$. From Lemma 1.1 the latter is a convex function and hence " bends forward " as shown in Figure 1.

By definition, $m(k, x) < m^G$ in the domain lying to the left of this curve. Then from Lemma 1.1, $m(k, x)$ is strictly increasing in k and decreasing in x over this domain. It follows that through any point $(\hat{k}_0, \hat{x}_0)$ with $\hat{k}_0 < k^G(\hat{x}_0)$ there is an iso-return contour

$$m(k, x) = m(\hat{k}_0, \hat{x}_0).$$

Given Lemma 1.1, this also bends forward as depicted in Figure 1.

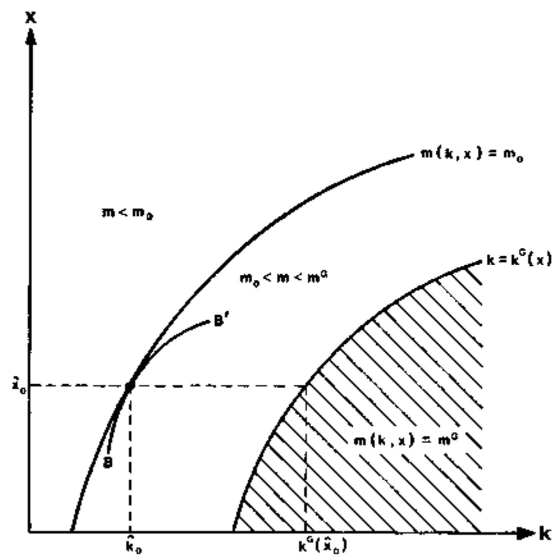

FIGURE 1
Iso-return contours

Suppose that given an initial $(\hat{k}_0, \hat{x}_0)$, the optimal allocation is $\{c_t^*, x_t^*, l_t^*, k_t^* \mid t \geq 1\}$. From (1.1) any alternative initial state (k_0, x_0) satisfying

$$x_0 = F(k_0, l_1^*) - c_1^* - k_1^* \qquad \ldots(1.8)$$

can also achieve the optimal allocation. Then for such points $m(k_0, x_0)$ is at least as great as $m(\hat{k}_0, \hat{x}_0)$. The curve defined by (1.8), depicted in Figure 1 as BB', must therefore lie below the iso-return contour $m = m(\hat{k}_0, \hat{x}_0)$. But such curves can be drawn for any point on the iso-return contour, therefore the latter can be thought of as the envelope of the BB' curves through various points (k_0, x_0). Then the iso-return contour must also be differentiable because it is quasi-concave and because it envelops a differentiable curve at each point. We therefore have:

Lemma 1.2. *Every iso-return contour $m(k, x) = m(k_t^*, x_t^*)$ is differentiable with slope at (k_t^*, x_t^*) equal to the gross marginal product of capital in period $(t+1)$.*

We now demonstrate that max-inf growth is both " maximin " and egalitarian for all initial states (k_0, x_0) with $k_0 < k^G(x_0)$.

Theorem 1.1. *If the initial state (k_0, x_0) satisfies $k_0 < k^G(x_0)$ the utility level attained by every generation is equal to the maximized infimum $m(k_0, x_0)$.*

The proof is by contradiction. Suppose that $m(k_\tau^*, x_\tau^*)$ were to exceed $m(k_0, x_0)$ for the first time in period τ. Since m is continuous it would then be possible to reduce the capital stock in period τ by some finite amount δ such that $m(k_\tau^* - \delta, x_\tau^*) > m_0$. This in turn could be achieved by reducing savings in each of the previous periods by δ/τ and allowing each generation to consume more and hence achieve a higher utility level. But such a conclusion contradicts the definition of m_0 as the maximized infimum.

An almost identical argument can be made contradicting the assumption that for some period $U_\tau > m(k_0, x_0)$. ‖

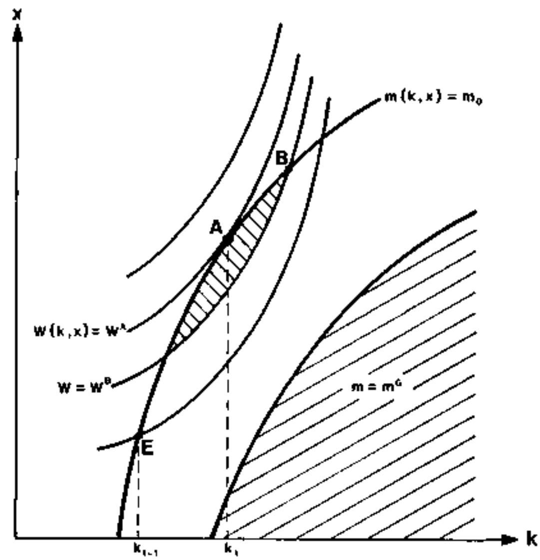

FIGURE 2
Optimal capital accumulation

We are now in a position to show that for all $k_0 < k^G(x_0)$ the optimal path is unique. Suppose that at time $t-1$ the economy is in the state represented by the point E depicted in Figure 2. Since $W(k_t, x_t \mid k_{t-1}, x_{t-1})$ is strictly quasi-concave there is one member of the family of W-indifference curves which touches $m(k, x) = m_0$ at a unique point A.

From Theorem 1.1 we know that (k_t^*, x_t^*) lies somewhere on $m(k, x) = m_0$. Suppose then that (k_t^*, x_t^*) is some point B different from A. Theorem 1.1 implies that

$$W(k_t^B, x_t^B) = m_0.$$

Since $W(k, x)$ is strictly quasi-concave and decreasing in k, and $m(k, x)$ is quasi-concave and strictly increasing in k (in the neighbourhood of $m = m_0$), there exists a non-empty set $\{(k, x) \mid W(k, x) > m_0, m(k, x) > m_0\}$. This is the heavily shaded region in Figure 2. But the existence of such a set contradicts Theorem 1.1. Then the optimal choice in period t must be the point A. We have therefore proved:

Theorem 1.2. *For all $k_0 < k^G(x_0)$ the optimal path is unique.*

Theorems 1.1 and 1.2 together imply that each stage in the dynamic programming problem can be represented in the following alternative form.

$$\max_{k_t, x_t} \{W(k_t, x_t \mid k_{t-1}, x_{t-1}) \mid m(k_t, x_t)\} = m(k_{t-1}, x_{t-1})\}.$$

Whether or not (k_t, x_t) is greater or less than (k_{t-1}, x_{t-1}) depends simply upon whether the slope of the W-indifference contour at (k_{t-1}, x_{t-1}) is greater or less than the slope of $m = m(k_{t-1}, x_{t-1})$.

Consider some point where the slopes are unequal. Without loss of generality suppose the slope of the W-indifference contour is less steep, as at the point E in Figure 2. Since W is a differentiable function the slope of the indifference contour varies continuously as the initial state (k_{t-1}, x_{t-1}) is moved from E, around $m = m_0$ in the direction of B. Either there is some point where the slopes are equal or the slope of the indifference contour through the initial points remains less steep. If the latter, it is implied that there exists δ such that $k^*_{t+1} > k^*_t + \delta$ for all future periods, contradicting our assumption of a bounded production set. We have therefore proved that there must be some point on the iso-return contour at which the slopes are equal. This result is summarized in the following Lemma:

Lemma 1.3. *For any " maximin " utility level less than m^G, there is at least one " stationary " initial state which is optimal for all t.*

We can now clarify the nature of the boundary curve $m = m^G$. Since the production set is bounded, there exists a finite " Golden Rule " state (k^G, x^G) which is maximal over all *stationary* states. From Lemma 1.3 we know that there is at least one stationary state on every iso-return contour. Therefore the maximal value of the return function m^G is in fact the *Golden Rule* utility level. The boundary curve $m = m^G$ is then a locus of " as-good-as-golden " initial states.

For initial states to the right of this locus, Theorem 1.1 no longer applies. Clearly, unless there are costs of disposal, it is always possible to jettison $k_0 - k^G(x_0)$ units of capital and move directly to the as-good-as-golden locus. Alternatively, half of the jettisoned capital can be consumed and half invested, yielding a higher total output in the following period. Continuing this process indefinitely yields (since $F_k > 0$) a sequence of utility levels all strictly greater than m^G. But from the above discussion the infimum of this sequence is m^G, therefore we have a sequence with no minimum utility which is strictly preferred by all generations to the sequence $\{U_t\} = \{m^G\}$. Combining these results we have:

Theorem 1.3. *Given free disposal and an initial state (k_0, x_0) such that $k_0 > k^G(x_0)$, the maximized infimum is the Golden Rule utility level m^G. There is no " maximin " solution for such states.*

For the remainder of the paper we focus on initial states strictly inferior to the as-good-as-golden states where $k_0 < k^G(x_0)$. We begin by further exploring the features of maximin growth paths.

2. CHARACTERIZATION OF THE OPTIMAL PATH

In the previous section it was shown that whether or not it is optimal to increase the capital stock in period t depends upon the relative slopes of the present generation's indifference contour and the iso-return contour. For example, in Figure 2 the latter is steeper at E, hence growth is optimal.

It is convenient at this point to introduce an alternative description of the iso-return contour through (k_0, x_0). Given our results on the form of such a contour we can express the relation between x and k as $x = x_0(k)$. Then from (1.8) the decision for generation t reduces to

$$\max_{k_t} W(k_t, x_0(k_t) | k_{t-1}, x_0(k_{t-1})).$$

From (1.4) this in turn can be rewritten as

$$W^* = \max_{k_t, l_t} U(F(k_{t-1}, l_t) - x_0(k_{t-1}) - k_t, x_0(k_t), l_t).$$

All the functions are differentiable and our assumptions preclude corner solutions. Therefore, the following first-order conditions must be satisfied.

$$\partial W^*/\partial k_t = -U_c(t)+U_x(t)x_0'(k_t) = 0 \qquad ...(2.1)$$

$$\partial W^*/\partial l_t = U_c(t)F_l(k_{t-1}, l_t)+U_l(t) = 0. \qquad ...(2.2)$$

From Lemma 1.2, the slope of the iso-return contour at k_t is the optimal gross marginal product of capital in period $(t+1)$. Therefore, the necessary conditions for optimality can be rewritten as:

$$-U_l/U_c = F_l(k_{t-1}, l_t) \quad U_c/U_x = F_k(k_t, l_{t+1}). \qquad ...(2.3)$$

Before examining the general solution we consider the simpler case in which individual labour supply is fixed ($l_t = \bar{l}$). For this (2.1) is the relevant first-order condition. To determine the implications of being further along the contour $x = x_0(k)$ at time $(t-1)$ we differentiate this expression with respect to k_{t-1}.

$$(\partial(U_c/U_x/\partial k_{t-1})[F_k(k_{t-1}, \bar{l})-x_0'(k_{t-1})] = F_{kk}(k_t, \bar{l})(dk_t/dk_{t-1})|_{m=m_0}. \qquad ...(2.4)$$

Again applying Lemma 1.2, the square bracket on the left-hand side of (2.4) is zero so that

$$dk_t/dk_{t-1}|_{m=m_0} = 0.$$

Therefore the capital stock after one period (also t periods) is the same for all initial states yielding a return m_0. Growth then, if it occurs at all, takes place only in the first period.

Summarizing, we have derived:

Theorem 2.1. *If the supply of labour is fixed, the Rawlsian economy reaches a stationary state after one period of adjustment in the state variables k and x.*[10]

This is at first sight a surprising result. Today's young can make an intergeneration " trade " of their consumption when young in return for increased consumption when old. Future generations gain through the increase in the capital stock thereby made available. What we have shown is that all opportunities for intergeneration transfer are exploited in the initial period.

The explanation is that with a fixed labour supply, the total resources made available to generation t (gross output less the debt owed to generation $(t-1)$) becomes a predetermined variable. From (1.1) we have:

$$c_t+k_t \leqq a_t = F(k_{t-1}, \bar{l})-x_{t-1}$$

Thus the " maximin " utility level depends simply on a_t rather than the bivariate initial state (k_{t-1}, x_{t-1}). Since utility opportunities increase with the total available resources, the Rawlsian economy is frozen into the stationary state $\{a_t\} = \{a_1\}$.

We now return to the general case with first-order conditions given by (2.1) and (2.2). From Theorem 1.2, these conditions define a unique maximum hence the following second-order necessary conditions must be satisfied.[11]

$$W_{kk}^*(t)<0, \quad \Delta = \begin{vmatrix} W_{kk}^*(t) & W_{kl}^*(t) \\ W_{lk}^*(t) & W_{ll}^*(t) \end{vmatrix} >0. \qquad ...(2.5)$$

Again we consider the implications of being further along the iso-return contour $x = x_0(k)$ at time $(t-1)$. Differentiating the first-order conditions with respect to k_{t-1} yields

$$\begin{bmatrix} W_{kk}^*(t) & W_{kl}^*(t) \\ W_{lk}^*(t) & W_{ll}^*(t) \end{bmatrix} \begin{bmatrix} dk_t/dk_{t-1} \\ dl_t/dk_{t-1} \end{bmatrix} = -\begin{bmatrix} \partial^2 W^*/\partial k_t \partial k_{t-1} \\ \partial^2 W^*/\partial l_t \partial k_{t-1} \end{bmatrix}, \qquad ...(2.6)$$

where

$$\partial^2 W^*/\partial k_t \partial k_{t-1} = (-U_{cc}(t) + U_{cx}(t)x_0'(k_t))[F_k(t-1) - x_0'(k_{t-1})]$$

and

$$\partial^2 W^*/\partial l_t \partial k_{t-1} = (-U_{cc}(t) + U_{cl}(t))[F_k(t-1) - x_0'(k_{t-1})] + U_c(t)F_k(t-1).$$

But (k_{t-1}, x_{t-1}) lies on the iso-m contour, therefore from Lemma 1.2 the square bracket in these last two expressions is zero. Then applying Cramer's rule (2.6) can be solved as follows:

$$dk_t/dk_{t-1}\,|_{m=m_0} = -U_c(t)F_k(k_{t-1}, l_t)W_{kl}^*(t)/\Delta \qquad \text{...(2.7)}$$

$$dl_t/dk_{t-1}\,|_{m=m_0} = -U_c(t)F_k(k_{t-1}, l_t)W_{kk}^*(t)/\Delta. \qquad \text{...(2.8)}$$

From (2.5) the right-hand side of the latter expression is strictly positive at all points along the iso-return contour. Therefore, if it is optimal for generation t to leave a greater capital stock than the *previous* generation it must be optimal for them to work shorter hours than the *following* generation. More generally we have:

Theorem 2.2. *Along the Rawlsian growth path*

$$k_t \gtreqless k_{t-1} \leftrightarrow l_{t+1} \gtreqless l_t.$$

That is, the two factors of production increase or decrease together.

For the implications of (2.7) it is necessary to examine $\partial^2 W^*/\partial k_t \partial l_t$. Differentiating (2.1) yields:

$$\partial^2 W^*/\partial k_t \partial l_t = (-U_c U_x U_{cl} + U_c^2 U_{xl} + U_l U_x U_{cc} - U_l U_c U_{xc})/U_c U_x, \qquad \text{...(2.9)}$$

where all evaluations are at t. Next consider the following Hicksian expenditure minimization problem

$$\min\,(p_c c + p_x x - wl) \quad \text{s.t. } U(c, x, l) \geqq m_0. \qquad \text{...(2.10)}$$

It is a straightforward matter to verify that the compensated demand functions have the property

$$\partial x/\partial w = -\partial l/\partial p_x = N/D,$$

where D is the usual bordered Hessian determinant, with negative sign since U is strictly quasi-concave, and N is the numerator of (2.9).

Combining these results with (2.9), we therefore have:

$$dk_t/dk_{t-1}\,|_{m=m_0} \gtreqless 0 \leftrightarrow \partial x/\partial w\,|_{\text{comp}} \gtreqless 0 \qquad \text{...(2.11)}$$

Since a compensated increase in w unambiguously increases hours worked, consumption in at least one period must rise. But the longer work time implies that there is less time for consumption when young. Therefore, if there were to be a complement to leisure it would surely be the latter. It is natural then to assume that leisure and consumption when old are Hicksian substitutes ($\partial x/\partial w\,|_{\text{comp}} > 0$).

We then have $dk_t/dk_{t-1} > 0$ at every point along an iso-return contour. It follows immediately that

$$k_{t-1} \gtreqless k_t \leftrightarrow k_t \gtreqless k_{t+1}. \qquad \text{...(2.12)}$$

That is, the optimal path is either stationary or strictly monotonic.[12]

In the previous section it was shown that on every iso-return contour there is at least one stationary state. Then either the economy begins at a stationary state or it must approach it monotonically and asymptotically. All this is summarized in the ensuing theorem.

Theorem 2.3. *Suppose that for all feasible allocations, leisure and old age consumption are Hicksian substitutes. Then for any initial state not on a locus of stationary states, the optimal sequence $\{(k_t, x_t) \mid t = 1, 2, \dots\}$ approaches this locus monotonically and asymptotically.*

The " well behaved " case in which the locus of stationary solutions intersects each iso-return contour only once is depicted in Figure 3.[13]

It is interesting to compare the Rawlsian path with a Ramsey-Weizsäcker or (Koopmans)-Modified-Utilitarian trajectory. It can be shown that the latter must lie below the locus of stationary solutions for all $k < k^{\gamma}$, the asymptotic capital stock. Therefore, if prior to time zero the economy had been on such a trajectory, the initial state (k_0, x_0) is below this locus. It follows that the introduction of the Rawlsian criterion results in growth of the capital stock towards some asymptote S.

Only if the economy *begins* on the stationary state locus, for example at the stationary Cassell-Harrod-Diamond laissez-faire equilibrium will there be no growth.

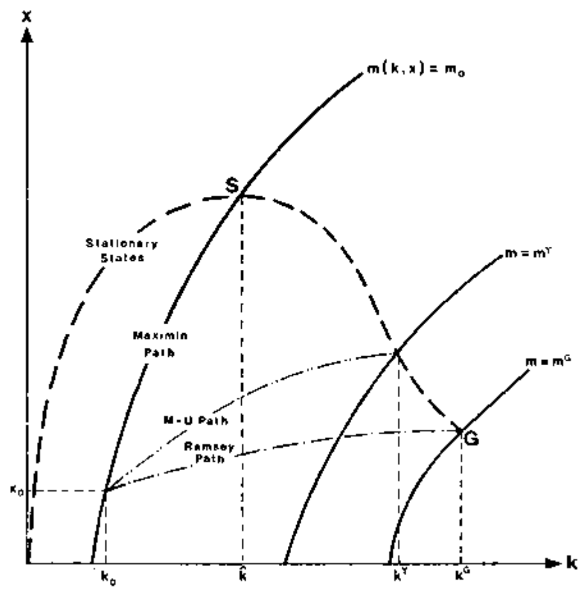

FIGURE 3
Optimal growth paths

3. COMPETITIVE MARKETS AND PUBLIC PLANNING

Having determined the optimal path, how might a planner utilize " perfect markets " plus maximizing behaviour on the part of all agents to implement that goal? From (2.3) we know that for every generation certain marginal rates of substitution and marginal productivities must be equated. Moreover, given the concavity of the production function and strict quasi-concavity of preferences these conditions define a unique solution which can be achieved by the introduction of a sequence of wages $\{\omega_t \mid t = 1, 2, \ldots\}$ and interest rates $\{\rho_t \mid t = 1, 2, \ldots\}$.

Since the production function is assumed homogeneous of degree one, equilibrium profits are zero in every period. Then consumer income is simply the sum of wage income plus demogrants less taxes.

When young, consumers must choose between consumption and saving where the latter is defined by

$$c_t + s_t = \omega_{t-1} l_t + \beta_t^1,$$

ω_{t-1} is the marginal product of labour associated with previously determined capital stock k_{t-1}, and β_t^1 is the net demogrant when young.

In old age, all income is consumed according to

$$x_t = (1+\rho_t)s_t + \beta_t^2. \qquad \ldots(3.1)$$

Since consumption of x_t takes place at the end of period $(t+1)$, $1+\rho_t$ is the marginal product of capital associated with k_t. Combining these two expressions yields the lifetime budget constraint

$$c_t + (1+\rho_t)^{-1}x_t = \omega_{t-1}l_t + \beta_t,$$

where $\beta_t = \beta_t^1 + (1+\rho_t)^{-1}\beta_t^2$ is the discounted value of demogrants received. Because only the choice of β_t is critical we shall consider the case $\beta_t^2 \equiv 0$.

In period $(t+1)$ firms borrow capital k_t and individuals save s_t. For equilibrium in the capital market the central authority must float public debt. Suppose it offers bonds paying one unit of consumption at the end of the present period. Then equilibrium requires that there must be an offering of d_t such bonds with market value v_t where

$$v_t = (1+\rho_t)^{-1}d_t = s_t - k_t. \qquad \ldots(3.2)$$

If the central authority raises receipts (new bond issues plus taxes) sufficient to just offset expenditures (old bond redemptions plus demogrants), Walras' Law automatically ensures equilibrium for consumers. To achieve the former the authority (at the end of period t) supplies a demogrant β_t equal to the difference between total new government borrowing v_t and the current value of the old debt d_{t-1} now due:

$$\beta_t = v_t - d_{t-1}. \qquad \ldots(3.3)$$

Combining (3.1) and (3.2), we have

$$x_t = (1+\rho_t)(k_t + (1+\rho_t)^{-1}d_t).$$

Since $1+\rho_t = F_k(k_t, l_{t+1})$ this can be rewritten as

$$x_t = d_t + k_t F_k(k_t, l_{t+1}).$$

Now suppose that the sufficient condition for monotonicity is satisfied and that initially the capital stock is below the corresponding stationary state level k. Then k_t lies to the right of k_{t-1} as depicted in Figure 4. From Lemma 1.2 the slope of the m-contour at (k_t, x_t) is the marginal product of capital $F_k(k_t, l_{t+1})$. It follows immediately that the size of the bond issue d_t is given by the intercept of the tangent with the x-axis.

Then to achieve the monotonic rise in the capital stock the government must float a monotonically increasing volume of debt. In the development depicted in Figure 4 the government is initially a net creditor, holding wealth claims against the private sector. Furthermore, in that development the government becomes a net debtor in period t and remains a debtor thereafter. However, it is quite possible that even in the asymptotic stationary state a net creditor position is optimal.[14]

By extending the tangent at (k_{t+1}, x_{t+1}) to the horizontal axis we obtain the market value $v_{t+1} = (1+\rho_{t+1})^{-1}d_{t+1}$, of bonds floated at the beginning of period $(t+1)$. From the diagram, a larger k_{t+1} implies a flatter tangent and a larger intercept with the negative k-axis. Hence, whenever it is optimal for the capital stock to increase, it is optimal for the *value* of the public debt to rise. We can summarize the above two results as follows.

Theorem 3.1. *To achieve a "maximin" increase (decrease) in the capital stock the central authority must increase (decrease) both the par value of the public debt and its market value at issue.*

Also shown on the negative half of the k-axis is the mirror image of the level of the previous periods debt. Thus, by (3.3), the difference between the two points is exactly the

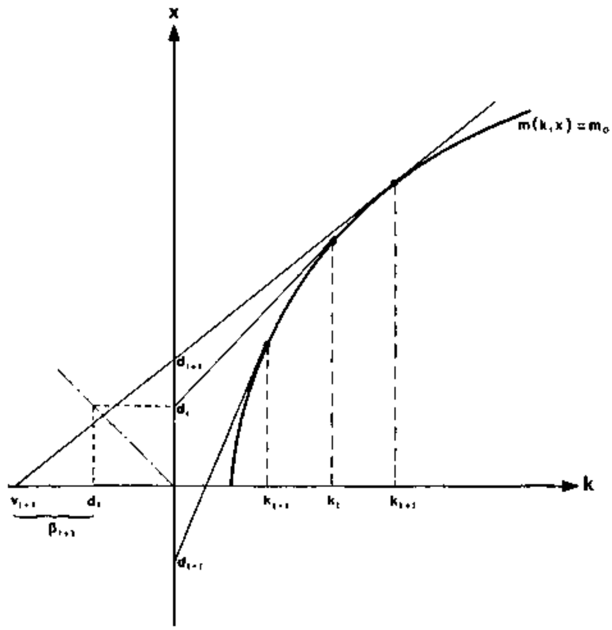

FIGURE 4
Optimal debt accumulation

present value of the demogrant paid to the $(t+1)$th generation. However, both terms in (3.3) increase as k increases and we cannot make general inferences about changes in the size of the demogrant over time.

4. INTERGENERATION TIES OF SENTIMENT

We have been studying the properties of " maximin " growth under conditions of generation egoism. Each generation, apart from its prior interest in justice, cares only about its own consumptions and leisure. One of these properties, we have shown, is that generation welfare is equalized. Another property is that the asymptotic state is completely sensitive (for $k < k^G(x)$) to the initial state. Both, especially the latter, have been regarded by some as unattractive implications of the " maximin " criterion. But we shall argue that they rely as much upon the postulate of *egoistic preferences* as upon the " *maximin* " *criterion*; they disappear (over most though not all of the domain) as soon as the former postulate is relaxed.

Before doing so, however, we should not let the objection to these properties pass without comment. One does not expect of a person who has grown up with less advantages in his formative years than someone else that he finally match the lifetime achievements of the other person, no matter how many years he is given to do it in. Why, then, should one expect of a society of egoistic generations that it strive for some asymptotic state that is as good as the destiny of another society more fortunate in its initial endowment? To make such a demand on the poorer society is to consign its early generations not only to bear a fair share of the burden of their society's bad luck but to repair the situation—as though it were their fault. Thus, justice does not seem to us to oblige a less fortunate society by dint of its *own* sacrifices to catch a more fortunate one.

If the failure of the disparity between the two societies to vanish in the limit does not accord with the intuition of some critics, Brandt and Solow for example, it is perhaps because they assume that *national or parental pride* would drive the less favoured society, given enough time, to erase its initial disadvantage or even to journey onward to some more absolute state of completeness. But surely such " values " are orthogonal to the " value " called justice, not the dictates of justice. An egoistic generation that lacked these drives yet heeded the " maximin " criterion might be called unaltruistic or uninspiring or abnormal. But if it made a " maximin " allocation, thus to assure for future generations the possibility of economic welfare at least as great as its own realized welfare, it could not reasonably be described as unjust.

Let us proceed to incorporate altruistic preferences into generation utility functions. We suppose that every (homogeneous) generation possesses altruistic preferences of a certain stationary or vintage-free type. The egoistic utility function $U(c_t, x_t, l_t)$ is replaced and incorporated by the altruistic utility function

$$V_t = V[U(c_t, x_t, l_t), V_{t+1}], \quad t = 1, 2, \ldots,$$

where the function V has positive and continuous first derivatives everywhere. Rawls's " ties of sentiment " are here like links in a chain. Each generation gives positive weight to its own interests and to the broad interests of the immediately succeeding generation, the latter calculated by the same function V. The chain creates a derived interest by any generation in the own interest (or self-interest) of subsequent generations indefinitely into the future.

Though the introduction of this altruism will generally alter the optimal allocation, the " maximin " criterion can still function in the same essential way. And some criterion of intergenerational justice is needed in order to obtain the " optimum " intertemporal allocations of a society. Otherwise there is no way to mediate the partially conflicting interests of generations.

In its technocratic version, putting aside fiscal implementability for a moment, the " maximin " problem becomes[15]:

$$\underset{\{c_t, x_t, l_t, k_t\}}{\text{maximize}}\; W(c_1, x_1, l_1, \ldots; k_0, x_0) = \inf\,[V_1, V_2, \ldots]$$

$$\text{s.t. } x_{t-1} + c_t + k_t = F(k_{t-1}, l_t)$$

given

$$k_0 > 0, \quad x_0 \geqq 0, \quad k_0 < k^G(x_0).$$

By considering the analogous " maximin " problem in the tth period under justice and upon defining

$$m(k_{t-1}, x_{t-1}) = \max_{\{c_t, x_t, k_t, l_t\}} W(c_t, x_t, l_t, \ldots; k_{t-1}, x_{t-1})$$

one obtains the dynamic programming equation

$$m(k_{t-1}, x_{t-1}) = \max\,[\min\,\{V[U(c_t, x_t, l_t), V_{t+1}], m(k_t, x_t)\}]$$

$$\text{s.t. } x_{t-1} + c_t + k_t = F(k_{t-1}, l_t).$$

Let us specialize the V function to the additive form with its implicit time discount:

$$V_t = U(c_t, x_t, l_t) + \gamma V_{t+1}, \quad 0 < \gamma < 1.$$

The solution to our altruistic " maximin " problem can then be seen to differ notably from that obtained earlier for the egoistic problem.

To obtain the solution to this additive altruistic " maximin " problem for some initial (k_0, x_0), we first find the intertemporal allocation that maximizes $V_1(\cdot)$. This sub-optimization problem can obviously be reduced to the familiar utilitarian problem of maximizing the geometrically weighted own-utility sum $\Sigma_t\, \gamma^{t-1} U(c_t, x_t, l_t)$.

There are two cases we have to distinguish. In the first case, the " sub-optimal " allocation yields a sequence of own-utilities U_t which are monotone increasing and which approach asymptotically some rest point utility level $U^{\gamma} < U^G$. In this case, that allocation must also be the full " maximin " solution. For if the U_t sequence is monotone increasing then so must be the corresponding V_t sequence; hence, noting that the V_1 maximum is unique, any other allocation could only lower the minimum V, namely V_1, and thus could not be " maximin ". Moreover, there will not arise any Strotz-Pollak problem of inconsistency causing generation 2 to select a different plan. For once (c_1, x_1, l_1) is given, the subsequent allocation maximizing V_2 also maximizes V_1. The quantity of capital in the asymptotic stationary state is determined by the familiar condition $F_k(k^{\gamma}, l^{\gamma}) = \gamma^{-1} > 1$ together with the usual marginal equivalence regarding the quantity of employment, l^{γ}.

In the second case, the allocation that solves the sub-optimization problem would make U_t decline monotonically and asymptotically down to U^{γ}. Then V_t would also be declining asymptotically down to $(1-\gamma)^{-1}U^{\gamma}$. The first generation under justice would be exploiting its position as first in the sequence of generation to award itself higher V_1 at the expense of subsequent generations thereby made worse off than it, which would not be " maximin ". In such a case the " maximin " solution must, from an initial state in the region $\{(k, x) \mid k < k^G(x)\}$ equalize generation utilities at the highest feasible level. Since constant V_t implies constant U_t over time, this " maximin " allocation is identical to the *egoistic* " maximin " allocation if, as we may suppose, the U functions in the two problems are identical.

The two cases are therefore illustrated in Figure 3. For initial states to the left of $m(k, x) = m^{\gamma}$ the optimal policy is to approach asymptotically the stationary state (k^{γ}, x^{γ}). For initial states between $m = m^{\gamma}$ and the as-good-as-golden locus, the optimal policy is to move along an iso-m contour to its associated stationary state. Outside the region $k \leqq k^G(x)$ some but not all generations can be assigned a V_t exceeding by a finite amount the maximum sustainable Golden Rule V^G. It follows that while there are many " max-inf " solutions there exists no " maximin " solution.[16]

5. CONCLUDING REMARKS

The principal messages of this paper are presumably clear. The application of the intergeneration " maximin " criterion is not generally a bar to the growth of capital. Unless the economy happens to be in an efficient stationary state initially, the " maximin " criterion will not lock the economy for ever in that state.

The Rawlsian criterion is not even a bar to the growth of utility. While it is hardly a defect, it is true that the " maximin " allocation (where it exists) is intergenerationally egalitarian with regard to utility *if* intergenerational externalities are excluded. That such intergenerational equality should result from the " maximin " criterion does not seem a telling objection to the use of that criterion when by hypothesis the generations, while just, are perfect egoists. In any case, the " maximin " criterion does not generally preclude the growth of utilities if initial capital is sufficiently scarce and if the generations possess an altruistic interest in the future utility possibilities.

Ethical theory, as Rawls has himself insisted, is uncertain and provisional like knowledge in general, especially the theory of human behaviour. Without being able to foresee the final verdict on the " maximin " criterion, we nevertheless find it significant that no anomalies or conundrums have been turned up by our study of " maximin " as a standard for the allocation of resources among generations—especially when " growth " has been considered a critical stumbling block for the " maximin " criterion. The only difficulty that the " maximin " criterion has encountered in our analysis occurs where the initial capital stock is so large that some generations can be allocated a utility exceeding the Golden Rule amount while by implication not all generations can be so favoured. Yet even this difficulty can be laid to the unboundedness of the time horizon rather than to the criterion itself. Moreover, it is a question whether our ethical principles should be asked to meet all manner of hypothetical conditions however counterfactual in actual experience. The unrestricted domain must always contain *terra incognita* so no criterion can ever be certified universally robust.

APPENDIX: CONTINUITY OF THE SUPREMUM

We wish to establish the continuity of

$$s(k_{t-1}, x_{t-1}) = \sup \{\inf_{\tau \geqq t} U_\tau\}$$
$$= \sup_{z_t} \{\min [U(z_t), s(k_t, x_t)]\},$$

where $z_t = (c_t, x_t, l_t, k_t)$ and $c_t = F(k_{t-1}, l_t) - x_{t-1} - k_t$.

1. Consider the subset of initial states

$$\Omega = \{(k_{t-1}, x_{t-1}) | F(k_{t-1}, 1) - x_{t-1} \geqq a, k_{t-1} \leqq A\},$$

where a, A are arbitrary positive numbers. Clearly by choosing a sufficiently small and A sufficiently large, *any* feasible initial state can be included in such a subset.

By assumption F is bounded from above, therefore Ω is compact. Moreover, from Section 1 the gross marginal product of capital satisfies $F_k(k, l) \geqq 1$ with the strict inequality if and only if k is less than some number $\bar{k}(l)$, positive for all positive l.

Then for each $w = (k_{t-1}, x_{t-1}) \in \Omega$ there exists a vector $z^s = (c^s, x^s, l^s, k^s)$ strictly positive with $l_s < 1$ satisfying

$$F(k_{t-1}, l^s) - x_{t-1} = \beta = (F(k_{t-1}, 1) - x_{t-1})/2 \qquad \text{...(a.1)}$$

$$x^s = F(\beta - c^s, l^s) - \beta = (F(\beta, l^s) - \beta)/2 \qquad \text{...(a.2)}$$

$$k^s = \beta - c^s. \qquad \text{...(a.3)}$$

Combining (a.2) and (a.3) we have

$$c^s = F(k^s, l^s) - x^s - k^s.$$

Therefore the conditions (a.1)-(a.3) define a feasible stationary sequence $\{z_\tau \mid z_\tau = z^s\}$.

Next define $U^s(k_{t-1}, x_{t-1}) = U(z^s)$.

Since U and F are continuous and Ω is compact

$$\exists \text{ some number } \bar{u} \text{ s.t.}$$

$$\bar{u} = \min \{U^s(k_{t-1}, x_{t-1}) \mid (k_{t-1}, x_{t-1}) \in \Omega\}.$$

2. Since $s(k_{t-1}, x_{t-1})$ is the least upper-bound, $\forall \varepsilon > 0$ and $\forall w = (k_{t-1}, x_{t-1}) \in \Omega$ there exists a sequence $\{z_\tau(w)\}$ feasible from w s.t.

$$\inf_\tau U(z_\tau(w)) > s(w) - \varepsilon/2 \quad \text{for} \quad \tau \geqq t. \qquad \text{...(a.4)}$$

But $s(w) \geqq U^s(w) \geqq \bar{u}$ therefore

$$\inf_\tau U(z_\tau(w)) > \bar{u} - \varepsilon/2 \quad \text{for} \quad \tau \geqq t. \qquad \text{...(a.5)}$$

Moreover, since F is bounded and Ω is compact all the sequences $\{z_\tau(w) \mid w \in \Omega\}$ are bounded from above by some vector $\bar{z} = (\bar{c}, \bar{x}, 1, \bar{k})$. Then all these sequences lie in the compact set

$$\Lambda = \{z \mid U(z) \geqq \bar{u} - \varepsilon, z \leqq \bar{z} + (\bar{c}, 0, 0, 0)\}.$$

Since U is continuous and Λ is compact, U is uniformly continuous on Λ. Then $\forall \varepsilon > 0$

$$\exists \delta, \bar{c} > \delta > 0 \text{ s.t. } |\hat{c} - c| < \delta \qquad c, \hat{c} \in \Lambda$$
$$\Rightarrow |U(z) - U(\hat{z})| < \varepsilon/2 \quad \text{where } \hat{z} = (\hat{c}, x, l, k)$$
$$z = (c, x, l, k).$$

In particular, this is true for all points on the boundary of Λ. Then all points in Λ within a distance δ of the boundary in the direction of the z_1-axis must satisfy *either* $U(z) < \bar{u} - \varepsilon/2$ *or* $c > \bar{c}$.

But from (a.5) inf $U(z_t(w) > \bar{u} - \varepsilon/2$, $\forall w \in \Omega$. Also $z_t(w) < \bar{z}$.

Thus all points within a δ-neighbourhood of $\{c_t(w) \mid w \in \Omega\}$ must lie in Λ. We have therefore shown that

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \text{ s.t. } \mid c_t(w) - \hat{c} \mid < \delta$$
$$\Rightarrow \mid U(z_t(w)) - U(\hat{z}) \mid < \varepsilon/2. \qquad \text{...(a.6)}$$

3. Let Λ^P be the projection of Λ onto $l \times k$.

Since F is continuous and both Ω and Λ^P are compact

$$C = c(k_{t-1}, x_{t-1}, l, k) = F(k_{t-1}, l) - x_{t-1} - k$$

is uniformly continuous on $\Omega \times \Lambda^P$.

Then for all pairs of initial states w', $w'' \in \Omega$ and $\forall (l, k) \in \Lambda^P$, $\forall \delta > 0$,

$$\exists \delta' > 0 \text{ s.t. } \| w' - w'' \| < \delta'$$
$$\Rightarrow \mid c' - c'' \mid < \delta,$$

where

$$c' = F(k'_{t-1}, l) - x'_{t-1} - k \qquad \text{...(a.7)}$$
$$c'' = F(k''_{t-1}, l) - x''_{t-1} - k. \qquad \text{...(a.8)}$$

In particular, this is true at $(l_t(w'), k_t(w'))$, in which case $c' = c_t(w')$.

Then from (a.6)

$$\mid U(c'', x_t(w'), l_t(w')) - U(z_t(w')) \mid < \varepsilon/2, \qquad \text{...(a.9)}$$

But the sequence $\{\hat{z}_\tau\} = \{(c'', x_t(w'), l_t(w'), k_t(w')), z'_{t+1}, z'_{t+2}, \ldots\}$ is feasible from the initial state w'' (since c'' satisfies (a.8)).

Then

$$\begin{aligned} s(w'') &\geqq \inf_\tau [U(\hat{z}_\tau)] \quad \text{for} \quad \tau \geqq t \\ &= \min [U(c'', x_t(w'), l_t(w')), \inf_\tau U(z_\tau(w'))] \quad \text{for} \quad \tau > t \\ &\geqq \min [U(z_t(w')) - \varepsilon/2, \inf_\tau U(z_\tau(w'))] \quad \text{from (a.9)} \\ &\geqq \inf_\tau [U(z_\tau(w')] - \varepsilon/2 \quad \text{for} \quad \tau \geqq t \\ &> s(w') - \varepsilon \quad \text{from (a.4).} \end{aligned}$$

Finally, making a symmetrical argument at $(l_t(w''), k_t(w''))$ we also have

$$s(w') > s(w'') - \varepsilon. \quad \|$$

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NOTES

1. It *could* be, contrariwise, that the initial capital deficiency is a transient phenomenon needing no surpluses for its elimination. Or it *could* be that only a temporary reduction of public debt is needed to jar capital loose from an unstable equilibrium. (The paper by Phelps and Shell [6] bears on these matters.) We are only recalling the flavour of the utilitarian message.

2. The substitution of a quasi-concave social-welfare functional for the old sum-of-utilities objective function does not alter the essentials of the utilitarian solution; it can affect only the speed of accumulation. The introduction of utility discounting has a critical effect, but such a modification has no evident justification.

3. A " maximin " allocation also has the merit, in the model we shall study, of being " fair " in the sense of Varian [12] (and others) in more recent work.

4. Rawls reaches the conclusion that there is no concept of justice between generations [9, p. 291]. The unhappy result follows from his ethical position that justice is a matter between parties who can gain from economic cooperation—no one is ever obligated to accept less than what he (or a nation?) can attain operating alone—and the economic premise that even adjacent generations cannot gain from economic cooperation. Whatever the merits and problems in the first postulate, Rawls has clearly made a (rare) slip in his economic premises—as this paper has demonstrated. " They " can benefit from our production of capital and " we " can later benefit from their working with it.

5. The misapprehension that " maximin " spells zero net saving has been held up by some utilitarians as a disqualification of that principle. It may well have been Brandt's criticism of " maximin " on this score that prompted Rawls, in his famously problematic section 44, to do without " maximin " in matters of distribution among generations.

6. A proof of that conclusion is given by Solow [11].

7. A different (and to our minds less satisfactory) way of representing this idea can be found in Arrow [1]. The incompleteness of the " no growth " conclusion found in the latter is clarified in Riley [10].

8. In the above formulation of our problem, x_0 is arbitrarily given. It is nevertheless possible to select x_0 in view of the past history of the old, (c_0, l_0), so as to adjust the lifetime utility of the old, U_0, to whatever feasible level may be desired. In particular, one could choose x_0 to maximise the minimum of $(U_0, U_1, U_2, \ldots)$, thus extending " maximin " justice to the old. Certainly the original expectations of the old need not be ruling.

9. It will be shown that this is the Golden Rule Utility level.

10. Even for this special case our result is in sharp contrast with those obtained previously by Arrow and Dasgupta. Their conclusion was that the economy would either remain at the initial state for ever, or would return at regular intervals. However, the economy described here leaves the initial state never to return.

11. One can extend the arguments of Section 1 to show that along any iso-m contour, $x'(k)$ is continuous. This leaves open the possibility of discontinuities in W^*_{kk}. However, even if this were the case, (2.5) would yield the same qualitative implications for both left- and right-hand derivatives.

12. If over some domain leisure and consumption when old are complements the inequality on the left-hand side of (2.12) is reversed. Oscillations in the capital stock may then be optimal.

13. It is quite possible for there to be multiple stationary states along an iso-return contour. In such cases the optimal policy is always to approach monotonically one of these states.

14. There is also a point on the stationary locus with zero public debt. This is the long-run equilibrium for a laissez-faire economy: see, for example, Diamond [4].

15. As in the egoistic case it is only for initial allocations within a certain domain that a " maximin " solution exists. For brevity's sake we exclude rigorous analysis of this and other technical issues.

16. A word about the implementation of the " maximin " allocation by taxes and transfers. Barro has argued that in a model (like ours) of perfect markets and foresight, any attempt to increase the volume of national saving by lump-sum tax policies would be exactly neutralised by offsetting changes in private saving " as long as current generations are connected to future generations by a chain of operative intergeneration transfers ". Of course, Barro has in mind the desire to help one's own descendants rather than a general concern for the opportunities of future generations (discussed by Marglin and others). In the region of our state space where utilities are rising, no taxes or transfers are required if the Rawlsian ties of sentiment operate entirely within the family unit.

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INTRODUCTION

I started work on Rawlsian economic policy sometime in 1972 as soon as my book and a companion paper on inflation policy were out of the way. The first paper in this part, on the maximin-optimal graduated taxation of wage income, was my premiere effort in that new line. The collaborations with Ordover and with Riley, which resulted in the joint papers contained in Parts I and III of this volume, soon followed. Two more recent papers, both stemming from the latter two, fill out the present group on taxation.

My study of wage-income taxation aimed to see whether some interesting results could be wrested from Mirrlees's model when the 'maximin' criterion is applied to it. While Mirrlees had used great ingenuity to solve his difficult problem in principle and to compute numerical examples, he had stopped short of looking for analytic results. One result I obtained is the formalism that a Rawlsian will want the tax schedule to maximize tax revenue—or, as we would now say, to reach the top of the Laffer curve. The chief result of a substantive sort was the so-called Phelps-Sadka proposition on the optimality of a zero marginal tax rate at the highest tax bracket (i.e., at the income of the highest earner). Sadka, in independent work, showed that the proposition held under the utilitarian criterion and its variants; my own demonstration was tailored to the 'maximin' criterion. (Under our mutual assumptions, in fact, reducing the last marginal tax rate to zero would permit a Pareto improvement.)

Is there an analogous and equally general proposition that can be demonstrated about the desirability of a zero marginal tax rate on interest-type income at the top bracket? That question nagged me while I was attempting a nontechnical piece for laymen on tax reform to celebrate the American bicentennial.[1] I worried that a cutback of such marginal tax

[1] Some of my thoughts for that occasion found their way into my essay, "Rational Taxation," *Social Research*, December 1978. Two of the proposals made there foreshadowed some results in the second Ordover-Phelps paper: levy a positive proportional tax on interest type income and a Phelps-Sadka kind of graduated tax on wage type income.

rates at the top—while possibly increasing tax revenue if done skillfully—might, by stimulating saving on the part of the top earners, so reduce the rate of return to saving as to do more harm than good to the present generation. More technical analysis was needed!

Yet it was not until studying the diagrams and Hamiltonians of the recent Ordover-Phelps analysis that a coherent picture of the matter came into view: If the government, out of a budget balance fetish or whatever, restricts the economy to an inefficient region of outcomes in which the social rate of interest for the present generation, sometimes called the social rate of discount, is less than the social rate of return to investment—which is measured by the market rate of interest if the next generation is optimizing and markets are perfect—then there is a presumption that the government should correspondingly tax away some of the private interest rate so as to drive the after tax return on saving down to the social interest rate; and this presumption is exactly borne out when all persons' utility functions satisfy (at least locally) the Corlett-Hague-Atkinson-Stiglitz separability condition regarding the effect of leisure on households' preferences for early versus late consumption. Under the separability condition, in fact, the tax on interest-type income should be proportional. That proportional tax rate should be zero, under separability, if and only if the aforementioned social interest rate equals the social rate of return.

It is interesting that the Phelps-Sadka finding with regard to wage taxation remains valid even in inefficient terrain to which second-best policy restrictions might confine the economy. Granted, there seems to be little likelihood that legislators will vote for diminishing marginal rates of tax on wage income—for "regressivity" of the marginal rates—in the near future. It does not seem so unrealistic, however, to expect that legislatures may soon contemplate the abandonment of graduated taxation in favor of proportional taxation of wage income. In that case we come up against the question of the linear tax mix first raised in the earlier paper with Ordover.

The aim of the second paper in the present group was to redo the original Ordover-Phelps analysis, this time under the single "restraint" of intergenerationally maximin-optimal growth in place of the original pair of steady-state restraints on capital and wealth (or capital and public debt). A critical step there in the derivation of the tax-rate formulae, a certain envelope theorem, proved to be a key step in the second Ordover-Phelps paper: One can always regard the present generation as doing its intrageneration maximizing subject to a *supporting* hyperplane restraint in capital-wealth space; in the case of maximin-optimal growth, though, that restraint is also a *separating* hyperplane, the slope of which is essentially equal to the social rate of return to investment. If the capital-wealth target of the present generation is intergenerationally maximin-optimal, therefore, the social rate of interest (or rate of discount) with which future goods are discounted by the present generation in relation to present goods is equal to the net marginal product of the capital it bestows on the next generation of workers. And if markets are perfectly competitive, a useful assumption for the sake of a preliminary analysis at least, the latter is conveniently measured by the market rate of interest.

With that result in hand it was possible to extract anew the Ordover-Phelps tax-rate equations in the nicest of cases—the golden case where the predetermined levels of capital and wealth which must be sustained are free to be chosen. Unfortunately, when writing that paper and the earlier one with Ordover I assumed that an intuitive explanation of the results would be something on the order of a literary proof of Cramer's rule. I now feel that much of the mystery can be taken out of the result for the optimal tax or subsidy on capital.

The formula for the tax rate on capital is sending out this message: In the maximization of total socially discounted tax revenue, taking as parameters the shadow interest rate and wage rates, we are to consider the consequences of each small change of the after-tax rate of interest in conjunction with an accompanying change of the after-tax wage to standardized labor that would leave total discounted revenue unchanged if there were no resulting incentive effects upon the supplies of labor and saving; hence the actual effects upon wage-tax revenue plus discounted interest-tax revenue of this peculiar tax-rate substitution are entirely those attributable to the incentive effects of that substitution. Of course, in the neighborhood of a zero tax rate on capital the incentive effects upon saving can be neglected because in that neighborhood the supply of saving is being taxed or subsidized at a zero or negligible rate. So it is understandable that it is the incentive effects of the tax-rate substitution on wage-tax revenue which is pivotal for the algebraic sign of the tax (or subsidy) rate on capital. Roughly speaking: If a small tax-rate substitution raising the after-tax rate of interest (thus lowering the after-tax wage rates) should have incentive effects upon labor supplies causing wage-tax revenue to increase, then a zero tax on interest income cannot be revenue-maximizing; such tax-rate substitutions should proceed until a point is reached where the marginal gain in wage-tax revenue is finally offset by the marginal loss of that revenue resulting from the induced rise of saving which has become the object of a subsidy. A rigorous account would take account of the possibility that the marginal wage-tax revenue and interest-tax revenue terms are not monotone, nor even one-signed.

The aforementioned kind of tax-rate substitution will remind readers familiar with the public-finance literature of the "compensated" tax-rate substitutions of utility theory. The former kind of tax-rate substitution leaves the "market as a whole" just able to buy the original aggregate basket of leisure, present consumption and future consumption. In retrospect, then, this kind of tax analysis does not contradict at all—rather it offers a kind of econometric slant on—the wisdom of Corlett and Hague: That unequal *ad valorem* tax rates on the various consumer goods are useful where there are unequal complementarities (or substitutabilities) between these consumer goods and leisure.

The taxation of capital and, more fundamentally, the meaning of optimal taxation are the focus of the last paper in the present group, my second collaboration with Ordover. Plainly this paper was provoked by the analysis of Atkinson and Stiglitz in which the Corlett-Hague viewpoint is brought to bear on the multi-period generation. Our difficulty was that this generation seemed to be disembodied, living apart from any contemporaneous ones; we were interested to see under what conditions their analysis would go through once their generation was imbedded in the overlapping-generations/diminishing-returns model to which we were accustomed. Our central finding is that there must be no "deficit constraint," or debt limit, if the Atkinson-Stiglitz results are to be applicable—quite contrary to what a casual reading of their paper would suggest. The presence of a deficit taboo, for example, will generally steer the present generation's capital-wealth target away from the "efficiency locus" and thus deprive the Atkinson-Stiglitz prices of any parametric "efficiency (or shadow) price" interpretation.

This last paper is still too fresh in my mind to allow me any chance for a considered reappraisal. It should be noted, though, that the balanced-budget locus seems to have been constructed from successive tangencies with the wrong family of iso-welfare contours! But, unless I am mistaken, the argument—that a generation "playing monopolist" against its successor will contract capital and wealth below the efficiency locus—survives the necessary correction. (There are other slips in this volume, beginning with the first paper, on the Golden Rule, that I have not felt it worth the reader's time to correct.)

The other correction is one of tone: The paper leans more toward the exemption of interest income from tax than it would have done had time and space been less constraining. There is no presumption in my mind in favor of the Atkinson-Stiglitz separability condition for equal complementarity and hence, provided the efficiency condition is also satisfied, a zero tax rate on capital. My guess is that an increase of effort in the "working period" would increase the marginal utility of working-age consumption more than that of retirement-age consumption—as if the former consumption were a kind of fuel, or balm, for the effort involved. On the other hand, I am venturing that hypothesis at a moment when the marginal utility of anything but one more sentence is rather high.

TAXATION OF WAGE INCOME FOR ECONOMIC JUSTICE

Sidgwick's principle of equity requires that "whatever action any of us judges to be right for himself, he implicitly judges to be right for all similar persons in similar circumstances."[1] With regard to tax systems if an individual whose situation and attributes are x ought to pay tax t, then, on the principle of equity, every other individual having the identical x ought to pay the same tax $t(x)$. It is clear that (horizontal) tax equity is incomplete as a criterion of just taxation, being only a necessary condition for fair taxation that many tax systems could satisfy, not all of which would otherwise be satisfactory.[2] Moreover, the requirement of equity becomes empty if the vector x is so lengthy or personal as to allow tailoring the tax to each person within wide limits. This suggests that the applicability of equity depends upon some deeper notion of impartiality or fairness.

Much of the economics of taxation is concerned with the principle of Paretian efficiency. If the purchases of some commodity (or commodities) were to be added to the tax base (to individuals' x's) and all tax rates reset as desired, would the resulting utility feasibiliy frontier lie somewhere outside the old one? In such regions

* My sincere thanks are due to the two dozen mathematical economists and philosophers, literally too numerous to identify, who gave generously of their time and knowledge during my efforts at this paper.

1. H. Sidgwick, *The Method of Ethics* (London: Macmillan, 1907), p. 379. See also the rule of universalizability in R. M. Hare, *Freedom and Reason* (Oxford: Clarendon Press, 1963), pp. 89–90.

2. Thus, Anatol France's remark that the laws of France are perfectly just, the rich and the poor having equal rights to sleep under the bridges of Paris.

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of the utility space, would some activities then become inefficient to tax at positive rates (and hence freely eliminable from the tax base)? It is interesting here that a range of natural candidates for an expanded tax base, such as school achievements and various child test scores, is tacitly excluded from consideration in the usual analysis. It is true that individuals would have an incentive to disguise their earning potentials by underachieving if their grades and scores were a basis for their future tax; but the analogous objection to the disincentive effects of graduated taxation of realized earnings has never been held to be fatal. The exclusion of such quasi-lump sum taxes would seem to be based in part on the view that, in a world of imperfect information where people's potential to earn income cannot be perfectly forecast, it would risk unfairness in some sense (or a loss of expected social welfare) to tax an individual according to the forecast of his earning power in the future as a substitute, in whole or part, for taxation of his actual earnings by that time.[3] This observation exemplifies the familiar point that efficiency is only a necessary condition and desirable exclusively relative to some criterion of social welfare or distributive justice. A tax system efficient in securing the wrong distributive results may be "worse" than some (improvably) inefficient tax systems. Neither equity nor efficiency then has any necessary merit apart from a satisfactory principle of distributive justice, at least in formal policy analysis.

The conception of distributive justice advocated by John Rawls, most comprehensively in *A Theory of Justice*, appears to be the first complete principle of social choice to command wide and serious interest since the time of sum-of-satisfactions utilitarianism.[4] My purpose in this paper is to derive the implications of the Rawls criterion for the graduated taxation of wage incomes within the context of two simple models of household earning decisions. The next section describes the criterion and briefly addresses his defense of this principle.

I. "Maximin" Justice

Rawls refers to the general concept of justice as the notion of a standard by which the distribution of the burdens and benefits from

3. This is at any rate plausible if individuals' observable earnings in the future are believed to show perfect, or at any rate much higher, rank correlation with their actual earning power in the future.

4. J. Rawls, *A Theory of Justice* (Cambridge, Mass.: Harvard University Press, 1971). By complete principle I mean an exact specification as distinct from a set of ethical postulates that narrow down the social ordering of social states or utility distributions to some restricted class.

cooperation by the individuals in society is to be determined. The particular conception of justice, the specific distribution criterion, argued for in *Justice* is what he calls the "difference principle": In the just economy the welfare of the worst-off is as large as is feasible. The principle does not imply the obliteration of all inequalities in well-being. Differences in liberty and opportunity, income, and other primary social goods are justifiable insofar as they benefit the least well-off.[5] It is the modern-day analogue of Aquinas: Everything for the greater utility of the poor.

The criterion is more aptly labeled "maximin" than "favor the least advantaged." The latter can be ambiguous when the individuals having least utility vary from state to state. We are to identify the smallest individual utility in each social state — the utility of the least well-off *in that state* — and choose the social state where this minimum utility is maximized. It is possible that this chosen state is not one the least well-off like best: if so, it is chosen, nevertheless, because their preferred state would make some others even less well-off than would be those who are least well-off in the chosen state.[6] Thus, the criterion does not necessarily make any individual a "dictator," even a postdetermined one, let alone one predesignated without regard to the eventual distribution of utilities. Yet the structure of social opportunities may very well be such that the maximin criterion will select the social state preferred by the individuals who are the least disadvantaged in that state. In such cases Arrow's "non-dictatorship" axiom is apparently not met. However, the "dictator" is not someone preordained according to proper names, but rather some individual determined impartially from ordinal comparisons of individual well-being.[7]

The Rawls criterion is lexicographic (or lexical, as he calls it). Of two or more social states tied for largest minimum utility, choose the one (or ones) giving the largest next-to-minimum utility and so on. (In the same spirit Rawls would have us treat the primary goods serially: liberty has the first priority, then the other social

5. Rawls typically refers to two principles, of which the first is that "each person is to have an equal right to the most extensive basic liberty compatible with a similar liberty for others," and the second (difference) principle applies to social and economic inequalities (*Justice*, p. 60). But he acknowledges (p. 83) that the difference principle really is to be applied to "all primary goods including liberty."

6. See Rawls's discussion of chain connection, *Justice*, p. 80.

7. K. J. Arrow, *Social Choice and Individual Values*, 2nd ed. (New York: John Wiley and Sons, 1964), p. 30. Of course, something in Arrow's system must be excluded if we are to obtain the social ordering Rawls wants. Arrow excludes interpersonal utility comparisons in the definition of the Arrow social welfare function (p. 23). Only the individual orderings are "fed in."

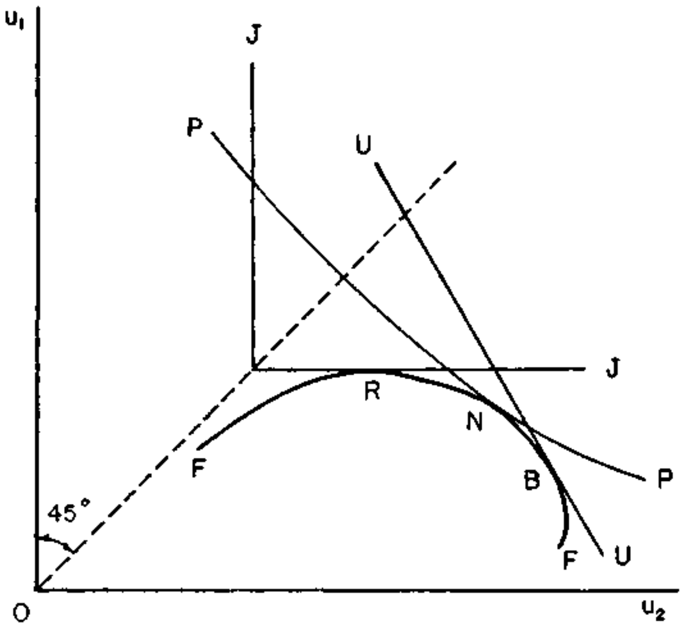

FIGURE I

The Rawlsian maximin criterion selects R on the per capita utility feasibility frontier FF. The product-of-per-capita-utilities criterion selects N, and the sum of utilities criterion selects B.

values.) Such an ordering of social states is not representable by a Bergson social welfare function, $W(u_1, u_2, \ldots, u_n)$. In the absence of ties the sense of the Rawls ordering is expressed by $W = \min(u_1, \ldots, u_n)$.

The working of the Rawls criterion is illustrated in Figure I, virtually an ideogram of Rawls's proposal.[8] It pictures a two-class economy in which, by assumption, the incentive effects of redistributive measures (say, graduated income taxation and transfers) cause the representative persons' utility feasibility frontier FF to slope upwards sufficiently near the egalitarian 45-degree line. The Rawls criterion, with its right-angled "contours" like JJ, picks out R on this frontier. The Benthamian sum-of-satisfactions criterion chooses B, and the Bernoulli-Nash product-of-utilities function selects N.

It is Rawls's conviction that this maximin criterion will emerge from a proper construction of social contract doctrine. To think about what is just, free from the known facts of his special interests,

8. *Justice*, p. 76. As Rawls notes, the Bentham contours in the diagram have a slope that depends upon the relative numbers of persons in the two classes.

a person may at any time figuratively "ascend" to the "orginal position" — a hypothetical situation of initial fairness in which the members of society are imagined to deliberate on the social and economic structure to be chosen without knowledge of the respective natural endowments, social advantages, and psychological propensities that they will possess, though with a general understanding of human behavior and an awareness that having more of the various primary goods would help anyone to realize whatever style or plan of life he may find he desires. "On the contract view, the theory of justice [as a general concept] is part of the theory of rational choice." [9]

The most prominent rival to the maximin rule is undoubtedly utilitarianism, modern or classical. Rawls gives three grounds for his contention that in the original position the maximin rule would be chosen over any utilitarian or neo-utilitarian rule.[1]

The first ground is that, because the agreement on a distributive principle is to be final and binding, any individual would want to protect against the worst imaginable eventualities. Now Rawls's original position is reminiscent of the neo-utilitarian approach of Harsanyi and Vickrey in which an individual's "ethical preference" is for that social structure, with its redistributive policies, which maximizes the mathematical expectation of utility when he believes it is equiprobable that he will be in each person's shoes and, knowing the technology and people's preferences, he can calculate the payoff (under each social structure) attaching to every pair of shoes.[2] A risk-neutral Harsanyi-Vickrey calculator would choose B on FF while a risk-averse individual would select a point left of B though, unless he is completely risk-phobic, *right* of R. In Rawls's construction of the original position, individuals do not have the numerical data for such calculations of expected utility. They do not, for example, know the relative numbers of who are top dogs and bottom dogs. Why not? One reason Rawls gives is that the original position is needed to resolve questions of justice among generations into the future, so that technologies and preferences cannot generally be known anyway. But were such data obtainable, they would not be wanted, for we seek an agreement on principle

9. *Justice*, p. 47.
1. See *Justice*, especially pp. 150–83.
2. J. C. Harsanyi, "Cardinal Welfare, Individualistic Ethics, and Interpersonal Comparisons of Utility," *Journal of Political Economy*, LIII (Aug. 1955), 309–21; and W. S. Vickrey, "Utility, Strategy, and Social Decision Rules," this *Journal*, LXXIV (Nov. 1960), 507–35. Neither author resolves the problem of a conflict among individuals' ethical preferences due to differing attitudes toward risk, while for Rawls this does not appear to be a difficulty.

that treats all equally as moral persons, not an agreement biased by arbitrary contingencies of nature and social advantage.[3] Might, I believe Rawls means, does not make right, including might that is the result of sheer relative numerical superiority. Under such informational constraints selection of the maximin rule does indeed seem natural.

The second ground is that, by their adoption of the maximin rule to outlaw policies and institutions that injure those with poor life prospects as the means to improve the life prospects of those more advantaged, people would be expressing their respect for one another in the very constitution of society. In contrast, recognition of the principle of utilitarianism (simple or generalized) may well entail some loss of self-esteem. For in allowing higher life prospects for some to counterbalance lower life prospects for others already less fortunate in natural and social advantages, it would have us exploit to a degree, rather than neutralizing, the contingencies of nature and social circumstance on behalf of the more fortunate — "as though we belonged to a lower order, as though we were a creature whose first principles are decided by natural contingencies" instead of as "free and equal rational beings with a liberty to choose."[4] Since a person's self-respect normally depends on the respect of others, and the more self-respect one has, the more likely it is that he will respect others, public acceptance of the maximin rule is likely to give greater support to people's self-respect all round. Thus, in choosing the maximin principle, people would be insuring their self-esteem as it is rational for them to do.

Third, the utilitarian principle would ask the less advantaged to view the greater advantages of others as a sufficient reason for accepting still lower life prospects than they could be allowed. This is an extreme demand that individuals in the original position would not feel it wise to commit themselves to. The difference principle, Rawls writes, seems to be one "on the basis of which those better endowed or more fortunate in their social position, neither of which we [sic] can be said to deserve, could expect the willing cooperation of others when some workable scheme is a necessary condition of the welfare of all."[5] The suggestion here is apparently that points to the right of R, on the top-most utility frontier at any rate, are not Nash equilibria if the poor can do better by concerted violations of such social contracts. Of course, no such Nash equilibria need occur

3. *Justice*, p. 137–42.
4. *Justice*, p. 256.
5. *Justice*, p. 15. See also pp. 496–504 on "relative stability."

anywhere on that frontier. Rawls argues only that *R* has the merit of being stable enough.

The maximin rule has also to be defended against neo-egalitarian proposals from the direction opposite to neo-utilitarianism. It has been argued that some point on the utility frontier to the left of *R*, even though below *R*, would be morally preferable in view of the "relative deprivation" of the poor at *R*.[6] On this view, the maximin criterion is too conservative — as conservative as the British Tory justification of a small reduction of tax burdens on the prosperous on the grounds it would so release incentives as to cause an upward movement toward *R* that would benefit all. Rawls resists the idea that any envy by the poor for the rich at *R* would induce the deliberators in the original position to prefer a leftward point that would put a crimp in the life prospects of all.[7] Resentment is a response to unjust treatment, while envy is one of Kant's "vices of hating mankind."

We shall hardly be able to decide these matters here and now. As Rawls remarks, the idea of the original position and of an agreement on principles there can serve only as the beginning.[8] Yet in a two-way contest between neo-utilitarianism and neo-egalitarianism, the Schellingesque salience of the Rawls point *R* recommends it as a point of obvious compromise. I find it highly appealing, especially when compared to utilitarianism, and believe that it merits the exploration of its applications that follow. To this Rawls would himself add that such exercises serve as checks on the acceptability of the distributive criterion itself, for the resolution of ethical principles comes about through a process of tatonnement in which postulates are revised when their implications are found unsatisfactory.[9]

II. Just Taxation in Two Earnings Models

We analyze here the implications of maximin justice for wage-income taxation in two market models of individual earnings. These

6. W. G. Runciman, *Relative Deprivation and Social Justice* (London: Routledge and Kegan Paul, 1966).

7. Rawls makes some qualifications in regard to "excusable envy" that I cannot explicate here. See *Justice*, pp. 534, 546.

8. *Justice*, p. 47.

9. See *Justice*, pp. 19–21. In a letter of comments on the present paper, Rawls writes: "We should not accept a standard, it seems to me, *whatever* the implications of it. Therefore how [the maximin criterion] applies to economic questions like taxation is not a matter of *mere* application. One is testing the viability of the conception of distributive justice itself, perhaps not as decisively in this sort of question as some others, but still one is testing it. The kind of exploration you present is necessary if we are to determine whether the criterion is really reasonable."

models highlight different effects upon incentives to earn wages, and hence to accrue tax liability, of the graduated (i.e., nonlump sum) taxation of wages. In the first model, which Sheshinski (1971) developed to study the implications for taxation of maximizing average utility, the disincentives from positive marginal tax rates fall on private education.[1] In the second model used by Mirrlees (1971) to study the maximization of an additive social welfare function, the corresponding disincentives fall on effort (say, hours worked).[2] Throughout this section we shall suppose with those authors that income other than wages is nonexistent or at least completely independent of wage earnings. Intertemporal and international aspects are ignored. Some questions concerning just taxation in a larger model — one in which the present ones can be imbedded — are discussed in the concluding section.

Some features common to both models can be indicated here. Individuals have identical preferences. They differ in opportunity or ability to earn income according to differences in a parameter n that ranges from 0 to $N \leqslant \infty$. Let $F(n)$ denote the proportion of individuals whose ability is less than or equal to n. It will be supposed that $F(n)$ is continuous, monotone increasing, and right-differentiable:

$$(1) \qquad F(n) = F(0) + \int_0^n f(s)ds, \quad F'(n) = f(n) > 0, \quad 0 \leqslant n \leqslant N,$$
$$F(0) \geqslant 0, \quad F(N) = 1.$$

There are not, therefore, any no man's stretches between 0 and N over which persons having such ability levels are nil. While we do not generally require it, differentiability of the density function f is needed for some propositions.

A person of type n exploits his opportunity by selecting a variable within his control, $x(n)$, which determines his before-tax wage earnings $y(n)$. In Mirrlees x is manhours worked per day in a competitive labor market and so, omitting the index where it is understood, we obtain

$$(2) \qquad y = nx.$$

In Sheshinski x is an index of time spent in private education that is supposed to augment individual earning power in the same multiplicative manner. A person's n is thus measurable by the wage he would receive when his $x = 1$. Every person's private marginal prod-

1. E. Sheshinski, "On the Theory of Optimal Income Taxation," Harvard Institute of Economic Research Discussion Paper No. 172, Feb. 1971.
2. J. A. Mirrlees, "An Exploration in the Theory of Optimal Income Taxation," *Review of Economic Studies*, XXXVIII (April 1971).

uct is a constant n independent of his x and of others. There are no externalities, so that we may interpret a person's n as his marginal social product as well.

The problem studied here is essentially finding the *net* tax function $k(y)$ or corresponding disposable-income function $z(y) = y - k(y)$, such that minimum utility is maximized. This maximization is constrained by the budgetary arithmetic that aggregate (gross) tax revenue net of transfers covers any fixed government expenditure γ and fixed desired budgetary surplus σ:

$$\int_0^N k[y(n)]\ dF(n) = \gamma + \sigma = \text{const.}$$

Without loss of generality we may view k as equal to a gross tax, $t(y)$ with $t(0) = 0$, *less* a "minimum-disposable-income" transfer or lump sum grant paid to all individuals g:

(3) $$z(y) = y + g - t(y),\ t(0) = 0.$$

Our problem then is to find the function $t(y)$ that maximizes minimum utility, subject to the relation,

(4) $$g = \int_0^N t[y(n)]\,dF(n) - \gamma - \sigma.$$

An individual of ability $n_2 > 0$ can and will earn more utility than persons of type n_1, $0 \leqslant n_1 < n_2$, for every g- and t-function. This is because an n_2-type individual can assure himself at least $z(n_1) = y(n_1) + g - t(y(n_1))$, namely by choosing $x = y(n_1)/n_2 < x(n_1) = y(n_1)/n_1$, and this smaller x leaves him with more utility than the n_1-type — either from less disutility of effort (Mirrlees) or less outlay for education and thus less consumption foregone (Sheshinski). It follows from such reasoning that the minimum utility for *every* t-function is that received by persons with $n = 0$. Their utility is an increasing function of g and a function of nothing else, given γ and σ. Thus, the maximization of minimum utility entails finding the tax function that maximizes g in (4). The problem of taxation for maximin justice in the present models, therefore, is the problem of maximizing aggregate (gross) tax revenue — of achieving *taxable capacity*.

The questions concerning the "maximin" tax function of greatest interest would appear to be these: Is the tax an *everywhere* increasing function of earnings? Is the tax uniformly progressive? That is, does the average tax rate, net or gross, rise with earnings throughout? Does the marginal tax rate, $t'(y)$ vary with earnings and, if so, is there a tendency for it to rise or fall with income? [3]

3. Before proceeding, I should acknowledge that income taxation *might,*

A. The Training Incentive Model

Each individual acts to maximize utility, which depends, given the configuration of government expenditure γ, only upon his private consumption:

(5) $$u = u(c),\ u'(c) > 0,\ c \geqslant 0.$$

Accordingly, hours worked may be imagined to be fixed.

While there is no disutility of education, there is a resource cost. The private (equals social) cost in terms of consumption foregone of an amount of education x for any individual is denoted $j(x)$ and is the same for all n.

(6) $$c + j(x) = y - t(y) + g.$$

This cost function is postulated to exhibit positive and rising marginal costs. More fully,

(7) $$j(0) = 0,\ j'(0) = 0,\ j'(\infty) = \infty$$
$$j'(x) > 0,\ j''(x) > 0, \text{ for } x > 0.$$

The maximization of utility by a type-n individual with respect to his x yields the first-order condition

(8) $$\frac{\partial c}{\partial x} = n(1 - t'(nx)) - j'(x) = 0,\ 0 \leqslant n \leqslant N$$

for an interior maximum. Let us assume provisionally that tax revenue maximization implies $t(y)$ to be twice continuously differentiable with marginal tax rate $m(y) \equiv t'(y) < 1$ for all y.[4] Then $x > 0$ for all $n > 0$, and $x = 0$ at $n = 0$ in any case. We also make the provisional assumption that the "maximin" tax function causes the second-order condition for a relative maximum to be satisfied:

(9) $$\frac{\partial^2 c}{\partial x^2} = -n^2 t''(nx) - j''(x) < 0.$$

Subject to the condition that $m < 1$ and (9) hold for *all* x, (8) gives the individual's global utility maximum. Then the individual's

from a formal point of view, be regardable as suboptimal. If the government could measure people's y/x at little or no cost — the model does not stipulate this one way or the other — then it could lump sum tax individuals according to their n values so calculated, thus to enlarge taxable capacity and to increase the maximum grant g. In a richer model where there is a variety of occupations of differing disutilities to choose from, individuals could disguise their abilities by opting for less well-paid jobs having more nonpecuniary compensations (at the cost to themselves of bidding down further the pay in these jobs). In view of these difficulties in ability measurement and possibly other obstacles and objections to lump sum taxation, the exploration of wage-income taxation seems amply justified at the present time.

4. If there were some interval over which $t'(y) \geqslant 1$ is better than $t'(y) < 1$, then a discontinuous jump of t at the beginning of the interval would be as good or better. Hence the question reduces to the continuity of t.

optimal x in (8) is free of "income effects" from taxation, being dependent on the marginal tax rate but independent of the level of the tax paid at optimal y, $t(y)$.

Equation (8), which we may rewrite as

$$(8') \qquad n[1-m(y)]-j'(x)=0,$$

makes x an implicit function of m and n, say $\chi(m, n)$, and makes y another function of m and n, $\psi(m, n)=n \cdot \chi(m, n)$. Differentiation of (8) yields

$$(10) \qquad \frac{\partial x}{\partial m} \equiv \chi_m(m, n)=\frac{-n}{j''}<0$$

$$\frac{\partial x}{\partial n} \equiv \chi_n(m, n)=\frac{1-m}{j''}>0$$

$$\frac{dx}{dn}=\frac{\chi_n(m, n)+xm'\chi_m(m, n)}{1-nm'\chi_m(m, n)}=\frac{1-m-ym'}{j''+n^2m'} \gtreqless 0$$

$$\text{as } ym' \lesseqgtr 1-m$$

and

$$(11) \qquad \frac{\partial y}{\partial m} \equiv \psi_m(m, n)=n\frac{\partial x}{\partial m}=\frac{-n^2}{j''}<0$$

$$\frac{\partial y}{\partial n} \equiv \psi_n(m, n)=x+n\frac{\partial x}{\partial n}=x+\frac{j'}{j''}>0$$

$$\frac{dy}{dn}=\frac{\psi_n(m, n)}{1-m'\psi_m(m, n)}=\frac{j'+xj''}{j''+n^2m'}>0.$$

Thus education would decrease with n if m rose with income sufficiently steeply, but income must rise with n in any case, provided $1-m=j'/n>0$, as we are assuming. As for consumption we have from (5) and (7')[5]

$$(12) \qquad \frac{dc}{dn}=(1-m)x+[(1-m)n-j']\frac{dx}{dn}=(1-m)x>0.$$

Our problem now is to find that distribution of tax burdens that maximizes aggregate tax revenue and thus also the lump sum grant so as to make minimum consumption (and hence utility) as large as is feasible. Here the taxes paid by individuals are to be a direct function of earnings, equal to $t(y)$; tax payments are only a derivable function of ability n, deducible from (11). It is natural, there-

5. It was already established that n-types could do better than n_1-types, for all $n>n_1>0$ and any admissible tax structure, $t(y)<y+g$. Equation (12) *measures* that advantage, and it states that the advantage from higher n is continuous in n on the assumption that t is continuous and $1-m>0$ everywhere.

fore, to express aggregate tax revenue as the integral over *income* of individuals' tax payments, rather than the integral over ability. In these terms our problem is

$$\text{(13)} \qquad \underset{\{t(y)\}}{\text{maximize}} \int_0^\infty t(y)\,dB(y) \text{ subject to } t(0)=0,$$

where $B(y)$ is the proportion of individuals with earnings below or equal to y. Of course, the distribution of income depends both upon the distribution of ability $F(n)$ and the tax function itself. To make this transformation of variables from n to y, we invert $y=\psi(m,n)$ to obtain a function $n=\phi(m,y)$, which gives the ability level that any individual has if he chooses to earn y and faces the corresponding marginal tax rate $m(y)$. That is, using

$$dy=\psi_m dm+\psi_n dn,\ \psi_n>0,$$

we obtain

$$\text{(14)} \qquad \begin{aligned} \frac{\partial n}{\partial m} &\equiv \phi_m(m,y)=\frac{-\psi_m(m,n)}{\psi_n(m,n)}=\frac{n^2}{xj''+j'}>0 \\ \frac{\partial n}{\partial y} &\equiv \phi_y(m,y)=\frac{1}{\psi_n(m,n)}=\frac{j''}{xj''+j'}>0 \\ \phi(m,0)&=0 \qquad (\text{if } m<1);\ \phi(m,Y)=N. \end{aligned}$$

Hence,

$$\text{(15)} \qquad \begin{aligned} &B(y)=F[\phi(m(y),y)] \\ &B(0)=F(0)\geqslant 0,\ B(Y)=F(N)=1 \\ &b(y)\equiv B'(y)=F'(\phi)\,[\phi_m m'(y)+\phi_y]=f(\phi)\cdot \\ &\frac{(n^2m'+j'')}{xj''+j'}>0, \end{aligned}$$

where $b(y)$ is the density of persons with income $y>0$ and Y is the largest earnings attained.

So armed, one's instinct is to proceed in the spirit of (13) with the problem:

$$\text{(16)} \qquad \underset{\{m'(y)\}}{\text{maximize}}\ R=\int_0^\infty t(y)f(\phi(m(y),y))\,[\phi_m\cdot m'(y)+\phi_y]\,dy,$$

$$\text{given } t'(y)=m(y),\ t(0)=0.$$

Here the *rate* of change of the marginal tax rate is the control variable, and there are two state variables, t and m. While the linearity of the integral in $m'(y)$ excludes a classical analysis in terms of an Euler equation in $m''(y)$, the methods of control theory yield the solution.[6]

6. See Appendix.

Shying away at first from the complexities of this maximization, the author fortunately stumbled onto a much more expedient formulation of the tax revenue maximand. Clearly, aggregate revenue equals the marginal tax rate on the first "dollar" of earnings times the number of persons earning a dollar or more, *plus* the marginal rate on the second dollar times the number earning two dollars or more, and so on to the last dollar of the highest earners. Hence,[7]

$$(17) \qquad \int_0^\infty t(y)\; b(y)\; dy = \int_0^\infty m(y)[1-B(y)]dy.$$

Therefore our problem can be cast in the simpler form,

$$(18) \qquad \underset{\{m(y)\}}{\text{maximize}}\; R(m) = \int_0^\infty m \cdot (1-F[\phi(m, y)])\; dy.$$

The first-order condition for revenue maximization is simply that at each y the corresponding m satisfy

$$(19) \qquad \frac{\partial R}{\partial m} = 1 - F[\phi(m, y)] - m\frac{\partial F[\phi(m, y)]}{\partial m} = 0,\; 0 \leqslant y \leqslant \infty;$$

that is,

$$(19') \qquad 1 - B(y) = m\phi_m(m, y) f[\phi(m, y)],\; 0 \leqslant y \leqslant \infty.$$

The left-hand side is the increment to aggregate revenue ("marginal revenue") from a small increase of m at given y owing to the presence of $1-B$ persons who would have their taxes increased by that amount given the m for each higher y. The right-hand side is the loss of revenue ("marginal cost") from the same small increase of m owing to the reduction of earnings it would cause that is to be multiplied by the marginal tax rate. These quantities are equal at the maximizing m for each y. The second-order condition is

$$(20) \qquad \frac{\partial^2 R}{\partial_m{}^2} = -2f\phi_m - m[f\phi_{mm} + \phi_m{}^2 f'] < 0,\; 0 \leqslant y \leqslant Y.$$

It is clear from continuity considerations that for each y in the interval $0 < y < Y$ there exists at least one finite m satisfying (19). If we restrict f' to satisfy the inequality in (20) for every m, then optimal m is unique for each such y. In particular, it is clear from (19′) that for all $y < Y$, where we have $1 - B(y) > 0$, the maximizing m is *positive*. Hence $t(y)$ rises monotonically with y.

7. The result in (17) is derivable from integration by parts:
(i) $\int uv'dy + \int vu'dy = (uv)_\infty - (uv)_0$.
Let $u \equiv t$, $v \equiv B$, where $t(0) = 0$, $B(Y) = 1$. Then the right-hand side of (17) is
(ii) $\int (1-v)u'dy = \int u'dy - \int u'v\,dy$
$= u_Y - u_0 - \int u'v\,dy$
$= u_Y v_Y - u_0 v_0 - \int u'v\,dy$ [using $v_Y = 1$, $u_0 = 0$]
$= \int uv'dy$ [from (i)].

It is also implied by (19′) that at $y=Y$, where $1-B=0$, the optimal m equals zero. This is because there is no additional revenue from raising m above zero at Y — there is no higher earning individual whose tax will thereby be raised — while for every $m>0$ there is a certain loss in revenue equal to $t'(Y)\ (dy/dm)>0$ (per largest earner) per unit of any rise of m on the last dollar earned Y. The whole rationale of positive $m(y_1)$ rests on the presence of persons with $y>y_1$ whose tax is thus made higher (given intervening m between y_1 and y) than it otherwise would be. When y_1 is so large that no such persons would be left even though m were to decline as sharply as desired, the case for $m>0$ vanishes.[8] Of course, because $m(y)>0$ for all $y<Y$, $t(Y)>t(y)$ for all $y<Y$; hence there is no inequality in the resulting tax function.

Now as we consider smaller and smaller y away from Y, does $m(y)$ increase throughout? Differentiation of (19) yields the Euler-like equation for the rate of change of m with respect to y:

$$(21)\qquad m'(y)=\frac{f\phi_y+m(f\phi_{my}+\phi f'\phi_y)}{-[2f\phi_m+m(f\phi_{mm}+\phi_m{}^2f')]}.$$

The denominator is negative according to (20). The numerator is positive for m sufficiently close to zero and hence, because m is continuous in y, for y sufficiently close to Y. For larger m we need the sign of ϕ_{my}, which depends upon the unsigned j'''. If, by way of example, we suppose that $xj''/j'=\lambda=\text{constant}>0$, then

$$\phi_{my}=\lambda[x(\lambda+1)(1-m)n]^{-1}>0.$$

In that case there is a presumption that $m'(y)<0$. Only where f' is sufficiently negative is the numerator possibly negative; but $f'<0$ is most likely, in fact, where n and y are large, and there m is small, which tends to make the numerator positive.[9] It should not be sur-

8. Equation (19) leaves no doubt that $m(Y)=0$ when $f(\phi(m,Y))>0$, as in (1). If f vanishes at Y, use l'Hôpital's rule to obtain

$$m(Y)=\lim_{y\to Y}\left[\frac{\dfrac{d}{dy}(1-B(y))}{\dfrac{d}{dy}(\phi_m f)}\right]$$

$$=\frac{-f\cdot\dfrac{dn}{dy}}{f\cdot\left(\dfrac{d\phi_m}{dy}\right)+\phi_m\cdot\dfrac{dn}{dy}\cdot f'}$$

$$=0 \text{ if } f'(\phi(m,Y))<0.$$

See also the control theory analysis in the Appendix.

9. Of course, (20) places a lower bound on f' in relation to optimal m, but this leaves room for a negative numerator, and hence $m'(y)>0$.

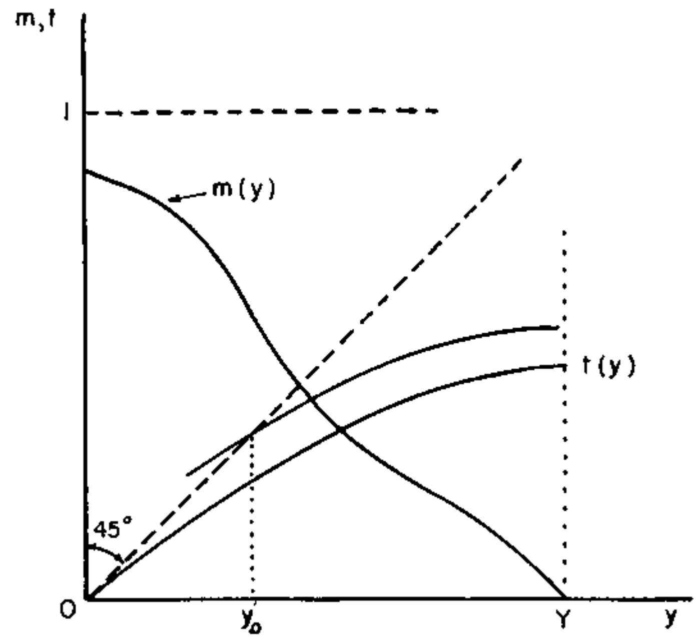

FIGURE II

In the well-behaved case, $m<1$ so $t<y$. If $m>1$ at small y, the tax schedule will cross the 45° line at some earnings level y_0.

prising that, in the absence of restrictions on f', we cannot show m to be *everywhere* decreasing in y.

Before turning to a concrete example of $f(n)$, let us consider the behavior of m as y goes to zero. Equation (19) admits the possibility that m approaches some $m(0)>1$. Then some initial interval will exist, $0<y<y^0$, over which $m(y)>1$ with $m(y^0)=1$, and a larger interval, $0<y\leqslant y_0$, over which $t(y)\geqslant y$, with $t(y_0)=y_0$. If such is the case, then $B(y_0)=B(0)$; no household will fall in this interval. Consequently, there would seem at first to be an unnecessary loss of earning incentives and of tax revenue in so reaching the corresponding $t(y_0)$. It would at first seem that there must be a better way to reach $t(y_0)$, say through still higher m at y near zero and lower m near y^0. If such could be argued, then by taking smaller and smaller y^0, we could thus show $m(y)$ to be always less than one for every $y>0$. But such an argument collapses when we realize that by reason of the conditions for the individual's global utility maximum, if $t(y_0)=y_0$, there will exist no $m(y_0)$ sufficiently small to induce n_0-types to earn $\psi(m, n_0)=y_0$ when $t(y_0)=y_0$. It does not seem possible to exclude the presence of a "shut-down" earnings zone,

$0<y\leqslant y_0$, such that $t(y)\geqslant y$ (with $m(y)<1$ typically for $y\geqslant y_0$). In this region $m(y)$ is indeterminate, though larger than one "on average." The usefulness of this zone is in providing a "running start" in the taxation of people whose abilities will induce them to earn incomes above y_0 when faced with low marginal rates.[1] Accordingly, we have to interpret (19) as valid only for the region, $y_0\leqslant y\leqslant Y$. Note that y_0 is not trivial to determine, since $m(y_0)$ is unknown until y_0 is known.[2] For each y_0 candidate, given $t(0)=0$, $t(y_0)=y_0$, and $m(y)$ from (19) for all $y\geqslant y_0$, one has to calculate maximized revenue, $g(y_0)=\max_m R(m;\, y_0)$, and then optimize y_0. Whether or not optimal $y_0>0$, equation (19) remains valid as a description of marginal tax rates at $y\geqslant y_0$.

Example. Consider the cost function,

$$j(x)=\left(\frac{a}{\lambda+1}\right)x^{\lambda+1},\ \lambda=\text{constant}>0,$$

where λ is the constant elasticity, xj''/j', of marginal cost, $j'=ax^{\lambda}$. Using (8), we have

$$\phi(m,\, y)=\left(\frac{a}{1-m}\right)^{\frac{1}{\lambda+1}}y^{\frac{\lambda}{\lambda+1}}$$

$$\phi_m(m,\, y)=\frac{\phi(m,y)}{(1-m)(\lambda+1)}.$$

In this example, formula (19) may be written

$$1-F(\phi)=\frac{m}{1-m}\phi F'(\phi)\frac{1}{\lambda+1}.$$

We see again that $F'>0$ implies $m=0$ at $F(\phi)=1$; that is, $m(Y)=0$. For $\phi<N$ we may divide by both m and $1-F$ to obtain

$$\frac{1-m}{m}=\frac{F}{1-F}\cdot\left(\frac{\phi F'(\phi)}{F}\right)\frac{1}{\lambda+1}.$$

Hence $0<m<1$ wherever $\phi>0$ and $0<F<1$, remembering that $F'(\phi)>0$. In the limit, as ϕ approaches zero, $(1-m)$ goes to zero. Hence $m(0)=1$. There is no $y_0>0$, such that $t(y_0)=y_0$.

In the extraordinary event that the elasticity of F is constant

1. In principle there could be a further zone, $(y_2,\, y_3)$, over which "ineffective" marginal rates would exceed one for the same kind of purpose if a discontinuous jump in $t(y)$ at y_2 were desirable but not allowed. No such jump in t is desired in the present model in view of (19) and our continuity assumptions.

2. With m continuous in (19) and assuming $m'(y)<0$, $t(y_0)=y_0$ implies $y_0>y^0$, the latter uniquely determined by
$$1-F[\phi(1,\, y^0)]=\phi_m(1,\, y^0)f[\phi(1,\, y^0)].$$

— this rules out $F(0) > 0$ — and equal to $(\lambda+1)$, we have $m = 1 - F$. One's marginal income-retention rate, $Z'(y) = 1 - m(y)$, is equal to the percentile score of one's income. In the case of the rectangular or uniform distribution, it is easily checked from the derivative dm/dF that m falls more slowly at first, beginning with slope $(\lambda+1)^{-1}$, and faster at the end, with final slope $\lambda+1$.

Generally, for any constant elasticity function F, m declines *monotonically* with increasing $F(\phi)$, that is, $m'(y) < 0$ for all y, $0 \leqslant y \leqslant Y$. This follows from

$$1 - m = \frac{\beta F}{1 - (1-\beta) F}, \quad 0 \leqslant F \leqslant 1,$$

where β denotes the ratio of the distribution elasticity to $(\lambda+1)$. There is no closed-form expression for m as a function of y, even in the rectangular case, and the differential equation in $m'(y)$ adds little to what has already been said. However, the cost-function example is seen to be very powerful *computationally*. By considering β a function of F, all manner of distributions yields to computation of $m(\phi)$, whence ultimately $m(y)$.

B. The Effort Incentive Model

This model is more complex than the previous one for its admission of income effects upon effort x of changes in the net tax $t(y) - g$. In place of (5) we write the ordinal utility function

(5B) $\quad u = u(c, x),\ u_1(c, x) > 0,\ u_2(c, x) < 0\ (x > 0).$

Each individual maximizes his utility subject to

(6B) $\quad c = nx - t(nx) + g \geqslant 0,\ 0 \leqslant n \leqslant N.$

We can deduce the implications

(7B) $\quad$ (a) $\{u_{11}u_2^2 - 2u_1u_2u_{12} + u_{22}u_1^2\}u_1^{-2} \equiv D < 0,$

(b) $u_{11}\left(\frac{-u_2}{u_1}\right) + u_{21} < 0,$

(c) $u_{22} + u_{21}\left(\frac{-u_2}{u_1}\right) < 0,$

from the postulates that each individual achieves an interior utility maximum for which the first-order condition is

(8B) $\quad \frac{du}{dx} = (1-m)nu_1(c, x) + u_2(c, x) = 0,$

and the second-order condition,

(9B) $\quad \frac{d^2u}{dx^2} = -n^2m'(y)u_1 + D < 0,$

with stipulated properties that $\psi(t, m, n; g) = n\chi(t, m, n; g)$ satisfy

$$\text{(10B)}\quad 0 < \psi_t \equiv \frac{\partial y}{\partial t} = n\left[u_{11}\left(\frac{-u_2}{u_1}\right) + u_{21}\right]D^{-1},$$

$$0 > \psi_m \equiv \frac{\partial y}{\partial m} = n^2 u_1 D^{-1}$$

$$0 < \psi_n \equiv \frac{\partial y}{\partial n} = \left\{-u_1\left(\frac{-u_2}{u_1}\right) + x\left[u_{22} + u_{21}\left(\frac{-u_2}{u_1}\right)\right]\right\} D^{-1}.$$

The first of these postulates states that leisure is a normal good, and the last that leisure decreases with n assuredly, no matter how strong the income effect on consumption (so that consumption must also be normal).

Then one can calculate

$$\text{(11B)}\quad \frac{dy}{dn} = \frac{\psi_n}{1 - m'\psi_m - m\psi_t}$$

$$= \frac{-(1-m)nu_1 + x\left[u_{22} + u_{12}\left(\frac{-u_2}{u_1}\right)\right]}{-n^2 m' u_1 + D} > 0,$$

and, using (6B) and (8B),

$$\text{(12B)}\quad \frac{du}{dn} = u_1\frac{dc}{dn} + u_2\frac{dx}{dn} = u_1(1-m)x > 0.$$

Our problem again is to maximize aggregate tax revenue so as to achieve the largest feasible minimum utility, $u(g, 0)$. For this purpose we again invert $\psi(t, m, n; g)$ to obtain $n = \phi(t, m, y; g)$ with the properties,

$$\text{(14B)}\quad \frac{\partial n}{\partial t} \equiv \phi_t(t, m, y; g) = \frac{-\psi_t(t, m, y; g)}{\psi_n(t, m, y; g)} < 0,$$

$$\frac{\partial n}{\partial m} \equiv \phi_m(t, m, y; g) = \frac{-\psi_m(t, m, y; g)}{\psi_n(t, m, y; g)} > 0,$$

$$\frac{\partial n}{\partial y} \equiv \phi_y(t, m, y; g) = \frac{1}{\psi_n(t, m, y; g)} > 0.$$

Hence,

$$\text{(15B)}\quad B(y) = F[\phi(t, m, y; g)];$$

$$B(0) = F(0) \geqslant 0,\ B(Y) = F(N) = 1;$$

$$b(y) \equiv B'(y) = F'(\phi)\,[\phi_t m + \phi_m m' + \phi_y] = f(\phi)\frac{dn}{dy} > 0.$$

Once again one could maximize

$$\text{(16B)}\quad R = \int_0^\infty t(y)b(y)dy,\ s.t.\ t'(y) = m(y),\ t(0) = 0,$$

where $m'(y)$ is the control variable. (See the Appendix.) But as in the Sheshinski model it is more expedient to cast the problem thus:

$$\text{(18B)} \quad \underset{\{m(y)\}}{\text{maximize}}\, R(m;\, t_0) = \int_0^\infty m \cdot (1-F[\phi(t, m, y;\, g)]dy$$

$$\text{subject to (a)} \quad t'(y) = m(y), \quad \text{(b)} \quad t(0) = t_0 = 0,$$

where it is understood that $g = \max R(m;\, 0) - \gamma - \sigma > 0$. This is a "dynamic" programming problem unlike (18), since $t(y)$ at any $y_1 > 0$ affects $B(y_1)$, and it depends on $m(y)$ at "earlier" y, $y < y_1$.

The first-order condition for maximizing m is the Euler differential equation:

$$\text{(19B)} \quad \frac{d}{dy}\{1-F(\phi) - m\phi_m f(\phi)\} = -mf(\phi)\phi_t, \quad 0 \leqslant y \leqslant Y.$$

The second-order requirement is the Legendre condition,

$$\text{(20B)} \quad \frac{\partial^2 R}{\partial_m{}^2} = -2f\phi_m - m[f\phi_{mm} + \phi_m{}^2 f'] < 0, \; 0 \leqslant y \leqslant Y.$$

Carrying out the differentiation in (19B) leads to the analogue of (21):

$$\text{(21B)} \quad m'(y) = \frac{f\phi_y + m(f\phi_{my} + \phi_m f'\phi_y + mf\phi_{mt} + mf'\phi_m\phi_t)}{-[2f\phi_m + m(f\phi_{mm} + \phi_m{}^2 f')]}.$$

It can be argued that the first integral of (19B) is

$$\text{(19'B)} \quad (1 - \int_0^Y m(s)f(\phi(s))\phi_t(s)ds) - F - m\phi_m f$$
$$= -\int_0^y m(s)f(\phi(s))\phi_t(s)ds$$

or, equivalently,

$$\text{(19''B)} \quad m\phi_m f(\phi) = 1 - F(\phi) - \int_y^Y m(s)f(\phi(s))\phi_t(s)ds.$$

It follows at once that $m(Y) = 0$ along the same reasoning as in the previous model. For all $y < Y$, $m(y) > 0$. Also if ϕ_m goes to infinity fast enough as m approaches one (as in the Example), then $m(0) \leqslant 1$, thus precluding complications of $t(y) \geqslant y$ near $y = 0$.

In model A where $\phi_t \equiv 0$, $1-F$ is interpretable as the "shadow price" of higher $t(y)$ at y, to which quantity the "marginal cost" of higher m is to be equated. Here, $1-F$ understates the shadow price because an increase of $t(y)$ at any $y < Y$ would reduce $\phi(s)$ for all $y \leqslant s \leqslant Y$ (recalling $\phi_t < 0$) and hence would increase $1 - B(s)$ so as to bring in the extra revenue yield indicated by the integral expression. Thus if ϕ_m is comparatively insensitive to $t(y)$ at $y = \psi(1, m, \phi)$ at each ϕ, (19''B) affords the presumption that m tends to be higher for every ϕ when $\phi_t < 0$ than in model A where $\phi_t \equiv 0$. If this presumption is correct, it accords with intuition: The income effect of higher taxation inducing greater effort, expressed by $\psi_t > 0$, presents an extra incentive for higher m — in the sense of m tending to decline with rising y more slowly. In this case, because the aggregate net tax collected is constrained to equal $\gamma+\sigma \geqslant 0$ in both models, g is higher in model B and $t(y)$ is higher for every $y > 0$.

III. Concluding Remarks

The main findings in this paper, I believe, are these:

1. If the least advantaged cannot earn income in the market, the maximin rule calls for the realization of taxable capacity.

2. At incomes and associated marginal tax rates where the density function of ability is nondecreasing, the marginal tax rate is a declining function of earnings under a plausible condition regarding the marginal costs of earnings.

3. If the largest ability level is finite and the slope of the density function does not go to zero as ability approaches this upper bound, then the marginal tax rate goes to zero as household earnings approach the corresponding largest earnings level.

4. The optimal taxation of low earnings and, possibly, the income effects of the governmental income grant may deter a range of low-earning-ability persons from earning income.

Questions will be (and have been) raised about the generality of these findings and properly so. Let me, in a very short space, record and respond to some of these doubts.

It may be objected that the least-favored class in society ought not to be identified with those persons totally incapable of earning any market income. If the persons designated "least advantaged" can earn income, their utility would not generally be maximized by the government's maximization of its tax revenue and resulting grants. Then proposition 1 would not apply, and proposition 2 might not hold. The definition of the bottom class may be the Achilles' heel of the Rawlsian system. It is not clear, however, why in the usual run of cases the victims of disaster should be viewed as a different species who are not to count in the census of utilities.

It has been objected regarding proposition 3 that there exists no ability level such that the probability of encountering a higher ability is zero. (The Pareto distribution, often employed in this context, possesses an infinite tail.) This paper analyzes today's maximin tax function with reference to the ex post function describing today's distribution of abilities; my critics have in mind the design today of a future tax system, and they refer to an ex ante function. My first line of defense is that if we are designing a maximin tax system for a tomorrow in which today's people will predominate, the relevant distribution is a *conditional* ex ante distribution that is largely shaped by today's ex post distribution amassed over some finite ability range; then today's ex post function is a better approximation of that conditional distribution than is the "ahistorical" or "stationary" ex ante function. The second line of defense is that surely tomorrow's GNP is bounded in view of the fixity of cooperating fixed resources and diminishing returns; hence tomorrow's individual personal incomes before tax and corresponding ability levels must be bounded as well.

Finally, à propos proposition 4, and here I enter a doubt of my own, it may be wondered whether the expansion of the class of persons finding it inoptimal to work for market wage income might not enlarge the set of persons in the next generation who, lacking the necessary models, would be unable to earn income. Presumably, we do not want to increase minimum utility within the present generation to the extent of reducing minimum utility in the next generation below present minimum utility. As Graaff and others saw decades ago, the various intertemporal linkages in an economy make the implications of intrageneration justice dependent upon the conception of justice among the generations.

Appendix

The maximization in model B as posed in (18B) is a variational problem that can be approached by the methods of control theory. The corresponding Hamiltonian is

$$\text{(i)}\quad H(t,m,y)=m[1-F(\phi(t,m,y))]+qm,$$

where t is the state variable and q the co-state variable. The necessary conditions for a maximum, subject to the constraint $t^*=m$, are

$$\text{(ii)}\quad q^*=\frac{-\partial H}{\partial t}=m\,f\,\phi_t,$$

$$\text{(iii)}\quad \frac{\partial H}{\partial m}=1-F-mf\phi_m+q=0,$$

$$\text{(iv)}\quad q(Y)t(Y)=0,\ \ q(0)t(0)=0,$$

where an asterisked variable denotes the total derivative of that variable with respect to y.

From (ii) and (iii)

$$\text{(v)}\quad 1-F-mf\phi_m=-q=-[c+\int_0^y mf\phi_t ds],$$

where c is the constant of integration. To determine c, it can be argued that $q(y) \geqslant 0$ in view of the favorable income effect upon work incentives of increased $t(y)$, that is, $\psi_t(t, m, y) > 0$. Hence, by (iii) $m(y) \geqslant 0$ with the strict inequality for some y. Therefore $t(Y) > 0$, and by (iv) $q(Y) = 0$. Hence,

$$\text{(vi)}\quad c = -\int_0^Y mf\phi_t ds$$

$$\text{(vii)}\quad mf\phi_m = 1 - F - \int_y^Y mf\phi_t ds.$$

The latter is (19″B) of the text, the first integral of the Euler equation.

In the best behaved cases in which $t(y) < y$ everywhere, (vii) appears to be adequate to compute a discrete approximation to the $m(y)$ function: For any trial value of $t(Y)$, calculate Y using $m(Y) = 0$ and $N = \phi(t(Y),\ 0,\ Y)$; then use (vii) to calculate $m(Y-1)$ and so on; then find the $t(Y)$ value such that the corresponding $\Sigma m(y) = t(Y)$. Or use the Euler equation to calculate $m(y)$ for each trial $m(0) \leqslant 1$ and choose the latter one that satisfies (vii).

These necessary conditions are said to locate a global maximum rather than merely to characterize relative maxima if the maximized Hamiltonian is concave in the state variable t. We have

$$\frac{dH}{dt} = -mf\phi_t + \frac{dm}{dt}\frac{\partial H}{\partial m} = -mf\phi_t$$

$$\frac{d^2H}{dt^2} = -m^2[\phi_t f' \phi_t + f\phi_{tt}] - m^*[f\phi_t + m(f'\phi_m\phi_t + f\phi_{tm})],$$

where for m^* we may substitute from the Euler equation (derivative of (vii)). It is by no means apparent that $d^2H/dt^2 \leqslant 0$ for all admissible f and ϕ functions. In model A where $\phi_t \equiv 0$, $d^2H/dt^2 = 0$, as desired.

The general problem in its original formulation, as expressed in (4), (13), and (16B), is

$$\text{(viii)}\quad \text{maximize } R = \int_0^\infty t\, f[\phi(t, m, y)]\phi^*\, dy$$

$$\text{subject to } t^* = m,\ t(0) = 0.$$

This may also be approached directly (without employing integration by parts to rewrite the integrand) using the methods of control theory.

Let $u(y) \equiv m^*(y)$ denote the control variable. The corresponding Hamiltonian function is

$$\text{(ix)}\quad H(t, m, u, y) = tf[\phi(t, m, y)](\phi_t m + \phi_m u + \phi_y) + q_1 m + q_2 u.$$

Here q_1 and q_2 are the co-state variables, and t and m are the state variables. (Because u is unbounded, m need not be continuous.) The necessary conditions for a constrained maximum are

(x) $q^*_1 = -\dfrac{\partial H}{\partial t} = -\left\{ f\phi + tf\dfrac{\partial \phi^*}{\partial t} + t\phi_t f'\phi^* \right\}$,
$0 \leqslant y \leqslant \infty$,

(xi) $q^*_2 = -\dfrac{\partial H}{\partial m} = -\left\{ t\left[\phi_m f'\phi^* + f\dfrac{\partial \phi^*}{\partial m}\right] + q_1 \right\}$,
$0 \leqslant y \leqslant \infty$

(xii) $\dfrac{\partial H}{\partial u} = tf\phi_m + q_2 = 0,\ 0 \leqslant y \leqslant \infty$

(xiii) $q_1(Y)t(Y) = 0,\ q_2(Y)m(Y) = 0$

(xiv) $q_1(0)t(0) + q_2(0)m(0) = 0.$

Differentiation of (xii) yields

(xv) $q^*_2 = -\{mf\phi_m + tf'\phi^*\phi_m + tf\phi^*_m\}.$

Equating the right-hand side of (xi) and (xv) and noting that

$$\frac{\partial \phi^*}{\partial m} = \phi_{tm}m + \phi_{mm}u + \phi_{ym} + \phi_t = \phi^*_m + \phi_t,$$

we obtain

(xvi) $mf\phi_m - tf\phi_t = q_1.$

Upon calculating that $\partial\phi^*/\partial_t = \phi^*_t$, we may write (x) as

(xvii) $q^*_1 = -[f\phi^* + t(f^*\phi_t + f\phi^*_t)]$
$$= -\left[\frac{dF}{dy} + \frac{d}{dy}(tf\phi_t) - mf\phi_t\right].$$

Hence, from (xvi) and (xvii)

(xviii) $mf\phi_m - tf\phi_t = c - F + \int_0^y mf\phi_t ds - tf\phi_t (= q_1),$

where c is the constant of integration. Differentiation of (xviii) gives (21B) of the text.

In model A where $\phi_t \equiv 0$, we argue that $t(Y) > 0$ so that by (xiii), $q_1(Y) = 0$. Then $c = 1$, and

(xix) $mf\phi_m = 1 - F,$

as in (19) derived classically in the text and as a special case of (vii) derived above.

In model B $q_1(Y) = 0$ implies

$mf\phi_m - tf\phi_t = 0 \quad$ at $y = Y,$

and $m(Y)q_2(Y) = 0$ implies that

$mtf\phi_m = 0 \quad$ at $y = Y.$

Together these equations imply that, as was to be shown,

$mf\phi_m = 0 \quad$ at $y = Y,$

but only at the cost of implying the perplexing further result,

$tf\phi_t = 0 \quad$ at $y = Y.$

The latter implication, it might be noted, also follows from the weaker transversality condition, as though $t(Y)$ or $m(Y)$ were bounded,

$$t(Y)q_1(Y)+m(Y)q_2(Y)=0.$$

Then, at $y=Y$,

$$0=t[mf\phi_m-tf\phi_t]+m[-tf\phi_m]=-t^2f\phi_t.$$

Thus this alternative proof of (19″B) is something of a puzzle.

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LINEAR 'MAXIMIN' TAXATION OF WAGE AND PROPERTY INCOME ON A 'MAXIMIN' GROWTH PATH

While theoretical welfare economists used to talk about social welfare, they long did very little with it – as students in the Sixties came increasingly to complain. Most studies, those in the line of Debreu and Koopmans, stayed cautiously with the notion of Paretian efficiency. And it was not apparent, not evidently to the Paretians, that the use by Samuelson and Graaff of the Bentham–Bergson social welfare function advanced the content of policy analysis much beyond the Paretian approach. The trouble lay in the neoclassical postulate of "perfect information" and its corollary, the lump-sum tax. Without allowance for disincentive effects, the distributive aspects of taxation (and public policy generally) were rendered an analytical triviality. For the maximization of social welfare, factors were to receive their marginal social products and transfers were to adjust incomes.

The face of welfare economics is now greatly changed in this respect. Mirrlees (1971) broke with the postulate of perfect information, investigating for the first time the optimal character of the graduated tax on wage income and the optimal level of the income supplement or "negative income tax". In the same year, Rawls (1971) put the finishing touches on his theory of distributive justice, which has done as much to stimulate economic research on redistributional public policies as it has to break the Benthamian habit of summing utilities. These two innovations have in a very few years loosed an outpouring of papers on the welfare theory of redistribution by actually available governmental mechanisms – such as the proportionate or graduated taxation of various kinds of income. Most of these new studies employ Mirrlees's premise of a one-dimensional continuum of abilities or advantages found predeterminedly within the population. Many also employ Rawls' "maximin" criterion – we are to maximize the opportunities of the least advantaged.

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The contribution here is the unavoidable offspring of two Rawlsian papers which study economies with overlapping two-period generations. A paper by Janusz Ordover and me has analyzed, for a society with exactly replicating Mirrleesian inequality of advantage, the "maximin" pair of linear tax rates on wage income and property income, respectively, on the condition that transfers and taxes combine to maintain the economy in a stationary state. This no-growth condition purposely side-stepped the question of intergeneration justice. Because the current generation is not allowed to "trade off" between the stocks of capital and public debt (the two-state variables) for mutual gain, the results obtained there must seem too special to be satisfying. A subsequent paper by John Riley and me resolved the trade-off problem by the relentless extension of the "maximin" criterion to justice among generations as well as to justice within a generation. But to solve the problem of *inter*generation "maximin" in full, the authors took refuge in the postulate that every generation is homogeneous, every member having the same productivity as well as the same tastes.

The present paper tackles one aspect of the "general" problem in which fiscal policy is Rawlsian *within* and *between* generations and each generation shows the same Mirrleesian inequality of advantages. Specifically, I develop the conditions for optimal taxation by the current generation in the manner of partial equilibrium analysis – taking as "given" in each time period the social rate of discount whose determination must await the ultimate intertemporal general equilibrium analysis.

1. Conditions for "Maximin" Taxation on a "Maximin" Path

Let $\Phi(m)$ denote the proportion of young workers in each and every generation whose ability or advantage is m or less, $m \geq 0$.

The natural specifications are

$$0 \leq \Phi(0) < 1 = \Phi(M), \qquad 0 < M \leq \infty,$$

and (1)

$$\Phi(a) < \Phi(b), \qquad \text{when} \quad 0 \leq a < b \leq M.$$

An individual of type m in generation v, $v \geq 0$, works an amount denoted e_v^m in his youth, consumes at the end of his youth an amount denoted c_v^m, and consumes in old age an amount denoted x_v^m. The corresponding amounts *per head* in this generation, all non-negative, are

$$e_v = \int_0^M e_v^m \, \mathrm{d}\Phi(m), \quad c_v = \int_0^M c_v^m \, \mathrm{d}\Phi(m), \quad x_v = \int_0^M x_v^m \, \mathrm{d}\Phi(m). \tag{2}$$

The "effective" labor supplied by an individual depends upon his m in a proportional way, namely $l_v^m = me_v^m$. Then aggregate effective labor *per head* is

$$l_v = \int_0^M l_v^m \, \mathrm{d}\Phi(m) = \int_0^M me_v^m \, \mathrm{d}\Phi(m). \tag{3}$$

Production per head by generation v is a neoclassical function, F of effective labor per head and existing capital per head left by the previous young generation, k_{v-1}. Generations are of constant size over time, so that this generation's deposit, k_v, for the next generation is simply this period's unconsumed gross output,

$$k_v = F(k_{v-1}, l_v) - c_v - x_{v-1} \geq 0. \tag{4}$$

The production function has the property that the net marginal products, the derivatives of the net product function $Y(k, l) = F(k, l) - k$, are everywhere positive, and Y exhibits constant returns to scale.

Members of generation v all receive a demogrant, g_v, in the first period, and a further installment, h_v, in their second period. Under linear taxation of wages and interest, the budget equation for households of type m in its two-part extensive form is

$$x_v^m = h_v + (1 + \rho_v)\sigma_v^m, \tag{5a}$$

$$c_v^m = g_v + \omega_v l_v^m - \sigma_v^m. \tag{5b}$$

Here $\omega_v \geq 0$ is the after-tax reward to one unit of effort by individuals of the "standard" $m = 1$ type, and $\rho_v \geq -1$ is the after-tax rate of return to anyone's private saving.

These rates of remuneration are to be contrasted to $F_l(k_{v-1}, l_v)$, the marginal product of standard labor, and $F_k(k_v, l_{v+1}) - 1 = Y_k(k_v, l_{v+1})$, the social rate of return to investment in period v. It is natural, though inessential, to suppose that before-tax rates of remuneration, denoted w_v and r_v, are equal to these marginal products,[1]

$$w_v = F_l(k_{v-1}, l_v) > 0, \qquad r_v = F_k(k_v, l_{v+1}) - 1 > -1. \tag{6}$$

At least, F_l and $F_k - 1$ are the shadow prices of labor and saving relative to which we shall define our unit taxes, or specific tax rates,

$$t_v^w = w_v - \omega_v, \qquad t_v^r = r_v - \rho_v. \tag{7}$$

[1]In any case, under conditions of perfect markets, the before-tax wage of m_1 types would be m_1/m_2 times the before-tax wage of any other type, m_2; and all types would face the same before-tax return to saving. Hence, under linear taxation (constant proportionate rates of taxation), the after-tax wage of m_1 types would also be m_1/m_2 times the after-tax wage of m_2 types, and all types would face the same after-tax return to saving.

Each household is supposed to choose its (c^m, x^m, l^m) so as to maximize a utility function, $U(c^m, x^m, e^m)$, which is independent of m, subject to the individualized budget restraint in (5) corresponding to its m. This leads to a labor supply function, $l^m(\rho, m\omega, g, h)$, and a supply-of-wealth (or saving) function, $\sigma^m(\rho, m\omega, g, h)$, for households of type m. We may write the corresponding aggregate supply functions as

$$l = l(\rho, \omega, g, h), \qquad \sigma = \sigma(\rho, \omega, g, h). \tag{8}$$

We will require, neglecting h for a moment, that

$$\omega > 0, \quad \text{if} \quad l > 0; \qquad \rho > -1, \quad \text{if} \quad \sigma > 0, \tag{8a}$$

$$l_g < 0, \quad \text{if} \quad l > 0; \qquad \sigma_g > 0, \quad \text{if} \quad \rho > -1, \quad 1 - \sigma_g > 0, \tag{8b}$$

$$\omega l_\omega + l > 0, \qquad (1+\rho)\sigma_\rho + \sigma > 0, \tag{8c}$$

$$\sigma_\omega > 0, \qquad \omega l_\omega + l - \sigma_\omega = \partial c/\partial\omega > 0, \quad \text{if} \quad l > 0. \tag{8d}$$

The first stipulation is that work and saving are functional, not compulsive (a tall assumption). The next two conditions state that leisure and present and future consumption are normal goods. The last condition states that the rise of "lifetime" consumption resulting from a rise of ω is divided positively between present and future consumption (a plausible assumption).

Each household's maximization also leads to its maximized utility, which quantity is some function of its parameters, $V^m(\rho, m\omega, g, h)$. The "maximin" society here is concerned in particular with the least of these utilities, the utility level achievable by the generation's poorest (non-working) members,

$$V^0(\rho, 0, g, h) = \max_{\{c^0, x^0, e^0\}} [U(c^0, x^0, e^0) \quad \text{s.t.} \quad x^0 = h + (1+\rho)(g - c^0)]. \tag{9}$$

Of course, the equations of the past two paragraphs apply to any generation so there is no need to subscript all variables by their v.

Obviously the least well-off, because they find it inoptimal to work, feel no direct effect, $\partial V^0/\partial\omega$, upon their utility from a change of ω. The latter will affect them only through its indirect effects upon g, h, and ρ. Now ρ will have a direct effect only in proportion to their saving, $\sigma^0 = g - c^0$. But the latter is arbitrary since the state is free to juggle g and h, subject to constancy of $(1+\rho)g + h$, without thereby altering people's decisions about (c^m, x^m, l^m). The larger h and the smaller g, the smaller is σ^0. For the sake of definiteness, and because the assumption appears the most natural one, I impose the condition that h is *fixed* according to

$$h = x^0(\rho, 0, g, h), \quad \text{i.e.,} \quad \sigma^0(\rho, 0, g, h) = 0, \tag{10}$$

in which the demand (or supply) functions are evaluated at the *optimal* ρ and g. Thus, for each generation, h_v is just large enough to cause zero saving by the poor (those with zero wage income) at the fiscal optimum. Hence $\partial V^0/\partial\rho = 0$ around that optimum. Finally, we have $\partial V^0/\partial g > 0$. The poor of generation v want any change of ω_v and ρ_v that offers greater g_v, given the properly fixed h_v. But too large a g_v will drive the maximum feasible $V^0(.)$ of subsequent generations below the level in generation v. Let us now formulate this intergeneration problem.

Define j_v as the maximum of the minimum $V^0(\rho_i, 0, g_i, h_i)$ over the current generation and those subsequent,

$$j_v = \max \min V^0(\rho_{v+i}, 0, g_{v+i}, h_{v+i}), \qquad i \geq 0. \tag{11}$$

Now it might be wondered whether there exists only an infimum, not a minimum, like the smallest piece of cake over infinitely many slices – size zero, though no actual piece need be so small. In that case, we would have a "max-inf" problem and an infinitude of solutions to it. However, on our assumptions of no capital saturation, $Y_k > 0$, and no utility satiation, $V_g > 0$, it is fairly obvious that the maximized minimum does indeed exist for every initial state, i.e., the infimum is attained by some or all $V^0(.)$ – provided that x_{v-1} is not too large to permit subsistence, or at any rate positive, levels of g and h for subsequent generations.

Of course, j_v is some function of the current or then-initial state, (k_{v-1}, x_{v-1}), say $j(k_{v-1}, x_{v-1})$, independent of v. Due to the happy commutative property of the minimum function, $\min(a, b, c) = \min(a, \min(b, c))$, one can write the following equation in the unknown function j:

$$j(k_{v-1}, x_{v-1}) = \max_{\{\rho_v, \omega_v, g_v\}} \{\min[V^0(\rho_v, 0, g_v, h_v), j(k_v, x_v)]\}, \tag{12}$$

subject to

$$k_v = F(k_{v-1}, l_v) - x_{v-1} - (g_v + \omega_v l_v - \sigma_v),$$

$$x_v = h_v + (1 + \rho_v)\sigma_v,$$

and h_v such that $\quad 0 = \sigma^0(\rho_v, 0, g_v, h_v) \quad$ at maximizing $\{\rho_v, \omega_v, g_v\}$.

In a "maximin" society, whatever the $\{\rho_v, \omega_v, g_v\}$ that generation v decides upon and the current $V(\rho_v, 0, \omega_v, g_v, h_v)$ that results, the pertinent minimum over the *subsequent* $V^0(.)$ is the *maximized* minimum achievable by subsequent generations, namely $j(k_v, x_v)$; current generation will take that into account in its "maximin" endeavor.

One property of the optimal path can quickly be seen: The lifetime utility of the poor is equalized across generations. Assume that, for every v, $V^0(\rho_v, 0, g_v, h_v) = j(k_v, x_v)$; then $j(k_{v-i}, x_{v-i}) = j(k_v, x_v)$ for every v, whence $V^0(\rho_{v+i}, 0, \omega_{v+i}; g_{v+i}; h_{v+i}) = j(k_{v-1}, x_{v-1})$ for all $i = 0, 1, \ldots$. To show that $V^0(\rho_v, 0, g_v, h_v) = j(k_v, x_v)$ we argue that $j_k(k_v, x_v) > 0$, $j_x(k_v, x_v) < 0$, again on grounds of non-saturation of capital and non-satiation of utility. Then, if $V^0(.) \neq j(.)$ *were* optimal for some v, one could always raise g_v, thus raising $V^0(.)$ and reducing $j(k_v, x_v)$, or else lower g_v, thus doing the reverse; one or the other action will raise the smaller of V^0 and $j(.)$. For example, if $V^0(.)$ were optimally greater than $j(.)$, it would be better to reduce g_v, which must reduce $V^0(.)$ but raise $j(.)$ in compensation because $\partial k_v/\partial g_v < 0$ and $\partial x_v/\partial g_v > 0$; so $V^0(.)$ could not optimally be greater than $j(.)$.

The above proposition is illustrated in Figure 1, where (ρ_v, ω_v) are fixed at their optimal values and h_v according to the convention in (10). The instrument g_v can be viewed as the equalizer. Correspondingly we may view (ρ_v, ω_v) as assigned to maximizing $j(k_v, x_v)$ when evaluated at the optimal g_v and conventional h_v. In short, (ρ_v, ω_v) are aimed at maximizing future utility possibilities while g_v offers the present generation and equal share in the outcome. Eventually I shall give an interpretation of optimal (ρ_v, ω_v) in terms of total tax revenue. A more specific interpretation presents itself if we assign to ρ_v, say, the task of determining the optimal

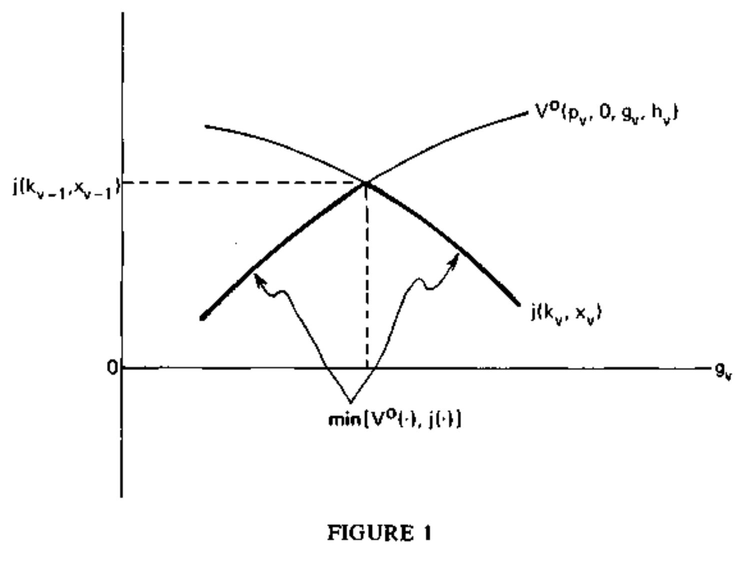

FIGURE 1

x_v: *given* that x_v, ω_v must be maximizing k_v (taking into account the accompanying variation in the required ρ_v), thus "maximizing growth" in order to maximize $j(.)$.

In view of the implied equalization of "least utility" over generations, we may formulate a new equation in our unknown function, j. In this equation, λ_v is a Lagrange multiplier; like $j(k_{v-1}, x_{v-1})$, λ_v is independent of $\{\rho_v, \omega_v, g_v\}$, being a function only of the predetermined state variables facing generation v, (k_{v-1}, x_{v-1}),

$$j(k_{v-1}, x_{v-1}) = \max_{\{\rho_v, \omega_v, g_v\}} \{V^0(\rho_v, 0, g_v; h_v) + \lambda_v[j(k_v, x_v) - j(k_{v-1}, x_{v-1})]\}, \tag{13}$$

subject to

$$k_v = F(k_{v-1}, l_v) - x_{v-1} - (g_v + \omega_v l_v - \sigma_v),$$
$$x_v = h_v + (1 + \rho_v)\sigma_v,$$
$$0 = \sigma^0(\rho_v, 0, g_v, h_v) \quad \text{at optimal } \{\rho_v, \omega_v, g_v\}.$$

The first-order conditions for an interior maximum of the right-hand side are

$$\frac{\partial V^0}{\partial \rho_v} + \lambda \left\{ j_k(k_v, x_v) \frac{\partial k_v}{\partial \rho_v} + j_x(k_v, x_v) \frac{\partial x_v}{\partial \rho_v} \right\} = 0, \tag{14a}$$

$$\frac{\partial V^0}{\partial \omega_v} + \lambda \left\{ j_k(k_v, x_v) \frac{\partial k_v}{\partial \omega_v} + j_x(k_v, x_v) \frac{\partial x_v}{\partial \omega_v} \right\} = 0, \tag{14b}$$

$$\frac{\partial V^0}{\partial g_v} + \lambda \left\{ j_k(k_v, x_v) \frac{\partial k_v}{\partial g_v} + j_k(k_v, x_v) \frac{\partial x_v}{\partial g_v} \right\} = 0. \tag{14c}$$

These conditions contain the derivatives of the unknown function j. Nevertheless we may differentiate (13) to obtain, at the fiscal optimum,

$$j_k(k_{v-1}, x_{v-1})(1 - \lambda_v) = \lambda_v j_k(k_v, x_v) F_k(k_{v-1}, l_v), \tag{15a}$$

$$j_x(k_{v-1}, x_{v-1})(1 - \lambda_v) = \lambda_v j_k(k_v, x_v)(-1). \tag{15b}$$

With $j_k(.) > 0$, therefore, we find

$$j_x(k_{v-1}, x_{v-1}) = -j_k(k_{v-1}, x_{v-1}) F_k(k_{v-1}, l_v)^{-1} < 0, \tag{16}$$

$$\frac{\lambda_v}{1 - \lambda_v} = \frac{j_k(k_{v-1}, x_{v-1})}{j_k(k_v, x_v) F_k(k_{v-1}, l_v)} > 0. \tag{17}$$

Hence, the "marginal utility" to future generations of reducing "our" x_v is equal to the "marginal utility" of our increasing "their" k_v *discounted* by

$F_k(k_v, l_{v+1})$, the gross marginal product of their initial k_v, evaluated at their optimal l_{v+1}. Also $0 < \lambda_v < 1$.

Recalling the notation $1 + r_v = F_k(k_v, l_{v+1})$, $w_v = F_l(k_{v-1}, l_v)$, the earlier results, $\partial V^0/\partial\rho = 0$, $\partial V^0/\partial\omega = 0$, and using (16) and (17), we can express the first-order conditions in the form

$$(w_v - \omega_v)\frac{\partial l_v}{\partial \rho_v} + (r_v - \rho_v)\frac{\partial \sigma_v^*}{\partial \rho_v} = \sigma_v^*, \qquad \sigma_v^* \equiv (1 + r_v)^{-1}\sigma_v, \tag{18a}$$

$$(w_v - \omega_v)\frac{\partial l_v}{\partial \omega_v} + (r_v - \rho_v)\frac{\partial \sigma_v^*}{\partial \omega_v} = l_v, \tag{18b}$$

$$(w_v - \omega_v)\frac{\partial l_v}{\partial g_v} + (r_v - \rho_v)\frac{\partial \sigma_v^*}{\partial \rho_v} = 1 - \frac{\partial V^0/\partial g_v}{\lambda_v j_k(k_v, x_v)}(<1). \tag{18c}$$

Of course the coefficients in parentheses are the tax rates, t_v^w and t_v^r.

2. Properties of the "Maximin" Tax Rates

The above result is recognizable as nominally identical to the conditions on taxation derived in Ordover and Phelps (1975), particularly in their Sections C and D where capital is variable. Much the same bag of tricks used there can be employed here to deduce the following propositions regarding the fiscal optimum. The first two of these are trivial and do not depend upon (18).

Proposition 1. *Optimal $\omega > 0$, thus $t^w < w$.*

Taxation of wage income is non-confiscatory because if $\omega = 0$ then $l_v = 0$, whence $k_v = 0$ which could not be "maximin".

Proposition 2. *The fiscal optimum is a corner solution, where $\sigma = 0$, iff $\rho = -1$.*

If $\rho = -1$ then $\sigma(.) = 0$ by hypothesis. And $\sigma(.) = 0$ implies $\rho = -1$, when $\omega > 0, g \geq 0$ and h large enough only to make $\sigma^0(.) = 0$.

Proposition 3. *At the optimum, $\partial l/\partial\omega > 0$.*

Use (18b) to write

$$F_l\frac{\partial l}{\partial \omega} = \left(l + \omega\frac{\partial l}{\partial \omega} - \frac{\partial \sigma}{\partial \omega}\right) + (1 + \rho)(1 + r)^{-1}\frac{\partial \sigma}{\partial \omega}.$$

As (8d) specifies, both terms on the right are hypothesized to be positive, like F_l, so $\partial l/\partial\omega$ must be positive.

Proposition 4. *Either $t^w > 0$ or $t^r > 0$ or both.*

In (18b), $\partial l/\partial\omega > 0$ by Proposition 3, $\partial\sigma/\partial\omega > 0$ by (8d) and $l > 0$ for maximin. Hence $\max(t^w, t^r) > 0$ for a positive left-hand side of (18b).

Proposition 5. *If the solution occurs at the propertyless corner, where $\rho = -1$, then the Hicks–Lucas effect of larger ρ, viz., $\partial l/\partial\rho > 0$, cannot be operative in that neighborhood.*

In the event of a corner maximum with respect to ρ, we have to replace (18a) by the inequality

$$(w_v - \omega_v)\frac{\partial l_v}{\partial\rho_v} + (r_v - \rho_v)\frac{\partial\sigma_v^*}{\partial\rho_v} - \sigma_v^* \leq 0. \tag{18a'}$$

Now $\sigma^* = 0$ at $\rho = -1$. Also (8c) implies $\partial\sigma/\partial\rho > 0$ at $\rho = -1$, there being no income effect to offset the substitution effect. Hence $\partial l/\partial\rho < 0$ if $w_v - \omega_v > 0$. From (18b), $(w_v - \omega_v)\,\partial l/\partial\omega = l$ at $\rho = -1$, so that $w_v - \omega_v > 0$ by Proposition 2. An interpretation: A rise of ρ, it is true, would raise x_v but only negligibly when $1+\rho$ is close to zero; thus, since $t_w > 0$, the only cost which can explain the inoptimality of $\rho > -1$ is that, around the $\rho = -1$ optimum, the anti-Hicks–Lucas effect prevails, $\partial l/\partial\rho < 0$.

Proposition 6. *If $\partial l/\partial\rho \leq 0$ at the fiscal optimum (non-Hicks–Lucas), then $t^r \equiv r - \rho > 0$. Corollary: If optimal $t^r < 0$, the explanation depends upon the Hicks–Lucas effect.*

If $r - \rho < 0$ while $\partial l/\partial\rho \leq 0$, then the rewriting of (18a),

$$(w - \omega)\frac{\partial l}{\partial\rho} = [(1+r)^{-1} - (1+\rho)^{-1}]\cdot\left[\sigma + (1+\rho)\frac{\partial\sigma}{\partial\rho}\right] + (1+\rho)^{-1}\sigma, \tag{18a''}$$

implies $(w - \omega) \equiv t^w < 0$. But Proposition 4 implies that if $t^r < 0$ then $t^w > 0$, a contradiction. Therefore, $r - \rho > 0$ when $\partial l/\partial\rho \leq 0$. *Interpretation*: A rise of x_v is better produced by a rise of $g + (1+\rho)^{-1}h$, in view of the needs of the poor, than by a rise of ρ. Hence if $\rho > r$ is optimal, the only explanation is *not* the effect of high ρ_v upon x_v – more $g + (1+\rho)^{-1}h$ could achieve that effect more desirably – but rather the effect upon labor supply, namely a sufficiently large $\partial l/\partial\rho > 0$.

Proposition 7. *If $\partial\sigma/\partial\rho \leq 0$ at the fiscal optimum (backward-bending segment), then t^w has the sign of $\partial l/\partial\rho$, which must be non-zero. Corollary: If $t^w \partial l/\partial\rho \leq 0$, then $\partial\sigma/\partial\rho > 0$.*

To show this, rewrite (18a) this time in the form

$$t^w \frac{\partial l}{\partial \rho} = (1+r)^{-1}\left[\sigma + (1+\rho)\frac{\partial\sigma}{\partial\rho}\right] - \frac{\partial\sigma}{\partial\rho}. \tag{18a'''}$$

Then $\partial\sigma/\partial\rho \leq 0$ implies $t^w \partial l/\partial\rho > 0$. *Interpretation*: At a trial solution on a backward-bending segment, the temptation is to reduce ρ, thus to reduce x_v and, with $\partial\sigma/\partial\rho \leq 0$, to increase k_v, given l_v. But if $t^w > 0$ and $\partial l/\partial\rho > 0$, for example, the reduction of ρ may apparently so decrease $F(k_{v-1}, l_v) - \omega_v l_v$ that k_v is reduced on balance by more than enough to offset the reduction of $x_v(1+r_v)^{-1}$.

Propositions 4, 6 and 7 may be summarized in the accompanying table:

	$\partial\sigma/\partial\rho > 0$	$\partial\sigma/\partial\rho < 0$
$\partial l/\partial\rho > 0$	$\max(t^w, t^r) > 0$	$t^w > 0$
$\partial l/\partial\rho < 0$	$t^r > 0$	$t^w < 0$ $t^r > 0$

Of course, a global analysis is required to establish rigorously that none of the cells in this matrix is in fact null, an unpopulated or empty box. Yet I see no signs that some of these cases are not true possibilities. In this connection, the following proposition is perhaps of some interest:

Proposition 8. *If first-period leisure and second-period consumption are "independent" goods or net complements for all (or sufficiently many) households at the fiscal optimum, then both tax rates are positive.*

A proof is given in the working paper preliminary to the article by Ordover and Phelps.

There is, finally, a sense in which, given the cross elasticities, the tax burden on *wage* income is comparatively greater the smaller is the wage-elasticity of labor supply. Solving for t^w and t^r from (18a) and (18b), we can obtain

$$\frac{t^r(1+\rho)^{-1}}{t^w\omega^{-1}} = \frac{\dfrac{\omega}{l}\dfrac{\partial l}{\partial\omega} - \dfrac{\omega}{\sigma}\dfrac{\partial l}{\partial\rho}}{\dfrac{1+\rho}{\sigma}\dfrac{\partial\sigma}{\partial\rho} - \dfrac{1+\rho}{l}\dfrac{\partial\sigma}{\partial\omega}}. \tag{19}$$

Note that while $\partial l/\partial\rho = 0$ is a tempting simplification, provided $\partial\sigma/\partial\rho > 0$ as Proposition 7 warns, it is not plausible to approximate (19) with the assumption that $\partial\sigma/\partial\omega = 0$.

3. Unsolved Problems

Section 1 showed that the government of generation v is to use (ρ_v, ω_v) to maximize $k_v - (1 + r_v)x_v$, evaluated at the "equalizing" g_v and given the no-saving by-the-poor convention determining h_v. An equivalent formulation of this maximand in terms of the generation's supplies of labor and wealth is

$$(w_v - \omega_v)l_v + [(1 + r_v) - (1 + \rho_v)]\sigma^*_v.$$

Thus the government belonging to the current generation can be regarded as taxing in order to maximize its socially discounted "revenue" – when labor and saving are assigned their correct shadow prices.

This observation suggests that the problem of the optimal *graduated* taxation of labor and saving, where marginal tax rates vary with the taxpayer's wage income and interest income, can be analyzed as a "straightforward" generalization of the one-dimensional wage-taxation problem. We let y denote the individual's wage income and z denote his interest income discounted by $(1 + r_v)$. Then aggregate revenue is the double integral

$$\int\int t(y, z)\, \mathrm{d}B(y, z)\, \mathrm{d}y\, \mathrm{d}z,$$

where B gives the proportion of people with earnings less than y and discounted interest less than z. $B(y, z)$ equals the proportion of people whose m's are less than the m corresponding to that (y, z). Note that the "corresponding m" is a function, say μ, involving the level and slopes of the tax schedule facing persons, thus:

$$B(y, z) \equiv \Phi[\mu(t, t_y, t_z, y, z; g, h; w, r)].$$

If I am right, we now have a well-determined problem: Find the tax function $t(y, z)$ which maximizes the revenue integral. In any case, I leave this as an intriguing – and important! – inquiry for others to pursue.

Another problem that this paper leaves outstanding is the final step in the solution of the intergeneration maximin problem. We have to determine the "right" r_v, which depends upon r_{v+1}, and so on *ad infinitum*.

There is little doubt that this problem is exceedingly difficult. Praise be to him or her who cracks it. If I were to try, I would seek to recast the intergeneration problem in terms of continuous time and a continuum of overlapping generations. It is a sheer speculation, but suppose that such a model can be reduced to two state variables, $k(t)$ and $x^*(t)$, the letter being the socially discounted volume of wealth claims by survivors at time t – including their prospective streams of $h(v, t)$ already and unrevokably committed to them. We wish to maximize the present value of the stream of transfers scheduled for new entrants, say $\beta(t)$. A "solution", for me, is an interesting characterization of the "policy functions", $\rho(k(t), x^*(t))$, $\omega(k(t), x^*(t))$ and $\beta(k(t), x^*(t))$, especially in relation to $w(k(t), x^*(t))$ and $r(k(t), x^*(t))$. (The dynamics of the resulting path, $\dot{k}(t)$ and $\dot{x}^*(t)$, are surely of interest, but not of primary interest.) This sort of problem is a differential game: The government player chooses the strategy $\{\rho(k, x^*), \ldots\}$ and households choose their strategy, both under perfect foresight regarding future states. Clearly this continuous-time problem is also going to be difficult to solve.

Fortunately, however, optimal tax analysis need not wait entirely for a full solution to the intertemporal problem, nor even for a computation of this generation's social discount, $1 + r$. Take any arbitrary (k_v^0, x_v^0) that generation v is heading for, optimally or not. More than one tax mix – more than one (ρ, ω) pair – will get it there. Which of these is optimal? What symptoms of inoptimality should be watched for?

The approach set out here makes obvious the answer. One may imagine assigning ρ_v to the target $x_v = x_v^0$, let ω_v maximize k_v while allowing g_v to rise to the point that the maximized $k_v = k_v^0$. Then "optimal" tax rates satisfy

$$
\begin{aligned}
0 = \max_{(\rho_v, \omega_v)} \{&F(k_{v-1}, l_v) - x_{v-1} - g_v - \omega_v l_v - \sigma_v - k_v^0 \\
&+ \theta[h_v + (1 + \rho_v)\sigma_v - x_v^0]\},
\end{aligned}
$$

where θ, a Lagrange multiplier, is a constant, like g_v, a function only of the state path $\{k_{v-1}, x_{v-1}, k_v^0, x_v^0\}$. The first-order conditions to which this "sub-optimization" leads are akin to those studied in Ordover–Phelps (1975), particularly their Section A where initial and subsequent capital and wealth are both fixed. The propositions that can be deduced about this constrained tax problem are of course fewer and weaker than those obtainable when (k_v, x_v) are "optimized", one or both. Nevertheless those weaker propositions serve to weed our certain tax mixes that could not be optimal even without revision of the economy's (k_v, x_v) heading.

4. Possible Policy Ramifications

Keynes wrote of the force of ideas, Marx of the forces upon ideas. Undoubtedly the Marxians are right to say that interests can often sustain an idea when better and more accurate conceptions are producible – though Marx was not the first nor the last to jeer at the cant and myths promulgated by vested interests.

But "induced ideology" raises doubts as deep as the notion of "induced invention". Keynes and others wisely saw that technology – including economic and ethical theory – is the prime mover. Ideas develop sequentially and cumulatively, are not wholly predictable and are difficult to suppress. So tomorrow's ideas are not altogether the convenient selection by today's powerful interests; they act, from time to time, more like stochastic disturbances, thus jarring the *status quo ante* into the *status quo post*. It is a commonplace that technological developments in physical science, and the resulting opportunities for physical engineering, often alter the balance of power between competing classes. It can also be presumed that the arrival of social and economic ideas, and the inventions in social institutions they indicate, can sway the distribution of income between rich and poor, young and old, healthy and sick, white and black, male and female – just as much as discoveries of material resources.

What I am leading up to, more ponderously than I had intended, is that we may be at the start of a technological revolution in the understanding of efficient tax collection. The growth of the public sector of the past quarter century was enabled and invited by the development of the graduated income tax. The social security system needed the invention of the limited payroll tax. I wonder now whether the institution of a greatly more substantial anti-poverty program does not require a similar innovation toward greater fiscal efficiency. And I suggest that far stiffer taxation of interest income, a tax on income from new saving if not a tax on already existing wealth, perhaps coupled with lighter taxation of low-wage incomes, is the invention needed.

It could turn out that heavier taxation of saving is not the answer, that efficient reforms in the taxation of wage income are all we can rely upon. But at least we now seem to have the conceptual apparatus with which to probe this question. Like a kid one thought would never grow up, "growth theory" seems finally to have come of age and able at least to address some problems of the time.

5. Appendix

At the risk of confusing matters that seem clear to the author, let me give another derivation of the first-order conditions obtained, by the method of freezing h at its magic value, in (18). I have cribbed freely from notes sent to me by John Riley on this approach. Errors and omissions are my own.

Define $q_v \equiv (1+\rho_v)^{-1}$, $p_v \equiv (1+r_v)^{-1}$ and $\beta_v \equiv g_v + q_v h_v$. In these terms we may express new demand functions,

$$c_v = c(\beta_v, q_v, \omega_v), \quad x_v = x(\beta_v, q_v, \omega_v), \quad l_v = l(\beta_v, q_v, \omega_v).$$

Then

$$k_v = F(k_{v-1}, l(\theta_v)) - c(\theta_v) - x_{v-1}, \quad \theta_v \equiv (\beta_v, q_v, \omega_v), \tag{A.1}$$

and our functional equation involving V^0, the indirect utility function of the poor, is

$$j(k_{v-1}, x_{v-1}) = \max_{\theta_v} \min[V^0(\beta_v, q_v), \quad j(k_v, x(\theta_v))], \tag{A.2}$$

subject to (A.1).

Now form the Langrangian

$$\mathscr{L} = j_{-1} + \lambda(V^0 - j_{-1}) + \mu(j(k_v, x_v) - j_{-1}), \tag{A.3}$$

where, using households' budget constraints,

$$k_v = F(k_{v-1}, l(\theta_v)) - x_{v-1} - \omega_v l(\theta_v) + q_v x_v - \beta_v. \tag{A.4}$$

Then, assuming $c_v > 0$,

$$0 = \frac{\partial \mathscr{L}}{\partial \beta} = \lambda V_1^0(\beta, q) + \mu j_k [F_l l_1 - \omega l_1 + q x_1 - 1] + \mu j_x x_1, \tag{A.5a}$$

$$0 = \frac{\partial \mathscr{L}}{\partial q} = \lambda V_2^0(\beta, q) + \mu j_k [F_l l_2 - \omega l_2 + q x_2 + x] + \mu j_x x_2, \tag{A.5b}$$

$$0 = \frac{\partial \mathscr{L}}{\partial \omega} = \mu j_k [F_l l_3 - \omega l_3 + q x_3 - l] + \mu j_x x_3, \tag{A.5c}$$

$$0 = \frac{\partial \mathscr{L}}{\partial j_{-1}} = 1 - \lambda - \mu. \tag{A.5d}$$

Upon assuming that $j_k \geq 0$ and $j_x \leq 0$ we deduce from (A.5a) and (A.5d) that $\mu > 0$, hence $j(k_v, x_v) = j(k_{v-1}, x_{v-1})$. Assume also that $\lambda > 0$. Then, as in the text, $j_x = -p j_k$, where $p = F(k_v, l_{v+1})^{-1}$. Hence,

$$(w - \omega) l_3 + (p - q) x_3 = l > 0. \tag{A.6a}$$

Also, recalling that $V_2^0/V_1^0 = -x^0$ and introducing $\alpha = \lambda V_1^0/\mu j_k$, we have

$$(w-\omega)l_1 + (q-p)x_1 = 1-\alpha, \tag{A.6b}$$

$$(w-\omega)l_2 + (q-p)x_2 = -x + \alpha x^0. \tag{A.6c}$$

Combining these two conditions gives a new condition on q to accompany (A.6a),

$$(w-\omega)(l_2 + x^0 l_1) + (q-p)(x_2 + x^0 x_1) = -(x - x^0) \leq 0. \tag{A.6c'}$$

Equations (A.6a) and (A.6c′) are recognizable as Ordover–Phelps equations once we recollect their convention that $x = h$ and realize that *their* x_2 is $\partial x/\partial q$ with *g and h* are held constant, not with β held constant; thus, *their* x_2 is $(x_2 + x^0 x_1)$ in the present notation.

Let us express (A.6c′) in the spirit of the Ordover–Phelps notation:

$$(w-\omega)\frac{\partial l}{\partial q} + (q-p)\frac{\partial x}{\partial q} = -(x-h). \tag{A.7}$$

Then, using $\mathrm{d}\rho/\mathrm{d}q = -(1+\rho)^2$, $h = x^0$, $x = (1+\rho)\sigma + x^0$, and $\sigma^* = p\sigma$,

$$(1+\rho)^2\left\{t_w \frac{\partial l}{\partial \rho} + (q-p)\frac{\partial x}{\partial \rho}\right\} = (1+\rho)p^{-1}\sigma^*, \tag{A.8}$$

where $\partial x/\partial \rho = p^{-1}[\sigma^* + (1+\rho)\partial\sigma^*/\partial p]$. This leads to (18a) in the text. Note the $\partial x/\partial \rho$ includes the downward effect upon β of a rise of ρ, while the induced rise of x^0, thus h, at the new β, has an equal and opposite effect on $(1+\rho)\sigma$, thus no net effect upon x.

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THE CONCEPT OF OPTIMAL TAXATION IN THE OVERLAPPING-GENERATIONS MODEL OF CAPITAL AND WEALTH

1. Introduction

Every meaningful welfare analysis of the fiscal choices before the current generation of workers and savers must somehow take into account the impact that each tax structure would have upon the productive capacity made available to the next generation and upon the after-tax wealth-claims to retirement consumption which are to be set against that future capacity. The 'maximin' tax analysis by Ordover and Phelps (1975) of a simple overlapping-generations model of heterogeneous workers, compounded from the settings in Diamond (1965) and Mirrlees (1971), and the corresponding numerical simulation analysis by Jha (1978), met this problem by requiring that the present generation endow the next one with at least the same capital stock per worker as it enjoys and burden the next generation with at most the same wealth per worker to be paid to the retired as it must pay.[1] Then

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[1]A modification of this requirement was considered by Ordover (1976) and additionally in our joint paper; the initial per worker quantity of wealth claims, and hence the per worker public debt held by the retired population, could be chosen by the present working generation subject to the condition that this quantity as well as the per worker quantity of capital be replicated for the next generation.

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the next generation, not being damaged, could not complain – or so we thought. Yet the next generation might well complain if the current generation in maintaining both capital and wealth at their initial levels thereby missed opportunities for a mutual welfare gain.

A traditional solution to the problem treats the choice of capital and wealth as an exercise in dynamic programming to achieve the optimal growth of social welfare from generation to generation. (Of course, the optimal path of capital and wealth will leave no unexploited opportunity for a mutual welfare gain if the intergeneration optimality criterion is of the Paretian type.) Two papers representing this classical approach, both on fiscal planning for 'maximin growth' in the overlapping-generations setting, found a Pareto-like condition that capital and wealth will necessarily satisfy on the intergenerationally optimal path. In their study of homogeneous populations of workers, Phelps and Riley (1978) showed that the marginal rate of substitution for the *next* generation between the capital left it by the retired generation, k, and the wealth-claim to consumption by the retired generation, x, is equal to the gross marginal product of that capital; to this number the *present* generation must equate *its* marginal rate of substitution between those same two variables for the growth of capital and wealth to be maximin-optimal over generations. In his subsequent study of heterogeneous worker populations, Phelps (1976) confirmed that this same marginal product provides the parametric shadow rate of interest, r^*, with which the fiscal planners in the present generation must discount its future consumption claim, x, in relation to the capital, k, it deposits at the end of its labors - and discount its future tax revenue from interest-income levies in relation to its present tax revenue from current wage-income levies. The present generation optimally taxes as if it were maximizing its own (maximin) social welfare subject to a single constraint on capital and wealth $(1+r^*)^{-1}x-k=\Delta^*$ $\equiv$constant. Here the second parameter Δ^* may be interpreted as the discounted value of the public debt to be sold to the present generation's savers along the maximin-optimal path; together with initial conditions k_{-1} and x_{-1}, it yields the optimal algebraic budgetary deficit to be allowed the current generation.

The present paper is a further contribution to the welfare economics of taxation in a dynamic setting – namely the overlapping-generations models with heterogeneous labor recalled in section 2. Our concept of optimal taxation is first developed in section 3 where we attribute to each generation any intragenerational social welfare function of the Bergson–Samuelson type; the same concept is redeveloped using the 'maximin' social welfare function in section 4. Two key elements of our concept of optimality may be noted. We do not require that the path of capital and wealth be maximin-optimal, or anything-optimal, over generations, nor do we return to our original requirement that capital and wealth per head be maintained; instead, we permit the present generation to set (within the limits of feasibility) an arbitrary target for the capital and wealth levels that its fiscal plan is constrained to meet; but we often require that the present generation confine its target to those consistent with the achievement of *Pareto efficiency* between it and the next generation – always taking as given whatever the next generation is (known to be) going to do toward its successors. We also permit each generation to employ *graduated* (nonlinear) taxation of its wage and interest incomes, making use of the Hamiltonian methods developed by Mirrlees (1971) and first applied to life-cycle worker–savers by Atkinson and Stiglitz (1976).

A methodological implication of our concept is the possibility of representing any pair of constraints, one on capital and the other on wealth, by a single linear constraint containing a shadow rate of interest and prescribed public debt; but that shadow interest rate is not generally the market interest rate. Those analyses of the taxation of life-cycle saving and interest that purport to hold for an arbitrary budgetary deficit and fixed rate of interest, such as the otherwise exemplary analysis by Atkinson and Stiglitz, are valid provided the algebraic deficit allowed and interest rate imposed are both associated with a target outcome (k, x) that is Pareto efficient; at any other target the shadow interest rate in the associated linear constraint, $(1+r^*)^{-1}x - k = \Delta^*$, will differ from the market interest rate with the consequence that the Atkinson–Stiglitz tax schedules will be inoptimal.

In section 5 we use the preceding conceptual apparatus to engage (again) in the mounting controversy over the taxation of interest income. It is first shown that the Phelps–Sadka injunction against a positive marginal tax rate 'at the top' extends to interest and wealth as well as wages provided that our Pareto-efficiency condition is met. The other issue is the optimality of interest-income taxation in general. Our 1973 and 1976 'maximin' tax analyses reduced the issue to the question: would some taxation of interest or wealth make the current generation's discounted tax revenue higher than otherwise possible? If so, no tax exemption of interest or wealth would be maximin-optimal for the present generation, since maximin taxation maximizes discounted tax revenues. This result is reaffirmed. But we also confirm a proposition of Atkinson and Stiglitz when properly qualified: under our Pareto-efficiency condition but not otherwise, it is Bergson–Samuelson optimal and maximin-optimal to exempt all interest income from tax if each worker's utility function is everywhere weakly separable between the two consumption goods (early and late) and leisure.

Yet section 6 concludes with a warning. Despite separability, exemption of interest income from tax may reduce the potential welfare of the next generation if unaccompanied by other tax-transfer changes; it may reduce the potential welfare of the present generation if the latter binds itself to keep the budget balanced.

2. The overlapping-generations model

The labor heterogeneity in the model is taken from Mirrlees (1971). The effectiveness of a person's labor is graded by an index m. The effective labor supplied by a worker of grade m, denoted l^m, is the worker's effort or labor-time, e^m, augmented by the factor m, i.e. $l^m = me^m$. Let us denote by $\Phi_v(m)$ the proportion of persons in generation v whose labor is m or less, $0 \leqq m \leqq M_v$, agreeing henceforth to leave the subscript v implicit. Then the postulated distributions of m all have the following properties:

$$0 \leqq \Phi(0) < 1 = \Phi(M), \qquad 0 < M \leqq \infty,$$

$$0 \leqq \Phi(a) \leqq \Phi(b), \quad \text{if } 0 \leqq a < b \leqq M.$$

The two-period life-cycle of overlapping generations is taken from Diamond (1965). In any generation v, $v \geqq 0$, every worker retires from paid work after the first period of economic life and no one bequeathes. The amount worked in the first period by a person born into generation v with labor of grade m is denoted by e_v^m, the consumption at the end of the first period is denoted by c_v^m, and the amount consumed at the end of the second and last period is called x_v^m. The corresponding amounts *per person* in generation v are

$$e_v = \int_0^M e_v^m \mathrm{d}\Phi(m); \qquad c_v = \int_0^M c_v^m \mathrm{d}\Phi(m); \qquad x_v = \int_0^M x_v^m \mathrm{d}\Phi(m).$$

The total effective labor supplied per person of generation v is then

$$l_v = \int_0^M l_v^m \mathrm{d}\Phi(m) = \int_0^M m e_v^m \mathrm{d}\Phi(m).$$

A neoclassical, one-product, constant-return-to-scale production function, F, is posited that makes output per person of any generation v depend on effective labor per person, l_v, and the predetermined per person quantity of capital, k_{v-1}, left by the previous generation, $v = 1, 2, 3, \ldots$. With absolutely no loss of generality we take the generations to be of unchanging size. Then the capital stock which generation v leaves per person of generation $v+1$ is the unconsumed output per person of generation v:

$$k_v = F(k_{v-1}, l_v) - c_v - x_{v-1} \geqq 0, \qquad v = 1, 2, \ldots. \tag{1}$$

It is assumed that both partial derivatives of the net production function, $Y(k_{v-1}, l_v) \equiv F(k_{v-1}, l_v) - k$, exist with $Y_k \geqq -1$ and $Y_l \geqq 0$.

All members of generation v receive a first-period demogrant, g_v, and a further installment, h_v, upon retirement. We assume that first-period consumption is untaxed: its consumer price is equal to the producer price and both are equal to one. All 'direct' taxation falls only on wage income, and indirect taxation falls only on second-period consumption; the latter taxation operates like a levy on saving or on interest income. For simplicity we require households to prepay their indirect taxes: a household that consumes x_v^m in its second period must pay $T_s(x_v^m - h_v)$ in indirect taxes in the first period; then there are no second-period revenue collections which need discounting back to the first period. The budget equation for household of type m in the extensive form is

$$x_v^m = h_v + (1 + r_v)\sigma_v^m, \tag{2a}$$

$$c_v^m = g_v + w_v l_v^m - T_l(w_v l_v^m) - T_s(x_v^m - h_v) - \sigma_v^m. \tag{2b}$$

The function $T_l(w_v l_v^m)$ gives the tax paid on earned income, and σ_v^m is the amount saved. Furthermore, w_v is the before-tax wage rate paid for one unit of effort by workers of the standard ($m = 1$) type, and r_v is the rate of return to private saving. For simplicity we suppose that w_v and r_v are equal to the corresponding marginal products:

$$w_v = F_l(k_{v-1}, l_v) > 0; \qquad 1 + r_v = F_k(k_v, l_{v+1}) \geqq 0. \tag{3}$$

There is a common utility function $u(c^m, x^m, e^m)$ which does not depend on m. Each household selects the triplet $\{c^m, x^m, e^m\}$ to maximize its utility subject to the individualized budget constraint (2) corresponding to its m. Formally, the household's problem is:

$$\max_{\{c^m, x^m, e^m\}} u(c^m, x^m, e^m) \tag{4}$$

subject to

$$wme^m + g + (h/F_k) - c^m - (x^m/F_k) - T_l(wme^m) - T_s(x^m - h) \geqq 0,$$

where we have omitted the generational subscript v. The first-order necessary conditions for this maximization program are:

$$e^m[u_e + \lambda wm(1 - T'(wme^m))] = 0, \tag{5a}$$

$$u_c = \frac{-u_e}{wm(1 - T_l')}, \tag{5b}$$

$$u_x = \frac{-u_e(1/F_k + T_s')}{wm(1-T_l')}, \tag{5c}$$

$$wme^m + g + (h/F_k) - c^m - (x^m/F_k) - T_l(\cdot) - T_s(\cdot) = 0. \tag{5d}$$

Conditions (5b) and (5c) are satisfied only for those who work. For those who strictly prefer full-time leisure to work, the term $-u_e/wm(1-T')$ must be replaced by λ. A simple envelope argument establishes that the *maximum direct utility function*,

$$u^*(m) \equiv \max u(wme^m + g + (h - x^m)/F_k - T_l(\cdot) - T_s(\cdot), x^m, e^m), \tag{6}$$

must satisfy the differential equation

$$\frac{\mathrm{d}u^*(m)}{\mathrm{d}m} = \frac{-u_e e^m}{m} = \frac{-u_e l^m}{m^2} \geqq 0. \tag{7}$$

If we take $u^*(m)$, x^m, and e^m as given, we can invert eq. (6) to obtain

$$c^m = z(u^*(m), x^m, e^m). \tag{8}$$

Eqs. (7) and (8) completely summarize the household sector.

The income statement of the v-government – the government that taxes generation v, that is – is given by

$$g_v + \frac{h_v}{F_k(k_v, l_{v+1})} = \int_0^M [T_l(w_v m e_v^m) + T_s(x_v^m - h_v)]\,\mathrm{d}\Phi(m) + (\Delta_v - d_{v-1}) + [F_k k_{v-1} - (x_{v-1} - d_{v-1})]. \tag{9}$$

Here d_{v-1} is the *face value* of the one-period discount-bonds issued by the government in period $v-1$ and Δ_v is the *market value* of the bonds issued in period v. If a bond entitles its owner to one unit of second-period consumption after tax (i.e. upon the payment of the appropriate tax), then, by arbitrage, its market value upon issue at the end of period v is $1/F_k(k_v, l_{v+1})$. This result reveals an interdependence between generations missing from the static Atkinson–Stiglitz model: the market value of a given debt issue by the v-government is a (decreasing) function of the effective labor supplied by generation $v+1$; and the v-generation supplies of labor, l_v^m, since they are not generally independent of $1+r_v$ given tax schedules, will also be some function of effective labor supplied by generation $v+1$. The presence of the last term in square brackets of the right-hand side reflects the fact that the v-government collects a surtax (positive or negative) from the $v-1$ generation equal to the excess of total earnings from capital over the claim that the $v-1$ generation has to those earnings. This surtax siphons off any windfall gain to that generation arising from the effects on earnings of unanticipated tax reform by generation v.

3. Optimal taxation: Bergson–Samuelson

Here we posit, for purposes of comparison with Atkinson and Stiglitz, that the v-government maximizes the Bergson–Samuelson social welfare function

$$\mathscr{W}^v = \int_0^M G^v[u(c_v^m, x_v^m, e_v^m)]\,\mathrm{d}\Phi(m).$$

What intergenerational constraints shall we place on this maximization? One is the ethical restraint that the government must fulfil the second-period consumption expectations of workers now retired from generation $v-1$. Hence x_{v-1} is a datum for the v-government.

The second restraint we impose is that the subsequence $\{k_{v+i}, x_{v+i} | i = 1, \ldots\}$ is predesignated. While the vector (k_v, x_v) is open to choice, all the subsequent vectors are fixed – perhaps optimally by some criterion of intergenerational justice – and the previous vectors are predetermined by history. The analytical rationale here is a familiar one: by the principles of Bellman and Euler, if the *whole* sequence including $i=0$ is to be a Pareto-efficient one, it is necessary that any particular vector $\{k_v, x_v\}$ be chosen efficiently when the other vectors are taken as given.

This latter restraint permits us to solve for the intragenerationally optimal l_{v+i}, beginning with $i=1$, as a function of the vectors $\{k_{v+i-1}, x_{v+i-1}\}$ and $\{k_{v+i}, x_{v+i}\}$, given the ith generation's social welfare function and all subsequent welfare functions. Hence the labor supply of generation $v+1$ is a function of the vector (k_v, x_v) chosen by the v-government, which function we denote by $\mathscr{L}(k_v, x_v)$, given the subsequent vectors $\{k_{v+i}, x_{v+i} | i = 1, 2, \ldots\}$ and social welfare functions $\{\mathscr{W}^{v+i} | i = 1, 2, \ldots\}$.

We can now formulate the welfare-maximization problem facing the government of generation v. For convenience of notation, we focus on the first generation, setting $v=1$. Following Atkinson and Stiglitz, we take x_1^m and e_1^m to be control variables, treat $u^*(m)$ as the state variable, and hence regard c_1^m as a function $z(\cdot)$ of $u^*(m)$, x_1^m, and e_1^m. In addition to the differential equation $\mathrm{d}u^*(m)/\mathrm{d}m$, the $v=1$ government faces two additional constraints. First, total second-period claims by its citizens cannot exceed some $\bar{x}_1$ and, secondly, the capital deposited cannot be less than some $\bar{k}_1$,

$$\bar{x}_1 - \int_0^M x_1^m \mathrm{d}\Phi(m) \geqq 0, \tag{10}$$

$$-\bar{k}_1 + F(k_0, l_1) - \int_0^M c_1^m \mathrm{d}\Phi(m) - x_0 \geqq 0. \tag{11}$$

(Of course, we are free to require that $(\bar{k}_1, \bar{x}_1)$ be chosen to yield Pareto-efficiency between generations 1 and 2.) The corresponding Hamiltonian function to be maximized is

$$\begin{aligned}\mathscr{H}^1(m) = {} & u^*(m)\mathrm{d}\Phi(m) \\ & + \left[\lambda^1(\bar{x}_1 - x_1^m) + \gamma^1\left(-\bar{k}_1 + F\left(k_0, \int_0^M l_1^m \mathrm{d}\Phi\right) - x_0\right.\right. \\ & \left.\left. - c_1^m\right)\right]\mathrm{d}\Phi(m) \\ & + \mu^1(m)\left[\frac{-e_1^m u_e^*}{m}\right],\end{aligned} \tag{12}$$

where λ^1 and γ^1 are the Lagrange multipliers associated with the two isoperimetric constraints on the control variables in (10) and (11), respectively, and are therefore independent of m, i.e. $\mathrm{d}\lambda^1/\mathrm{d}m = \mathrm{d}\gamma^1/\mathrm{d}m = 0$; $\mu^1(m)$ is the co-state variable associated with the constraint (7).

We can give an interpretation of the multipliers λ^1 and γ^1 that will prove useful in the derivation of the optimal tax formulas. Using a well-known envelope theorem in control theory[2] we note that $\partial \mathscr{W}^{*1}/\partial \bar{x}_1 = -\lambda^1$ and $\partial \mathscr{W}^{*1}/\partial \bar{k}_1 = \gamma^1$ where $\mathscr{W}^{*1}(k_1, x_1)$ is the maximum social welfare that generation 1 can attain given its target vector $(\bar{k}_1, \bar{x}_1)$. If both of our constraints are binding at this maximum, then $\lambda^1 < 0$ and $\gamma^1 < 0$. Hence,

$$\left(\frac{\mathrm{d}k_1}{\mathrm{d}x_1}\right)_{\mathscr{W}^{*1}} \equiv -\frac{\partial \mathscr{W}^{*1}/\partial \bar{x}_1}{\partial \mathscr{W}^{*2}/\partial \bar{k}_1} = \frac{\lambda^1}{\gamma^1} > 0 \tag{13}$$

along any predetermined isowelfare contour. Two such contours for generation 1 are depicted in fig. 1.

One could also find from eq. (12) the marginal rate of substitution for generation 1 between k_0 and x_0, making another application of the envelope argument. By the same analysis we find that *for generation 2* the marginal

[2]Bryson and Ho (1969, pp. 90–91).

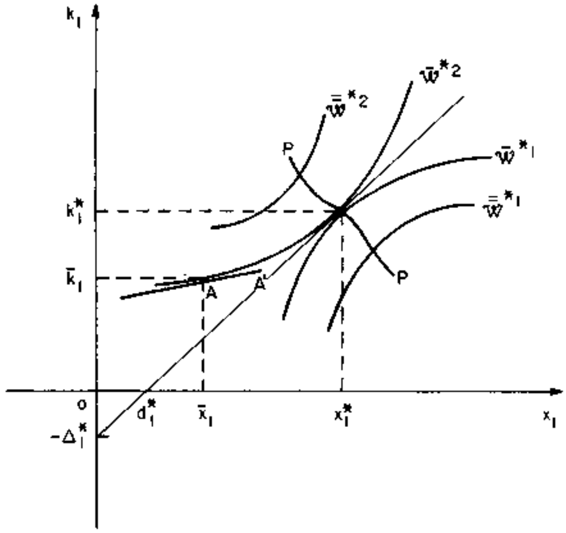

Fig. 1

rate of substitution between k_1 and x_1 is the reciprocal of one-plus-the-before-tax-rate-of-interest:

$$\left(\frac{\mathrm{d}k_1}{\mathrm{d}x_1}\right)_{\mathscr{W}^{*2}}=\frac{1}{F_k(k_1,\mathscr{L}(k_1,x_1))}. \tag{14}$$

This result, first obtained by Phelps and Riley (1978), is of wide generality, not depending on the form of the u^2 and $\mathscr{W}^2$ functions. Fig. 1 depicts two isowelfare contours for generation 2.

As shown in fig. 1, the isowelfare contours $\mathscr{W}^{*1}$ and $\mathscr{W}^{*2}$ are strictly quasiconcave in (k_1, x_1).[3] It follows from these respective quasiconcavities that there exists an interior locus, labelled *PP* in fig. 1, of intergenerationally Pareto-efficient vectors along which the two marginal rates of substitution are equal.[4]

[3]The requisite proof for $\mathscr{W}^{*1}$ is given in Phelps and Riley (1978, pp. 105–106). The proof for $\mathscr{W}^{*2}$ is as follows. Consider a vector $(\bar{k}_1, \bar{x}_1)$ as at point *A* in Fig. 1. The slope of $\mathscr{W}^{*2}$ at that point is $1/F_k(\bar{k}_1, \mathscr{L}(\bar{k}_1, \bar{x}_1))$. Now assume that generation is faced with a different vector (k_1', x_1') lying on the line segment *AA'*. If generation 2, behaving passively, does not adjust its tax rates and consequently the values of c_2, x_2, and l_2, it will violate its own investment constraint analogous to eq. (11) of the text. The reason that this constraint would be violated at *A'* is that $F_{kk}<0$. Since the constraint on k_2 must be met, c_2 must be reduced, or l_2 must be increased, or both. In either case aggregate social welfare must decline.

[4]To show that there must be a stretch of the *PP* locus in the positive quadrant note that $\partial\mathscr{W}^{*1}/\partial x_1\to\infty$ as $x_1\to 0$ whenever $\partial u/\partial x^m\to\infty$ as $x^m\to 0$. On the other hand F_k is bounded above zero, at least for some values of k_1, since l_2 is finite. That is, F_k does not tend to zero as $x_1\to 0$.

The Hamiltonian formulation in (12) is well suited as it stands to the analysis of intragenerationally optimal taxation in an intergeneration setting in those instances when no particular constraints are placed on the target vector $(\bar{k}_1, \bar{x}_1)$. When we speak of our concept of optimal taxation we mean the above formulation in its full generality. Yet there is surely special interest in those cases where the target vector meets the condition of Pareto-efficiency between the generations. For that special case a somewhat different development of the optimization problem, as follows, is a little more expeditious.

We noted earlier that at any interior Pareto-efficient vector (k_1^*, x_1^*) the marginal rate of substitution between k_1 and x_1 must be the same for the two adjacent generations. Therefore, we have $\lambda^1 = \gamma^1 / F_k(k_1^*, \mathscr{L}(k_1^*, x_1^*))$ and consequently the Hamiltonian in (12) simplifies to

$$\begin{aligned}
\mathscr{H}^1(m) &= u^*(m)\mathrm{d}\Phi(m) \\
&\quad + \gamma^1 \Big\{ [F_k(k_1^*, \mathscr{L}(\cdot))^{-1} x_1^* - k_1^*] - \Big[F_k(k_1^*, \mathscr{L}(\cdot))^{-1} x_1^m \\
&\quad - F\Big(k_0, \int_0^M l_1^m \mathrm{d}\Phi(m)\Big) + c_1^m + x_0 \Big] \Big\} \mathrm{d}\Phi(m) \\
&\quad + \mu^1(m) \left[\frac{-e_1^m u_e^*}{m} \right].
\end{aligned}$$

If we then define r_1^* and Δ_1^* by

$$1 + r_1^* = F_k(k_1^*, \mathscr{L}(k_1^*, x_1^*))$$

and

$$(1 + r_1^*)^{-1} x_1^* - k_1^* = \Delta_1^*,$$

we obtain a new Hamiltonian formulation of the maximization problem:

$$\begin{aligned}
\mathscr{H}^1(m) &= u^*(m)\mathrm{d}\Phi(m) \\
&\quad + \gamma^1 \Big[\Delta_1^* - (1 + r_1^*)^{-1} x_1^m + F\Big(k_0, \int_0^M l_1^m \mathrm{d}\Phi\Big) - c_1^m \\
&\quad - x_0 \Big] \mathrm{d}\Phi(m) \\
&\quad + \mu^1(m) \left[\frac{-e_1^m u_e^*}{m} \right].
\end{aligned} \tag{15}$$

Thus, the social welfare of generation 1 is to be maximized subject to the deadweight debt constraint *and* shadow rate of interest, both parameters being associated with a preselected Pareto-efficient vector, as well as the differential-equation condition expressed in (7). It cannot be stressed enough that the former constraint specifies *two* parameters, Δ_1^* and r_1^*, and thus $d_1^* = \Delta_1^* \cdot (1+r^*)$, not just the budgetary deficit, and both parameters are derived from a prior choice of some target on the Pareto locus.

We remark that, as the formulation of the optimal tax problem in (15) makes clear, any point on the Pareto locus can be supported by a proper combination of public debt and the gross rate of return on saving, $F_k(\cdot)$. To attain the point (k_1^*, x_1^*) in fig. 1, for example, the government offers the gross rate of return of $F_k(k_1^*, \mathscr{L}(k_1^*, x_1^*)) \equiv F_k^*$ at which rate of return households save $\sigma^1(\cdot, F_k^*)$. Firms will demand that amount of capital which has a marginal product, when calculated at the optimal $l_2^* = \mathscr{L}(k_1^*, x_1^*)$, equal to the same rate of return to saving which was fortuitously chosen to be F_k^*. To accommodate private saving the government will have to issue the amount d_1^* of bonds where $d_1^*/F_k^* = \sigma_1(\cdot, F_k^*) - k_1^*$. The market value of those bonds, Δ^*, can be read-off from fig. 1 by extending the tangent to $\mathscr{W}^{*2}(k_1^*, x_1^*)$ to its intersection with the vertical axis.[5]

In principle, given certain transversality conditions on $\mu(\cdot)$ and the differential equation $\mathrm{d}u^*(m)/\mathrm{d}m$, the whole optimal tax schedule $T_s(\cdot)$ should be deducible from the first-order condition for a maximum of $\mathscr{H}^1(m)$ in (15):

$$\frac{\partial \mathscr{H}^1}{\partial x_1^m} = \gamma^1 \left[-(1+r_1^*)^{-1} - \left(\frac{\partial c_1^m}{\partial x_1}\right)_{u^*(m)} \right] \mathrm{d}\Phi(m) + \mu^1(m) \left\{ \frac{-e^m}{m} \left[u_{ec}^m \left(\frac{\partial c_1^m}{\partial x_1^m}\right)_{u^*(m)} \right] + u_{ex} \right\} = 0. \tag{16}$$

In practice, however, little can be said about the shape of this schedule other than at its endpoints or when the term in the curly brace is zero. We take up those implications in section 5.

4. Optimal taxation: 'maximin'

Since the 'maximin' social welfare function can usually be represented by the limiting Bergson–Samuelson social welfare function, as the degree of concavity of the functional G is increased without bound – for example,

[5]See Phelps and Riley (1978, fig. 4). Actually the argument does not require that the point being supported lie on the Pareto locus.

Atkinson (1973) – we should expect the above analytics to carry over to 'maximin' without essential differences. We show here that this is indeed the case, and we point out some special features of maximin-optimal taxation.

The minimum life-time utility scored by persons from generation 1, which is to be maximized, must equal $u^*(0)$ since no person of generation 1 can have a smaller utility than that; of course, $u^*(m)$ may equal $u^*(0)$ for positive m so small that such persons choose not to work. Thus, our 'maximin' problem is to maximize $u^*(0)$ subject to the x_1 and k_1 constraints in (10) and (11), respectively, and to the differential equation constraint in (7). This problem is equivalent to the problem of maximizing k_1 subject to (10), (7), and the constraint that $u^*(0) \geqq \bar{u}(0)$, where $\bar{u}(0)$ is the maximum minimum utility in the former problem. The Hamiltonian corresponding to the latter formulation is

$$\mathscr{H}^1(m) = \left[-x_0 + F\left(k_0, \int_0^M l_1^m \mathrm{d}\Phi\right) - c_1^m \right] \mathrm{d}\Phi(m) + \pi^1[\bar{x}_1 - x_1^m] \mathrm{d}\Phi(m)$$

$$+ \mu^1(m) \left[\frac{-e_1^m u_e^*}{m} \right] + \alpha^1 [u^*(0) - \bar{u}(0)], \tag{17}$$

where π^1 is the Lagrange multiplier associated with the constraint in (10), $\mu^1(m)$ is the co-state variable associated with (7), and α^1 is the Lagrange multiplier associated with the constraint that $u^*(0) \geqq \bar{u}(0)$; both π^1 and α^1 are independent of m.

It is clear at once that π^1 measures the marginal rate of substitution for generation 1 between k_1 and x_1 since this multiplier measures the increase of maximum k_1 per unit of increase in the allowable x_1 at fixed $u^*(0)$, say $\partial k_1^*/\partial x_1$ by way of notation. Furthermore, the marginal rate of substitution for generation 2 between k_1 and x_1 is $F_k(k_1, l_2)^{-1}$ if that generation maximizes its Bergson–Samuelson welfare, as was shown in section 3, or if it maximizes its own minimum utility. To show the latter one only has to note that the maximand of generation 2, on the present formulation, is

$$-x_1 + F(k_1, l_2) - c_2 \equiv k_2^*,$$

subject to constraints in none of which do x_1 and k_1 appear; by the familiar envelope theorem, therefore,

$$\left(\frac{\mathrm{d}k_1}{\mathrm{d}x_1}\right)_{k_2^*} = \frac{1}{F_k(k_1, l_2)}.$$

Now consider any interior vector (k_1^*, x_1) that is Pareto efficient between generations 1 and 2, and denote such a Pareto optimum by (k_1^*, x_1^*). At any such point the above two marginal rates of substitution are equal, and consequently $\pi^1 = F_k(k_1^*, \mathscr{L}(k_1^*, x_1^*))^{-1}$. Hence, if x_1 has been set at some value x_1^* that will secure Pareto efficiency when k_1 is maximized subject to the three constraints, the corresponding Hamiltonian reduces to

$$\begin{aligned}\mathscr{H}^1(m) = \Big[&-x_0 + F\Big(k_0, \int_0^M l_1^m \mathrm{d}\Phi\Big) - c_1^m - F_k(k_1^*, \mathscr{L}(k_1^*, x_1^*))^{-1}(x_1^m \\ &- x_1^*)\Big] \mathrm{d}\Phi(m) \\ &+ \mu^1(m)[-e_1^m u_e^* m^{-1}] + \alpha^1[u^*(0) - \bar{u}(0)]. \end{aligned} \tag{18}$$

As before, we are to choose l_1^m and x_1^m to maximize this Hamiltonian, subject to the relation in (8) determining c_1^m as a function of the control variables, l_1^m and x_1^m, and the state variable $u^*(m)$. Thus, the optimal (l_1^m, x_1^m) for each m yield a constrained maximum of the quantity

$$F\Big(k_0, \int l_1^m \mathrm{d}\Phi\Big) - c_1^m - F_k(k_1^*, \mathscr{L}(\cdot))^{-1} x_1^m.$$

This Pareto-efficient maximization has an interesting interpretation. At each m, l_1^m and x_1^m are doing their bit to maximize $-x_0 + F(k_0, l_1) - c_1 - F_k(k_1^*, \mathscr{L}(k_1^*, x_1^*))^{-1} x_1$, which is just $k_1 - F_k(k_1^*, \mathscr{L}(\cdot))^{-1} x_1$, subject to the constraints. But x_1 is $F_k(k_1^*, \mathscr{L}(\cdot))k_1 + d_1$; so the maximand is just $-d_1/F_k(k_1^*, \mathscr{L}(\cdot))$, the negative of the market value of the government debt sold to generation 1 savers. Thus, the paths $(l_1(m), x_1(m))$ must maximize the algebraic budgetary surplus of the government.

More formally, we may examine the integral of the maximand, with the constants omitted, and see from the household budget equations, the assumption of competitive factor rewards, and Euler's theorem, that this integral satisfies

$$\begin{aligned}&\int_0^M \Big[F\Big(k_0, \int_0^M l_1^m \mathrm{d}\Phi\Big) - c_1^m - F_k(k_1^*, \mathscr{L}(\cdot))^{-1} x_1^m \Big] \mathrm{d}\Phi \\ &= F_k(k_0, l_1)k_0 + F_l(k_0, l_1) l_1 \\ &\quad - \int_0^M \Big[w l_1^m + g_1 + \frac{h_1}{F_k(\cdot)} - T_l(\cdot) - T_s(\cdot) - \frac{x_1^m}{F_k(\cdot)} \Big] \mathrm{d}\Phi - \frac{x_1}{F_k(\cdot)} \\ &= F_k(k_0, l_1)k_0 + \int_0^M [T_l(\cdot) + T_s(\cdot)] \mathrm{d}\Phi - \Big[g_1 + \frac{h_1}{F_k(\cdot)} \Big]. \end{aligned} \tag{19}$$

Now let $\bar{r}_0$ denote the net rate of return on k_0 that the old must receive if their total receipts, $d_0+(1+\bar{r}_0)k_0$, are to add up to their old-age consumption claim, x_0. Then the algebraic excess earnings, $(Y_k(k_0,l_1)-\bar{r}_0)k_0$, constitute surtax revenue additional to the revenues from $T_l(\cdot)$ and $T_s(\cdot)$. Hence the maximization of our integral in (19) implies the maximization of *total* tax revenues *net* of the present value of g_1 and h_1 required to bring $u^*(0)$ up to the mandated $\bar{u}(0)$.

It is now a short, but subtle, step to the conclusion that total tax revenues are maximized – gross of g_1 and h_1. Clearly, if g and h were arbitrarily set, tax revenues could not generally be maximized without causing $u^*(0)$ to be excessive or else deficient in relation to $\bar{u}(0)$. However, the availability of the demogrant, $g+h/F_k(\cdot)$, provides a degree of freedom with which to compensate those persons having the least utility for any taxation of their savings, i.e. their second-period consumption, and thus to maintain their utility at $\bar{u}(0)$.[6] By contrast it would be empty to say that taxation to maximize Bergson–Samuelson welfare causes the revenue collected to be maximized; for while it is true that the deadweight burden on the next generation is minimized, and thus also the algebraic budgetary deficit, when the Bergson–Samuelson social welfare level is taken as given, total taxes collected could be simultaneously at a maximum only if g and h were *graduated* according to each person's m – a graduation which, if the required information were available, would permit lump-sum taxation.

There is a technical point about the revenue maximum worth noting. While the aforementioned surtax revenues figure in the total revenue collected, it is nevertheless true that at the optimum the quantity

$$\int_0^M [T_l(w_1^* l_1^m)+T_s(x_1^m)]\mathrm{d}\Phi$$

is being maximized where w_1^* is a *parameter* equal to $F_l(k_0,l_1)$ when evaluated at the optimal l_1. The fact that larger l_1^m over any interval of m between 0 and M would reduce $F_l(k_0,l_1)$, and thus lower both the before-tax wage bill and wage-tax revenue on that account, is just offset by an equal rise of the surtax revenue since, by what amounts to the factor-price-frontier relation,

$$\frac{\mathrm{d}}{\mathrm{d}l_1}[k_0F_k(k_0,l_1)-(1+\bar{r}_0)k_0+l_1F_l(k_0,l_1)]=F_l(k_0,l_1).$$

[6]At least in the present formulation, therefore, it is unnecessary to adopt the device employed by Ordover and Phelps (1975) of supposing that g and h are so distributed as to obviate private saving by the poorest persons to achieve their optimal consumption plan. See also Ogura (1977).

The foregoing analysis has no immediately apparent implications for the maximin-optimal taxation of wages and second-period consumption by persons with utilities at or near the minimum level, $u^*(0)$. However, it seems to us improbable that the marginal tax rate on the first few dollars of wage earnings would optimally be negative, i.e. $T_l'(0) < 0$. Such a property of the tax schedule would raise the eventual height of the marginal tax rate needed to extract any given revenue from middle-wage and higher-wage workers, and the heightened disincentives for lower-middle-wage workers would presumably cost more revenue than what might be gained from low-wage workers. It seems more probable that $T_l'(0) > 0$. Then, with marginal tax rates unchanged at some wl_1^m and beyond, all persons earning that much more will have to pay a greater tax on their inframarginal earnings than if marginal tax rates started at zero.

We ought to add in closing that our tax-maximization proposition does not extend to a rather different conception of social welfare – a notion of economic justice more in the spirit of Rawls. One might restrict the demogrant, to be set at some humanitarian level, to those persons who do not work, whether unable or unwilling, and seek to maximize the after-tax wage rate – more naturally, the after-subsidy wage rate – of those working persons with the lowest such wage rate. Then revenue maximization would be the objective only at the high-wage-income brackets.

5. Two implications for taxation of interest and wealth

Here we discuss two consequences of our concept of optimal taxation for the appropriate tax or subsidy on second-period consumption. The first implication pertains to the optimal marginal tax rate at the highest attained level of second-period consumption, in effect the marginal tax rate that applies to the top interest-income bracket. Phelps (1973) showed the maximin optimality, and Sadka (1976) the Bergson–Samuelson optimality, of driving the marginal rate of taxation of *wage* income to *zero* at the top level of wage earnings attained – provided, as is natural, that m has some upper bound $M < \infty$ so that there exists a highest earnings level for each wage-tax schedule. Other derivations can be found in Cooter (1978), Ogura (1977), and Seade (1977) which studies also the marginal tax rate at the bottom wage-income bracket. The question we pose is whether the analogous result extends to the optimal taxation of second-period consumption.

There is an easy heuristic argument that if the vector (k_1, x_1) is required to be *Pareto efficient*, then for both Bergson–Samuelson and maximin optimality the marginal tax rate $T_s'(x_1^m)$ must indeed be zero at the highest x_1^m. For suppose, contrariwise, that $T_s'(\max_m x_1^m)$ were positive, say, at the fiscal optimum. Then persons with the highest x_1^m would be driven to a point like y

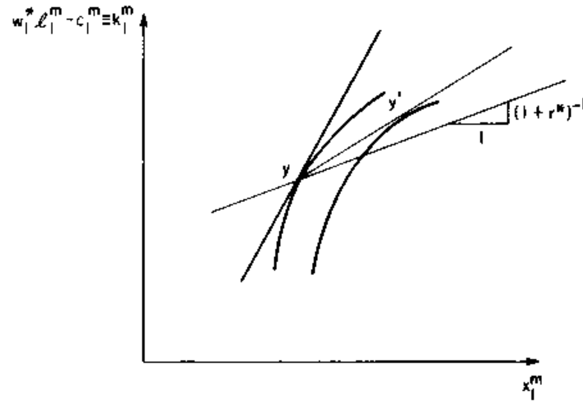

Fig. 2

in fig. 2. In this diagram each indifference curve shows the amount of x_1^m needed to compensate them for a given amount of private plus public saving extracted from them for a given wage tax function and demogrant. At y the corresponding indifference curve is steeper than the straight line with slope $(1+r_1^*)^{-1}$ because their 'budget line' has locally a slope $(1+r_1^*)^{-1}+T_s'(x_1^m)$. An infinitesimal reduction of the marginal tax rate on this x_1^m and all higher x_1^m would bend down the budget line beyond y and cause some point y' to be chosen instead of y. At y' the top savers would necessarily be saving more, through lesser c_1^m or greater l_1^m, and consuming larger x_1^m. They and possibly some near-top savers would be better off, their opportunities having been expanded; no one from generation 1 need be left worse off, since the government could introduce *ad valorem* surtaxes and subsidies to maintain the market wage rate at w^*, the rate of return at r^*, and the retired persons' receipts at x_0 despite the rise of k_1^m and perhaps l_1^m; and generation 2 would enjoy a rise of social welfare because that generation, whether of the 'maximin' or Bergson–Samuelson type, would be willing to accept the burden of additional x_1 at the rate $(1+r_1^*)$ per unit of increase in k_1 in the neighbourhood of (k_1^*, x_1^*) while the move from y to y' offers better terms than that. Therefore y could not have corresponded to a Pareto optimum – a contradiction. Hence $T_s'(\max_m x_1^m)$ cannot optimally be positive. By a similar argument it must not be negative either.

Our proposition regarding the marginal tax rate at the top is simply proved by appeal to the mathematics of the Hamiltonian formulation. If the target vector is a Pareto-efficient one, (k_1^*, x_1^*), the Hamiltonian becomes (15) in the Bergson–Samuelson case and (18) in the maximin case. In either case, it is a known property of the solution to the maximization problem that the co-state variable, $\mu^1(m)$, reaches zero at $m=M<\infty$.[7] At lower m, $\mu^1(m)>0$ to

[7]Gelfand and Fomin (1963, sec. 1.6).

signal that, by reason of the differential equation in (7), the faster utility is made to rise around m the greater will be the utility in store for persons with higher m and hence the smaller will be the remaining room to meet the target vector, (k_1^*, x_1^*); but at $m=M$ there are no implications for utility at still higher m to worry about, so $\mu^1(M)=0$.

Hence, at $m=M$ the first-order condition $\partial\mathscr{H}^1/\partial x_1^m=0$ for a maximum of the Bergson–Samuelson Hamiltonian in (16) reduces to

$$\frac{\partial\mathscr{H}^1(M)}{\partial x_1^M}=\gamma^1\left[-(1+r_1^*)^{-1}-\left(\frac{\partial c_1^M}{\partial x_1^M}\right)_{u^*(M)}\right]\mathrm{d}\Phi(M)=0.$$

Since $\gamma^1<0$, we have $(\partial c_1^M/\partial x_1^M)_{u^*(M)}=-(1+r_1^*)^{-1}$. And since $1+r_1^*=F_k(k_1^*,\mathscr{L}(k_1^*,x_1^*))$ when the target vector is Pareto efficient, it follows that $T_s'(x_1^M)=0$. Of course, a similar argument involving $\partial\mathscr{H}^1/\partial l_1^M$ shows that since $\mu^1(M)=0$, $T_l'(wl_1^M)=0$ also. Readers may verify that maximization of the maximin Hamiltonian in (18) at $m=M$ has the same implications.

It ought to be acknowledged, as has been conceded before, that the deterministic framework here makes no provision for random influences upon interest-type income some insurance against which might be Bergson–Samuelson or even maximin optimal for the government to provide. Some extreme examples, based on an *ex ante* homogeneous population, where optimal 'regressivity' of marginal tax rates occurs only beyond the experienced range have been constructed by Varian (1978). However, it must be wondered whether the insurance function of everywhere-positive marginal tax rates is in fact very important to the optimal graduation of marginal tax rates. It may also be noted that, surprisingly, the above proof does not assume that it is the workers with the largest m who have the greatest x_1^m; but perhaps this relation is implied.

Now consider the optimal taxation of second-period consumption, hence of interest income, at *any* $m>0$. If again we impose the condition that the target be some *Pareto efficient* vector, (k_1^*, x_1^*), a first-order condition for the maximum of the corresponding Bergson–Samuelson Hamiltonian in (15) is

$$\begin{aligned}\frac{\partial\mathscr{H}^1}{\partial x_1^m}=&\gamma^1\left[-(1+r_1^*)^{-1}-\left(\frac{\partial c_1^m}{\partial x_1^m}\right)_{u^*(m)}\right]\mathrm{d}\Phi(m)\\&+\mu^1(m)\left\{\frac{-e^m}{m}\left[u_{ec}^m\left(\frac{\partial c_1^m}{\partial x_1^m}\right)_{u^*(m)}\right]+u_{ex}\right\}=0.\end{aligned}$$

The terms in the curly braces also turn up in the conceptually distinct maximization problem of Atkinson and Stiglitz. As is shown there, these terms sum to zero for every m if the utility function is of the weakly separable form $u[\psi(c,x),e]$ so that each worker's marginal rate of substitution between the two consumer goods is independent of the amount of first-period effort or leisure. On this further *separability condition*, then, our first-order condition implies $(1+r_1^*)^{-1} = -(\partial c_1^m/\partial x_1^m)_{u^*(m)}$, whence $T_s'(x_1^m)=0$ for every m, not just for $m=M$. Interest income must then be tax-exempt.

Does this result extend, under the conditions of Pareto-efficiency and separability, to the maximin case? One might guess not on the thought that a tax structure designed to maximize total tax revenue would shun any form of tax exemption as nature abhors a vacuum. Yet, it is a straightforward calculation to show that the first-order condition, $\partial\mathscr{H}^1/\partial x_1^m=0$, for a maximum of the maximin Hamiltonian in (18) yields essentially the same equation as above – with γ^1 replaced by 1.

An intuitive explanation of this result is that if the taxation of labor income has been preset at its first-best optimum, there is no more to be gained in revenue from a surtax on x_1^m than from a surtax on c_1^m since, under the separability condition, leisure is no more complementary to (or less substitutable for) c_1^m than to x_1^m; apparently either surtax or both together would further weaken the incentive to work and thus cause wage-tax revenues to fall by at least the amount of the surtax revenue collected. Moreover, either surtax would cause the target (k_1^*, x_1^*) to be achievable, if at all, only through a lower $u^*(0)$ since households would not be trading x_1^m for c_1^m at the r^* associated with $F_k(k_1^*, \mathscr{L}(k_1^*, x_1^*))$ and consequently second-best tax and transfer measures would have to be taken.

It must be clear that the above tax-exemption proposition fails to hold once the Pareto-efficiency condition is dropped. Indeed, the determination of the optimal tax or subsidy to second-period consumption when the target vector is Pareto inefficient is quite transparent under the separability condition. We note that an inefficient target, $(\bar{k}_1, \bar{x}_1)$, makes the marginal rates of substitution between k_1 and x_1 unequal between generations. In the Bergson–Samuelson Hamiltonian of eq. (12), therefore, the ratio of the Lagrange multipliers must satisfy $\lambda^1=\gamma^1[1/F_k(\bar{k}_1, \mathscr{L}(\bar{k}_1, \bar{x}_1))+\delta]$, where δ, which is a function of $(\bar{k}_1, \bar{x}_1)$, is either positive or negative. Hence the Hamiltonian in (12) can now be written as

$$\begin{aligned}\mathscr{H}^1(m) &= u^*(m)\mathrm{d}\Phi(m)\\ &\quad+\gamma^1\left[\left(\frac{1}{F_k}+\delta\right)\bar{x}_1-\bar{k}_1+F\left(k_0, \int_0^M l^m \mathrm{d}\Phi(m)\right)-x_0\right]\mathrm{d}\Phi(m)\\ &\quad-\gamma^1\left[\left(\frac{1}{F_k}+\delta\right)x_1^m+c_1^m\right]\mathrm{d}\Phi(m)+\mu^1(m)\left[\frac{-e_1^m u_e^*}{m}\right].\end{aligned}$$

On the separability assumption, then, the necessary first-order condition with respect to x_1^m is simply

$$\frac{\partial \mathscr{H}^1}{\partial x_1^m} = -\gamma^1\left[\left(\frac{1}{F_k}+\delta\right)+\left(\frac{\partial c_1^m}{\partial x_1^m}\right)_{u^*(m)}\right]=0 \tag{20}$$

whence, by the consumer equations (5b) and (5c),

$$-\gamma_1\left[\left(\frac{1}{F_k}+\delta\right)-\left(\frac{1}{F_k}+T_s'(x_1^m)\right)\right]=0. \tag{21}$$

From this we can deduce that $T_s'(\cdot)$ is constant for all values of x_1^m. The optimal indirect tax schedule must be linear in x – *for all inefficient* $(\bar{k}_1, \bar{x}_1)$ *vectors.* In fact we can be more exact than that: if the target vector $(\bar{k}_1, \bar{x}_1)$ is to the left of the Pareto locus, a positive tax on saving is called for; if the target vector is to the right of that locus, a subsidy is appropriate to the target.[8] The reader can confirm that these results carry over to the maximin case.

6. Piecemeal reform and 'second best'

The first-best optimality of exempting interest from taxation, under the separability condition, is an arresting result. The same proposition reached before us by Atkinson and Stiglitz, whose argument we have corrected with the needed Pareto-efficiency condition, has already aroused wide interest. In effect it restores the doctrine of Corlett and Hague (1953) which states that a tax on present consumption, equivalently a subsidy to saving, would be optimal if leisure were more complementary to present consumption than to future consumption; and a tax on saving or interest, equivalently a subsidy to present consumption, would be optimal if the differential complementarity is the reverse. Although the theorems here do not strictly apply to any of our own earlier work on maximin taxation, because our tax structures were linear rather than graduated, we somehow missed (or repressed) an important observation: there is *no utility-theoretic presumption* that the taxation of interest income would gain more in interest-income levies collected than it would lose in revenue from wage income – just as there is no presumption that a surtax on present consumption would raise revenue on balance – in those cases studied where wage-income taxation, as well as interest-income taxation, is being set to maximize total revenue collected.

That said, it must still be emphasized that the sort of piecemeal taxation reform which the above conclusions may encourage is not without risks.

[8]See also Atkinson and Sandmo (1977) for the corresponding point in relation to steady-state paths.

Even though we take separability for granted, it is not a corollary of our findings that the economy would be led (as if by an invisible hand) to the Pareto locus were interest-income taxation to be abolished; that efficient outcome would be assured only if wage-income taxation were already, or was going simultaneously to be, optimized as well. And even if exempting interest from taxation would move the economy nearer to that locus, other steps might have to be taken simultaneously to prevent a fall of social welfare for one generation or the other.

The risk that lighter taxation of interest-type income carries for the *next* generation is straightforward – assuming, as heretofore, that the next generation will tax optimally and redeem in full the public debt. Each increase of the after-tax rate of return to saving will burden the next generation with larger x_1 – necessarily if second-period consumption is a noninferior good – while the consequent k_1 may fail to increase (if it increases at all) by a compensating amount. It is immediately evident from the geometry of fig. 1 that k_1 must rise per unit of increase in x_1 at a rate at least as great as $F_k(k_1, l_2)^{-1}$ in order that the maximized welfare of generation 2 not fall as x_1 is increased. In algebraic terms, the next generation will be injured if and only if the constraint under which it operates,

$$F(k_1, l_2) - x_1 - c_2 = k_2, \tag{22}$$

is contracted on balance by the change in (k_1, x_1), i.e. if and only if $\mathrm{d}k_1/\mathrm{d}x_1 \leqq F_k(\cdot)^{-1}$.

This result has an interesting (though deceptive) interpretation. In the linear tax case, with the after-tax rate of return to saving denoted by ρ and the after-tax wage of a standard ($m=1$) unit of labor denoted by ω.

$$k_1 = F(k_0, l_1) - \omega l_1 - g + \sigma - x_0,$$

$$x_1 = h + (1+\rho)\sigma;$$

hence the derivative with respect to ρ of the left-hand side of the constraint equation in (22) is

$$F_k(k_1, l_2) \cdot \left\{ [F_l(k_0, l_1) - \omega] \frac{\partial l_1}{\partial \rho} + \frac{\partial \sigma_1}{\partial \rho} \right\} - \left[\sigma_1 + \frac{\partial \sigma_1}{\partial \rho}(1+\rho) \right].$$

Evidently that derivative is non-negative, so that generation 2 is not being harmed by rising ρ, if and only if

$$(F_l - \omega)\frac{\partial l_1}{\partial \rho} + \frac{1}{F_k}\left\{ [F_k - (1+\rho)] \frac{\partial \sigma_1}{\partial \rho} - \sigma_1 \right\} \geqq 0. \tag{23}$$

This 'marginal revenue' condition means that total tax revenue, discounting future revenues by $F_k(k_1, l_2)^{-1}$, is locally nondecreasing in ρ.

The proper deduction to be drawn, apropos discrete changes in ρ, is that the next generation will surely be harmed by a given increase of ρ if there results a rise of the par value of the *notional* debt as calculated at the *original* or no-reform rate of return, $F_k(k_1^0, \mathscr{L}(k_1^0, x_1^0))$, i.e. a rise of $x_1 - k_1 F_k^0(\cdot)$; as fig. 3 clearly demonstrates, every (k_1, x_1) outcome that is as

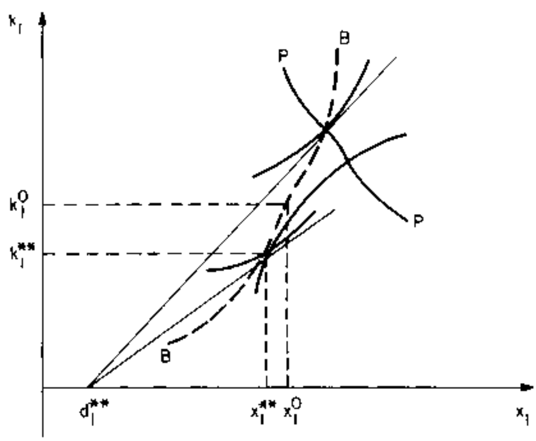

Fig. 3

good or better for the next generation than the no-reform outcome, (k_1^0, x_1^0), makes $x_1 - k_1 F_k^0(\cdot)$ smaller than the original $x_1^0 - k_1^0 F_k^0(\cdot)$. Yet it is also clear from the geometry of fig. 3 that the *actual* debt associated with the *new* and lower rate of return might rise without harm to the next generation; for example, a rise of x_1 for which the increase of k_1 just compensates, causes $x_1 - k_1 F_k(\cdot) > x_1^0 - k_1^0 F_k^0(\cdot)$ owing to the curvature of generation 2's welfare contours. The producer's surplus of the next generation is

$$F(k_1, l_2) - d_1 - k_1 \cdot F_k(k_1, l_2),$$

the derivative of which

$$-\frac{\partial d_1}{\partial \rho} - F_{kk}(k_1, l_2) \cdot k_1 \frac{\partial k_1}{\partial \rho},$$

can be positive whether or not $\partial d_1 / \partial \rho > 0$; for even if the increase of k_1 is due only to a ρ-induced rise of *private* saving, tax revenue and thus public saving being down by assumption, the next generation gains on that account by paying out a diminished marginal product on the original quantity of k_1.

We have shown that a steadily rising ρ, ultimately meeting $F_k(\cdot)-1$, will continue not to harm the next generation as long as the 'marginal revenue' condition in (23) is satisfied. The expression in the curly braces there may be written as

$$\left[\left(\frac{1+r}{1+\rho}-1\right)\frac{\partial\sigma_1}{\partial(1+\rho)}\cdot\frac{1+\rho}{\sigma_1}-1\right]\sigma_1\,;$$

its algebraic sign gives the direction of interest-income tax revenue as ρ is rising. To calculate that sign one wants something like the 25-year rates of return to saving and the corresponding elasticity. Boskin (1978) has estimated the *annual* after-tax interest-rate elasticity to be 0.4. Using his average *annual* after-tax rate of return of 0.03, we then calculate[9] the 25-year $(1+\rho)$-elasticity to be 0.76. Furthermore, taking his average *annual* before-tax rate of return of 0.07, we find the 25-year interest-factor ratio, *our* $(1+r)/(1+\rho)$, to be 2.67. Then the term in the brackets above is positive, equal to 0.27. But high confidence in the 0.4 estimate is not widespread; a mere one-quarter reduction of Boskin's elasticity and thus ours as well just about reduces the bracketed expression to zero. And we have no estimates of the wage-tax term.

It is especially uncertain whether the wholesale elimination of interest taxation would raise tax revenue figured at the original rate of return. If ω is so large that σ is substantial then, as ρ also becomes large, it may well be that $\partial l/\partial\rho$ and $\partial\sigma/\partial\rho$ will become small or even negative because the income-effects of further increases of ρ become large enough to outweigh the substitution effects. And if ω is close to $F_l(k_0, l_1)$ there can be little help from $\partial l_1/\partial\rho>0$ anyway. As ρ is increased all the way to $F_k(k_1, l_2)-1$, therefore, x_1 may eventually increase so steeply against k_1 as to jeopardize all the gain (if any) to the next generation from the first increments of ρ.

The risks to the *present* generation posed by lighter interest-income taxation raise more delicate issues. At first it might seem that the present generation will lose nothing from increasing its ρ since, as assumed above, it can protect its after-tax wage rates through lighter wage taxation if needed. It is true that the present generation need lose nothing as long as its private saving is large enough to accommodate both the desired public borrowing and the minimum capital that the next generation will need to meet old-age claims. However, the next generation may similarly avoid injury up to a point by denying the retired population some of their anticipated consumption claims – if it does not thus prompt the succeeding generation to do

[9]The 25-year $(1+\rho)$-elasticity is

$$\left(\frac{\partial\sigma_1}{\partial\rho}\frac{\rho}{\sigma_1}\right)\left(\frac{1+\rho}{\rho}\right)=(0.4)\left[\frac{(1.03)^{25}}{(1.03)^{25}-1}\right].$$

the same. Any risks to the present generation from raising ρ – even by the *fait accompli* of tax credits for saving made deductible from wage tax liabilities – therefore hinge on the presence of one or more self-imposed constraints it may place on its actions *vis-à-vis* the next generation in recognition of the latter's deterrent powers of retaliation. Then lighter interest-income taxation may induce heavier wage-income taxation and/or lower transfer payments than would otherwise have been possible in order still to meet the constraints, and the net result may be a reduction of the present generation's welfare.

One constraint that the present generation might adopt is the one-parameter restraint of budget balance:

$$x_1 - k_1 F_k(k_1, \mathscr{L}(k_1, x_1)) \leqq d_1^{**}, \; d_1^{**} = d_0.$$

While we shall see that this restraint is inappropriate for the attainment of any Pareto-efficient or 'first-best' optimum – for that one wants the appropriate two-parameter restraint, $x_1 - k_1(1 + r_1^*) \leqq d_1^*$ – the balanced budget constraint has been of long-standing interest and it may be realistic.[10] If that is the only self-restraint exercised by the present generation, and if we put aside the next generation's willingness and ability to redeem the public debt, the government can then select any point of the opportunity locus, labelled *BB* in fig. 3, that is traced out by clockwise rotation of a straight line around the point $(x_1 = d_1^{**}, k_1 = 0)$. We shall assume that *BB* makes x_1 a single-valued function of k_1 – at least there is a largest x_1 for every $k_1 > 0$; its slope relative to the k-axis,

$$\left(\frac{\mathrm{d}x_1}{\mathrm{d}k_1}\right)_{d_1^{**}} = \frac{F_k(\cdot) + k_1\left[F_{kk}(\cdot) + F_{kl}\dfrac{\partial l_2}{\partial k_1}\right]}{1 - k_1 F_{kl}\dfrac{\partial l_2}{\partial x_1}},$$

necessarily starts positive and approaches 1 minus the rate of depreciation as $k \to \infty$. Travelling onward along *BB* from its origin corresponds to improving potential welfare for generation 2. Hence *BB* slices the next generation's isowelfare contours from below. At each intersection *BB* is steeper than the local isowelfare contour.

If the present generation maximizes its own social welfare subject only to the balanced-budget constraint, besides the omnipresent Mirrleesian differen-

[10] In Phelps (1965) it is speculated that a generation might embrace the 'fiscal neutrality' of lifetime budget balance out of fear that unbalancing the budget would provoke the next generation to repudiate the increment of the debt. Some results on the second-best optimality of taxing interest income in this setting were recently derived by Krohn (1978). As noted earlier, Atkinson and Stiglitz also postulate budget balance.

tial equation in (7), it will choose a 'second-best' target (k_1^{**}, x_1^{**}) where – assuming an interior solution – BB is tangent to the best of its intersecting isowelfare contours, as illustrated in fig. 3. It follows immediately that at this second-best target the isowelfare contour of generation 1 is steeper than that of generation 2. Under the separability condition, as we showed, the slope of the former is precisely the optimal $F_k(\cdot)^{-1} + T_s'(x_m)^{**} \equiv (1+\rho^{**})^{-1}$, for all m, and the slope of the latter is $F_k(\cdot)^{-1}$. At the second-best optimum, therefore, $T_s'(x_m)^{**} > 0$, i.e. $1+\rho^{**} < F_k(\cdot)$. To prove this proposition more directly we would remark that, on the separability assumption, the first-order condition for an interior maximum of the second-best Hamiltonian, $\mathscr{H}^{**}(m)$, corresponding to the balanced-budget constraint is

$$\begin{aligned} 0 &= -1 + k_1 F_{kl}(k_1, \mathscr{L}(k_1, x_1)) \frac{\partial \mathscr{L}}{\partial x_1} \\ &\quad + \left(-\frac{\partial c_1^m}{\partial x_1^m} \right)_{u^*(m)} \left\{ F_k(\cdot) + k_1 \left[F_{kk}(\cdot) + F_{kl}(\cdot) \frac{\partial \mathscr{L}}{\partial k_1} \right] \right\} \\ &\equiv \frac{\partial \mathscr{H}^{**}(m)}{\partial x_1^m}, \end{aligned}$$

after dividing by the associated Lagrange multiplier. It follows that $(-\partial c_1^m / \partial x_1^m)_{u^*(m)}$ is independent of m and equal to the slope $(\mathrm{d}x_1/\mathrm{d}k_1)_{d_1^{**}}$ of BB at the second-best maximum.

It has just been shown that a generation bound only by budget would, for its self-interest, drive ρ below the net market rate of return to investment, moving inward along BB until the optimum degree of 'monopolistic' exploitation is reached. If we hypothesize that its adherence to traditional tax practice would be second-best optimal for the present generation, therefore, and if the present generation is in fact bent on preserving budget balance, then any tax reform on its part that abolished or merely lightened the taxation of interest income would necessarily reduce that generation's social welfare – given that the next generation will maximize its welfare independently of any precedents or innovations made by the present generation. However, the hypothesis of a second-best *status quo ante* is obviously overstrong.

A weaker hypothesis is that, owing to the traditional practice of taxing interest income, the no-reform outcome (k_1^0, x_1^0) would fall short of the intersection of BB with the Pareto locus – it would be a point on BB southwest of PP – though it would not lie farther down BB than the second-best (k_1^{**}, x_1^{**}). Even if the economy had been overtaxing interest income from the second-best standpoint, to take one contrary example, it might have been undertaxing wage income and thus not tending to fall short of its second-best optimum. Such a no-reform outcome, (k_1^0, x_1^0), is illustrated in fig. 3. At that (k_1^0, x_1^0), generation 1 cannot move outward along BB without reducing its own potential welfare; and it can never move inward without injuring generation 2. If we made the additional hypothesis that the present generation will constrain itself not to reform its taxation in ways that injure the next generation, whether out of a sense of fairness or (again) out of fear of retaliation, then the present generation will not wish to deviate from (k_1^0, x_1^0).

But if a point like (k_1^0, x_1^0) becomes the present generation's target vector, the corresponding taxation that is third-best optimal makes $\rho^{***} < F_k(k_1^0, \mathscr{L}(k_1^0, x_1^0)) - 1$. This follows from the fact that at (k_1^0, x_1^0) the isowelfare contour of generation 1 slices that of generation 2 in the same way as at (k_1^{**}, x_1^{**}) – because both target vectors lie on BB at points southeast of the Pareto locus. Therefore the outright abolition of interest-income taxes by the present generation would make its potential welfare smaller than it could be – and perhaps smaller than it would be if instead the generation maintained ρ at its initial level.

Of course it is not denied that there exists a subset of target vectors different from (k_1^0, x_1^0) each of which would permit a mutual welfare gain for the two generations over that available with (k_1^0, x_1^0). On the foregoing hypothesis about the no-reform (k_1^0, x_1^0), however, the set of mutually superior target vectors – including an interval of the Pareto locus – lies entirely beyond the BB locus. Hence, each of these targets would require an increase of the public debt for its support. Until this implication is understood and accepted in fiscal policy-making, then there is a danger that exempting interest income from tax would ultimately reduce the welfare of the present generation.

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INTRODUCTION

This last paper is not so much an even-handed survey, which is the sort of paper I was probably meant to prepare when invited, as a partisan defense of Rawls's vision of the just economy against the last-ditch defenders of utilitarianism in its various shapes. As such, it seems to me, it makes a fitting epilogue to this volume which, as will have been evident, comes down (after much searching) firmly in Rawls's camp—or at any rate at a not terribly distant approximation thereto.

If there is anything I remember about this paper with special warmth it is its awareness of the structuralism contained in "Rawls"—it matters who does what with whom and what are the available alternatives—compared to the "welfarism" (to use Sen's apt term), characteristic of utilitarianism, according to which the desirable distribution of utilities is not dependent upon persons' contribution to their society. That economic justice is what is owed to those who contribute is a thought that will be reverberating profoundly for many years.

An implication of that view is that the welfare economics of taxation, as developed in the essays of Part IV (for example), is an inexact model of the Rawlsian view: It is the reward to the least-rewarded contributor that we should seek to maximize.

RECENT DEVELOPMENTS IN WELFARE ECONOMICS: JUSTICE ET ÉQUITÉ

It was widely thought, at the time, that Arrow's 1951 Impossibility Theorem sounded the death-knell for welfare economics in its Edgeworth–Pigou and Pareto–Bergson versions. Yet the social-welfare theory of social choice behavior is today the subject of more research than ever. It is as though a couple once written off as incompatible had turned up again, older and wiser yet still going at it with the same hopefulness as before. Herewith a guide, selective and informal in view of the space and time available, to this renascent field.[2]

1. *New and resuscitated conceptions of social welfare*

Samuelson says somewhere that the cash value of an idea lies in its vulgarization (thus vulgarizing James?) The hardest part of this survey for its writer has been to reconstruct the shape of welfare economics in the early post-war period from which "recent developments" (the phrase in the mandated title of this survey) may be said to have "developed." No doubt I have vulgarized quite a lot in my effort at a concise and simple history.

[1]I am grateful to the discussants at the 3rd World Congress, Kenneth J. Arrow and Peter J. Hammond, and to the chairman, Armartya K. Sen, for their comments and suggestions.

[2]A sample of the papers mentioned in this survey is contained in my reader, *Economic Justice* (1973).

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1.1. *Welfare theory without interpersonal comparisons? Expanded social choice.*[3]

Harrod asked in 1938 how an economist, anticipating that it would cause uncompensated damage to landowners, could prescribe repeal of the Corn Laws unless "... individuals are treated in some sense as equal. Otherwise how can the loss to some ... be compared with the general gain?" Robbins (1938) then commented that the economist advocating a social decision ought to make his value judgments explicit and recognize that they are beyond economics.

The compensation tests formulated by Kaldor (1939) and Hicks (1940) were evidently an attempt to find a middle course between the arid wasteland of Positivism and the overgrown swamplands of Subjectivism.

Suppose that the collective decisions of society obey the rational-choice axioms of revealed preference. If point P in utility space is the point that was chosen before the discovery of a new potential policy, then there was no other point on the old Utility Feasibility Curve, labelled PP′ in fig. 1, which was (strictly) preferred to P. Supposing that society's preferences are Paretian, so that more utility is preferred to less for any individual in society, there was in particular no point available on PP that

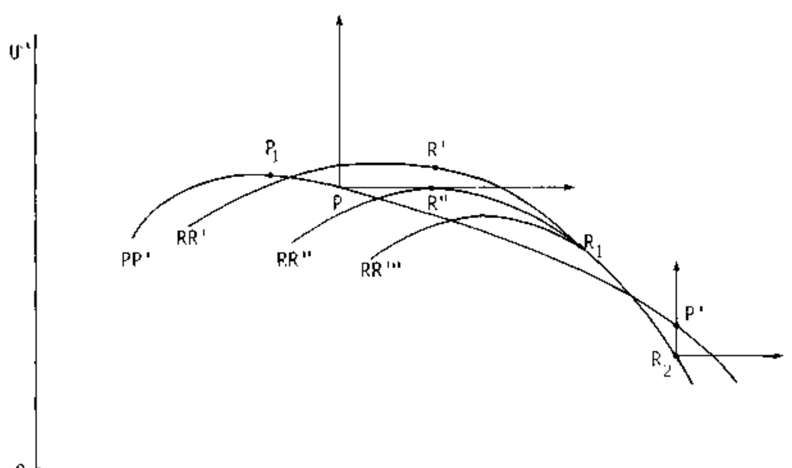

Figure 1. Diagram in utility space.

[3]See also the excellent treatments by Chipman and Moore (1973), Graaff (1959), and Little (1950).

was north-east (nor north nor east) of P, whatever else we may know about PP.

Now if the new policy would establish a new UFC, like RR′, which still contains the old point P and further opens up new territory north-east of P, like point R′, then we should be willing to say that, under existing social-choice behavior or social preferences, society's *real income* has increased: there has been an expansion of social choice so far as this community is concerned. If society actually choses R_1 south-east of P, we do not regard this choice behavior as contradicting the above proposition. R_1 is revealed to be preferred (or indifferent) to a point like R′ which is preferable to P. The increase in real income is "taken out" in so large a gain to the beta-people that the alpha-people actually suffer some loss of utility.

Kaldor realized that real income could still be said to have increased, whether or not the old point P is still available, if the new UFC makes available any point which is Pareto-better than P. Thus, if it is inferrable that the new UFC makes available a point like R″ east of P then we can again say that there has been an increase of real income (in this society's eyes) despite the choice of R_1 over this R″. In Kaldor's terms, the gainers could compensate the losers, by relinquishing R_1 for R″, and still come out ahead. The objection by some that this is no consolation to the losers, since society is not requiring the compensation, is difficult to understand: It is not an attribute of any of the familiar welfare functions that society not move north-west or south-east whenever it can move north or east.[4]

The argument prompted two complaints, both invoking the possibility of intersections between the old and new Utility Feasibility Curves.

Scitovsky (1941) challenged the adequacy of the Kaldor test for an increase of real income by posing this question: What if society were to choose not R_1 which is north-east of PP but instead a point like R_2 which is south-west of PP? Then it would seem, by the application of a second Kaldorian test around R_2, that there could be said to be an increase of real

[4]What if later the UFC slipped down to R‴ while R_1 is still chosen? Having learned by then that R_1 was preferred or indifferent to R″ which is better than P, we would not presumably scrap our declaration that real income had increased between the old situation affording P and the new situation (still) affording R_1. (Hicks on the other hand would declare that real income was lower under RR‴ than PP; that seems unwarranted.) On the other hand, had the new policy immediately produced the new UFC RR‴ rather than RR″, then the Ideal Econometrician would calculate that the gainers could not compensate the losers. Because we could not then infer that R_1 is revealed preferable to some point Pareto-better than P, Kaldor would be unwilling *under the available information* to declare that real income had increased.

income if society went back (or could go back) to PP, because the latter makes available a point like P′ which is Pareto-superior to R_2. But the objection strikes a dissonant note: to make sense of the first Kaldor test, we had to suppose that the point originally chosen (P) was preferred or indifferent to all other feasible points and that preferences are Paretian, so that a Pareto-inferior point is never preferred or indifferent to a Pareto-better one; and this not by accident, but because society makes rational choices. Hence R_2 cannot be preferred or indifferent to P; it must be worse. So the society simply would not utilize the movement of the UFC to RR″ in order to choose R_2 (worse than P) when it could choose R″ (better than P). Of course we could suppose, contrary to the spirit of the whole approach, that collective decisions occur with an "error," thus admitting the possibility of R_2. In that case, though, passing Scitovsky's double-Kaldorian test, as RR″ does under the choices P and R_1, would hardly assure us that "welfare" (to use Scitovsky's term) is greater at R_1 than at P because R_1 may still be a serious mistake even though not so visible a one as R_2.

An objection raised by Samuelson (1950) appeals not to the possibility of errors in social choice but to the possibility that social preferences will change at some future time in a way that is now unforeseeable. Suppose that society, having opted for RR″ over PP, finds later that its social preferences make P_1 preferable to P. Since RR″ fails to offer a point east (or north-east) of the higher P_1, the economist cannot say that RR″ will then offer more real income than PP would have done under these new social preferences. Or suppose that P′ and R_2 later become the points of choice. Then PP would seem to offer an increase of real income over RR″ under the new preferences. According to Samuelson, the new Utility Feasibility Curve can be declared to offer an increase of *potential real income* – i.e. an increase of real income for all logically possible social preferences (of the Paretian type) – if and only if for *every* (Pareto-optimal) point on PP the new Utility Feasibility Curve offers at least one point Pareto-superior to it, hence lies "uniformly outside" (in an appropriate sense) the old PP.

Let us suppose that a new policy would be confidently predicted to offer expanded social choice – either in Kaldor's sense of increased real income or in Samuelson's more demanding sense of increased potential income. Should that attribute of the new policy be deemed sufficient for the economist to recommend or prescribe the new policy? Going farther, does that attribute obligate the economist to make public his discovery of the new policy? That was, after all, the original problem: to find grounds on which the economist might prescribe without having to appeal to any but the most widely accepted value judgments (such as Paretianism).

The answer, I should think, is no. Though it expands social choice, as RR″ does, that would surely not warrant nor compel an economist to recommend or endorse the new policy if he disliked its actual (predicted) distributive consequences, for example, a move to R_1. Further, why should those economists advocating the new policy regard it as a merit, or a qualification for their supporting it, that it makes R″ feasible if it is recognized that R_1 would be chosen in any event – under RR‴ quite possibly as well as R″? To make one's advocacy of the new policy turn on the feasibility or not of a Pareto-improvement when none is going to occur smacks of an "irrelevant alternative." (Perhaps it will be replied that this objection is another application of mistaken "end-state principles" to borrow Nozick's (1974) phrase: to some, the conditions under which, or the processes by which, R_1 is selected over P make a difference.)

There is the further problem: If economists did agree to make passing-the-Kaldorian-(or Samuelsonian)-test at least a necessary condition (whether or not also a sufficient condition) for their professional endorsement of any new policy, they might never find a policy they could endorse. The now prevailing wisdom is that only rarely if ever does an opportunity arise for a Pareto-improvement (like R″ from P). Where there are many people in society and they are heterogeneously endowed with many different kinds of property and ability, every policy innovation appears to reduce some individuals' before-tax wealth and thus, under graduated taxation, their after-tax wealth as well. How could they all be identified? If identified, could they be compensated through either lump-sum like adjustments of tax credits or ordinary tax rates without the likelihood or at any rate the risk that some other individuals would then suffer net losses of after-tax wealth (if only because some of these latter losses would go unidentified or else be misattributed to other causes)?

The suspected irrelevance of the Kaldor test and the suspected paucity of policy proposals that would pass that test has led some writers to a revised position: Why not regard the function of the welfare economist simply to acquire and disseminate information that would push outward, anywhere or everywhere, the outer envelope of the Utility Feasibility Curves? Graaff (1957) concluded that the social function of the economist is simply to discover and make known his findings on how the economy works. Lancaster (1958) postulated that any expansion of the outer envelope of the Utility Feasibility Curves which leaves the original point still available constitutes expanded social choice and can thus be approved. The thrust of "theoretical welfare economics" (the use by Samuelson (1956), Graaff (1957), and others of the Bergson social welfare function) appears to be the same: Whatever the welfare function that society is maximizing – the welfare econometrician may not be able to estimate it and in any case the modeler need not specify it beyond the assumption that it is Paretian – society would not want to operate in the interior of the outer envelope (or frontier) of the Utility Feasibility Curves (each one corresponding to a set of policies and institutions).

Yet, that conception of the role of the economist has two defects possessed also by the compensationist approach. First, we have no assurance, nor much evidence to suggest, that our political system makes choices as if to maximize a welfare function, *any* welfare function. It is not known (or widely supposed) that the social choice-behavior of governments satisfies the axioms of revealed preference. (We know only, from Samuelson (1956), that if society were maximizing a quasi-concave welfare function of quasi-concave utilities then, under lump-sum taxation at any rate, the social choices would obey those axioms.) Indeed Arrow's theorem (discussed below) shows that where there is a perfect democracy of "ideas" (no Dictator to impose his ideas) and these potentially conflicting "values" of individuals are not unified by some agreed-upon distributive ethic – people agree only to abide by the Pareto Principle and Independence of Irrelevant Alternatives – then there cannot exist generally (for every configuration of the individual preference orderings) a social ordering which satisfies the transitivity requirement of "collective rationality."

The other defect of this conception is that the economist who is told it is his social function to report any discoveries that would push out the utility feasibility frontier (anywhere) will find this role unsatisfactory if he expects that some of his researches, if reported, will be used to worsen social welfare as *he* sees it (whether or not social choice satisfies the requirements of rational choice). So the decision merely to release or to disseminate his information, much less advocate or prescribe, presupposes the value judgment on the economist's part that it will not be dangerous for social welfare (as he sees it) to do so.

But in order to defend to others (or even perhaps to justify carefully to oneself) one's decision to impart information, to advise, one has to make one's value judgments "explicit" (to use Robbins' term). And to be able to make them explicit one has to have a set of concepts in terms of which to communicate. The basic purpose of welfare theory, I take it, is to develop and to study the relations among such concepts.

1.2. *Welfare theory without interpersonal comparisons? Arrow's theorem*

Even before the finally aborted exploration of the compensation test, Harrod (1938) and Bergson (1938) and probably most other theorists of the thirties took it for granted that we could not expect to construct a ranking or (social-preference) ordering of alternative social states (corresponding to alternative social policies) of the individualistic type – a ranking which somehow gives weight to the well-being of every individual in society – without our having to make interpersonal comparisons (of some kind or another) of those individuals' well-being. Arrow's path-breaking book (1951) was also path-ending in this respect: it produced an explicit and rigorous argument for the truth of that intuition.

Every individual is postulated to have a preference ordering over all social states. An individual's ordering is subject to change; in particular, it could vary independently of other individuals' orderings; for, while a person's ordering reflects his "values," we want to allow for opposing self-interests or opposing values or both. Then we may define a Generalized Social Welfare Function to be a mapping from the set of all logically possible lists or censuses of the individuals' preference orderings to the set of all logically possible social-preference orderings over the alternative social states. Hence, in ranking social states x and y we are provided with information of the sort: i prefers x to y and j prefers y to x. We do not know (perhaps cannot know) that we would prefer not to be in i's shoes in state y compared to j's shoes in x.

Arrow then shows that no Generalized Social Welfare Function exists if we require that such a function satisfy the requirements of Unrestricted Domain, Non-Dictatorship, the Pareto Principle, and Independence of Irrelevant Alternatives. The difficulty is that for some configurations of the individuals' preference orderings, our attempt to meet the above requirements generates a sequence of social preferences between pairs of social states that displays cyclicity or intransitivity – in the manner of Condorcet's paradox of voting.

Would knowledge of the intensity of each person's preferences among social states provide a way out of the difficulty? Sen (1970) has produced an argument proving that the existence of cardinal utility indicators (without interpersonal comparisons of them) does not suffice to escape the difficulty.

Why fret over the non-existence of Arrow's Generalized Social Welfare Function, one Welfare Function for each configuration of the individuals' respective preferences, when it may be the case in fact that the actual configuration of preferences does not lead to cyclicity under, say, majority voting or some other procedure satisfying Arrow's requirements? Arrow's position presumably is that it cannot be satisfactory to use a procedure for "aggregating" individuals' preferences which only by luck fails to produce crazy results.

Some have wondered whether Arrow's "difficulty" was not a technical coup de grace dealt to a method of social decision-making that is faulty in its first premise: that it is satisfactory (or would be satisfactory if it were possible) to base social decisions on information about individuals' preferences without our making judgments about those individuals' comparative well-being within and across social states. This is a misunderstanding of Arrow's meaning, however, as he made clear enough as far back as the 1951 *ur*-text. Arrow interprets individuals' preferences as already reflecting their own respective interpersonal comparisons of well-being and consequent personal social-welfare judgments; his persistent use, even in his book's title, of the term *values* in connection with individuals' preferences over states emphasizes this interpretation.[5] But if we place that interpretation upon the individuals' preferences, then why is it plausible of Arrow to demand that a Social Welfare Function exist for all logically possible configurations of individuals' (social) preferences? Maybe they are bound to be identical, everyone being (say) an Edgeworth–Pigou utilitarian making identical interpersonal comparisons of cardinal utilities. And if the interpersonal comparisons that individuals make are heterogeneous, do we not still lose the logical possibility that my (social) preferences over states can vary in every (logically possible) way without others' preferences showing concomitant changes of some sort or another?[6] Peter Hammond voiced (at the Congress) his conclusion that Arrow's framework does not apply to the problem of likeminded individuals making conflicting judgments as to whether i in state x is better or worse off than j in y. His point is that even if the interpersonal comparisons are heterogeneous, they give us more information than is imparted by the individuals' social orderings alone; so a wider kind of social welfare function than Arrow's seems to be appropriate.

Whatever the resolution of this matter, Arrow's book has produced two effects of lasting importance in addition to the substance of the theorem itself: Its methodological innovations have enriched the concepts in terms

[5] Professor Arrow has confirmed this interpretation in conversation.

[6] Arrow replies that one needs only to come up with one triple of states displaying cyclicity. Hammond alerts us to an impossibility theorem for aggregating *fixed* individual orderings in a forthcoming paper by Kemp and Ng in *Economica*.

of which we can discuss social welfare functions. And, perhaps ironically, by finally discouraging the quest for welfare functions without interpersonal comparisons it has at last faced us with the question of the kinds of interpersonal comparisons we are willing to make and the sorts of welfare function each such kind of comparison will allow.

1.3. *Classical utilitarianism up to date*

The classic in social preference orderings is, of course, the utilitarians' welfare function, the sum of individuals' "utilities:"

$$\begin{aligned} W &= B(U^1(C^1), \ldots, U^n(C^n)) = U^1(C^1) + \cdots + U^n(C^n) \\ &= S(C^1, \ldots, C^n) \end{aligned}$$

where C^i stands for the ith person's consumptions or experiences, $U^i(\cdot)$ the resulting utility for that person, B is for Bentham, and W is social welfare. State x is better than state y if and only if $S(C^1_x, \ldots C^n_x) > S(C^1_y, \ldots C^n_y)$. Obviously it will not matter for the function S if we replace the Us in the function B by Vs where $V^i = a_i + bU^i$, $b > 0$ and independent of i, $i = 1, \ldots, n$. Further, once it is specified what the ordinal properties of S are, the shape of its contours, then the underlying cardinality of Bentham's conception is of no further significance. (Bergson, Samuelson and Graaff always emphasized that any ordinal transformation of the Us, if accompanied by an appropriate transformation of the B function, will leave the contours of S undisturbed. Why did they?)

Sidgwick (1907) was uneasy over the linearity of Bentham's function B. He seemed to favor replacing B by some function $W(U^1, \ldots, U^n)$ which is increasing, strictly concave (or quasi-concave) and symmetrical. Because, in blackboard space, the contours of the function W are strictly convex and symmetrical around the forty-five degree line from the origin, such a function is sometimes called "equality-preferring;" but whether its maximization will in fact give equal Us depends also upon the shape of the opportunity set (UFC or whatever) in that space. Because Sidgwick gives importance to the pattern of utility levels realized in any two states, not just the changes in utilities when moving from one state to the other, Sidgwick is more cardinal than Bentham. He needs both to sign $U^1_x - U^i_x$ and to compare its magnitude to that of $U^i_x - U^i_y$ for every i. Even the absolute level of U^1_x may make a difference to Sidgwick. Any monotone transformation of the Us, even $a + bU^i$ or (unless W is homothetic) bU^i,

$b > 0$, would require redrawing the W contours in order to keep the corresponding S contours unchanged.

Why should the marginal rate of substitution between U^1 and U^2 (say) depend upon U^i ($i \neq 1, 2$)? Making the obvious separability postulate, Fleming (1952) showed that W and S can be written as an additive (additively separable) function:

$$\begin{aligned} W(U^1, \ldots, U^n) &= w(U^1(C^1)) + \cdots + w(U^n(C^n)) \\ &= f^1(C^1) + \cdots + f^n(C^n) \\ &= S(C^1, \ldots, C^n), f' > 0, w' > 0. \end{aligned}$$

Suppose that $f^{i\prime\prime} < 0$ for all i (as would have to be the case if W is symmetrical *and* the utility functions are all alike). An ordinalist-utilitarian would say that it is without consequence to say that $f'' < 0$ "in part because $w'' < 0$ (Sidgwick's concavity) and in part because everyone's $U'' < 0$ (concavity of U)." One can agree (I think) and still insist that, *cet. par.*, an increase in inequality-averseness – an unambiguously closer approximation to right-angledness by the contours of W – would reshape the f function, presumably intensifying its "curvature." The shape we give to f is influenced by both our "socio-psychology" and our "ethics."

Sen (1973) has attacked Bentham because the latter would give a person whose marginal utility is greater than another's at every equal level of consumption the larger share of the pie – even if, because the first person is so efficient at extracting utility from consumption, his total utility would also be larger than the other's at every equal consumption level. In his 1973 book, Sen proposes what he calls the Weak Equity Axiom:

> Let person i have a lower level of welfare than person j for each [equal] level of income [between them]. Then in distributing a given total of income among $\underline{n}$ individuals including i and j, the optimal solution must give i a higher level of income than j.
> [Bracketed words are mine.]

It is immediately evident that Sidgwick's introducing just a little bit of concavity in the W function would not save the resulting optimal allocation of the fixed pie from Sen's objection; that allocation too, being close to Bentham's optimal allocation, would violate the Weak Equity Axiom. Sen works out the example $U^2(C) = mU^1(C)$ with $m < 1$ and

$$W = \frac{1}{\alpha}[(U^1)^\alpha + (U^2)^\alpha]$$

which is to be maximized subject to $C^1 + C^2 = k > 0$. While strict concavity of W requires only $\alpha < 1$, the allocation will show $C^1 < C^2$ (as required by the Weak Equity Axiom) if and only if $\alpha < 0$. (The borderline case, $C^1 = C^2$, occurs where $\alpha = 0$, which is interpretable as the logarithmic W function.) As Sen Notes: "There is an analogy here [when $\alpha < 0$] with Ramsey's 1928 social welfare picture with [its upper bound] level of 'bliss'."

1.4. *The neo-utilitarians: Vickrey and Harsanyi*

What I have labelled neo-utilitarianism (aptly or not) is developed in papers by Vickrey (1945, 1960) and Harsanyi (1955). This doctrine would have us interpret the concept of impartiality in our ethical preferences (over alternative social states) by supposing that we had an equal chance of being each person in the society. Each social state is then a lottery and, under the von Neumann–Morgenstern approach to choice under risk (as later axiomatized by Savage, Marschak, and others), I will prefer one lottery to another (hence one social state to another) if and only if the expected utility (for me) of the one exceeds that of the other. This expected utility is the mathematical expectation of the utilities (all equi-probable) of the "prizes" in the lottery; the utility function is determined up to a linear (or affine) transformation and its curvature reflects my "risk aversion."

It has seemed to many students of this approach that it founders at three points.[7] Diamond (1967) objected that two persons' having an equal chance in life of winning something is surely not morally equivalent to their having had the hypothetically equal chance of being the person pre-designated to win something – contrary to the reduction-of-compound-lotteries axiom.

Second, it is possible that I would rank lottery A above B because, being very risk averse, I would hate to be in your shoes in either case and A is a little better for you than B; and, simultaneously, you would rank the lotteries oppositely because you are risk-neutral and I am so much better off under B and A that you are willing to take the chance of losing (if you turn out to be in your own shoes) in the hope of that large gain (if you turn out to be in my shoes). Are these ethical preferences of ours useful or material in this case? If not, are they useful in the other cases?

[7]Sen (1970) remains the best single reference for criticism of neo-utilitarianism and also of Rawls.

In any case, it is clear that different persons, having different risk aversion, will not generally rank social states alike. Now there may always be some differences in our ethical beliefs, of course. But the feeling does seem to be fairly widespread that differences in our ethical rankings of states should not hinge so critically on the single attitude called risk aversion. If I, a risk lover, am to consider the possibility that I might have been anyone else (with equal probability), should I not reflect too that I might also have been born with everyone else's intense risk aversion?

Even if attitudes toward risk were all alike, Rawls (1971) questioned whether it is right to make the ethical weight given to the "utility" of a person of a certain type depend upon their relative numbers. It seems to be a characteristic (Rawls would call it a defect) of utilitarianism (see the second half of this paper) that some persons are called upon to suffer for the greater convenience of others for no other reason than their numerical inferiority. It may be that neo-utilitarianism will emerge in a revised form in which the probabilities are known only to be bounded above zero and below one and where accordingly one looks for certain boundaries on the social indifference curves, the contours of S in the earlier notation. But the other objections are less easy to meet.

1.5. *Rawls and the Rawlsians*

Anyone who grew up under generalized utilitarianism might have been forgiven for wondering how one could reject it, there being (like motherhood) no alternative to it. But there is also the possibility that the social preference ordering over states (yours, mine, ours) is lexicographic – heeding certain individual rights, first one, then the next in importance, and so on – and hence not representable by a proper welfare function W just as lexicographic consumer preferences are not representable by a utility function. (There may remain some room for a kind of sub-utilitarianism that would be applied subject to constraints that certain prior rights be satisfied.) In fact, classical utilitarianism originated in dissatisfaction with Kantian arguments for the existence of certain natural rights that ought not to be traded off for the general public convenience – for example, my right to be protected against certain murder, however costly the protection to others (short of their death).

The alternative to utilitarianism (and neo-utilitarianism) presented by Rawls is distinctive in several respects, among them: What the society can produce through the cooperation of its members exceeds the totals of goods producible when each member works and invests alone, Crusoe style; the resulting problem of justice is to determine the terms under which a person's cooperative effort will be rewarded. If a person works in cooperation with the others in society, he is assured of at least as "much" as he could gain from working alone; indeed every person is free to opt out. Rawls's system does not (yet) address the question of optimal charity toward those who do not work in cooperation with society – either because they are incompetent or because their product is higher if kept apart.

Second, Rawls dispenses with the co-cardinal utility functions of Bentham and Sidgwick. To "think Rawls" it suffices to introduce a new kind of "preference relation," $(x, i)R(y, j)$ in Sen's notation, which means that the ith individual in state x is no worse off than (is at least as well off as) the jth individual in state y. If convenient we may represent this relation by a co-ordinal utility function, determined up to a common monotone transformation, having the property that $u^i(x) \geq u^j(y)$ if and only if $(x, i)R(y, j)$. In fact, Rawls appears hopeful of dispensing with the word utility altogether: he does not imagine our comparing people's unhappiness which might (like blood pressure) be labile and mysterious (if we could measure it). Instead he asks whether a person's advantages or opportunities to carry out his life plans – his objectives regarding career, family, etc. – or, if you like, the prospects of his self-realization, are greater, smaller or the same as another person's. For Rawls this boils down to differences in after-tax real income, although it has to be conceded that the index number problem if and when it arises cannot then be brushed aside. (It will not generally be true that for every state x and y each person's budget set either weakly dominates or is weakly dominated by every other person's.)

To jump to the conclusion: Rawls argues for a social welfare criterion to govern the rewards of workers (and savers?) cooperating in production that he dubs the Lexical Difference Principle:

> A state x is strictly preferred to a state y if a worst-off person in the former would be better off than a worst-off person in the latter; or, when all the worst-off are equally well-off, if a next-to-worse-off person in the former would be better off than a next-to-worse off person in the latter, and so on "lexicographically."

A state x is optimal, of course, if there is no other state strictly preferred to it. In utility terms, the optimal state x^* has the property that $\min_i$

$u^i(x^*) = \max_x [\min_i u^i(x)]$ with the lexicographic principle being used to break ties among two or more states having that property.

Before discussing Rawls's arguments on behalf of this principle, I should indicate that it has been "derived" – in 1975 papers by Hammond and by Strasnick (which I have not yet seen) – from a set of axioms in which the distributional or equity axiom is significantly weaker than the lexical principle itself.

Hammond defines the Generalized Social Welfare Function (GSWF) to be a mapping from the set of all logically possible interpersonal-interstate well-being orderings, which convey the information (x, i) $R(y, j)$, to the set of social preference orderings on X, the set of possible states. This function is required to obey axioms analogous to Arrow's requirements: Unrestricted Domain, Independence of Irrelevant Alternatives, the Strict Pareto Criterion, and an axiom related to Suppes' Grading Principle (involving the requirement of "anonymity" or "nondictatorship"). In addition the welfare function is to satisfy the following Strong Equity Axiom (SEA):

> Suppose that j is better off than i in both states x and y, and that i strictly prefers x, j strictly prefers y, and everyone else is indifferent between x and y. Then x is at least as good as y in the social preference ordering.

Hammond shows that there is at least one GSWF satisfying all these axioms. Further, the lexical difference principle of Rawls is the only such function obeying these axioms when the number of feasible states (options in X) is three or more.

From what standpoint, by what method of inquiry, is one to judge the acceptability of the difference principle – or for that matter, any other distributional principle such as the Strong Equity Axiom of Hammond? Rawls asks anyone having to make such judgments to ascend (figuratively) to the "original position" where he does not yet know "who he is going to be." That much is similar to the approach proposed by the neo-utilitarians. But Rawls jettisons the neo-utilitarians' postulate that one knows the probabilities of turning out to be this or that type of person. Now Rawls cannot prove that a person (you, me, . . .) would opt for the difference principle in that hypothetical choice situation; he regards it as an empirical question. But he thinks it plausible that most or all of us who can get our heads into that original position would indeed focus as he proposes upon the plight of the worst off in each social state.

Two protests against Rawls's position are made by the "neo-utilitarians." Harsanyi (1974) and Samuelson (1976) complain that if a person truly believes (from the original position) there is some positive probability of not turning out to be the worst off, he would not play so conservatively as to choose among states *as if* he were sure he would be among the worst off in each. Now consider the Strong Equity Axiom. Suppose I am told (in the original position) what the "real incomes" of i and j in x and y are, and j has the larger stake in the choice between x and y. I might play conservatively, opting for x as the SEA indicates, if I were "afraid" the probability of my turning out type i is greater, perhaps substantially greater, than the probability of my turning out type j. And this no matter how large j's consumption gain from getting y instead of x if my utility function is bounded from above. The difference between Rawls and the neo-utilitarians necessarily turns up, however, as i's gain from x vanishes. Rawls would defend SEA "to the limit" while the neo-utilitarians would reject it at a sufficiently shrunken gain.

Vickrey (in conversation) raises another objection (one also involving continuity) which I shall paraphrase as I recall it: Suppose another class of persons is added who are just epsilon worse off (in all states, say) than the previously worst off under the previously maximin policy. Imagine, for example, that Europe discovers America populated by Indians poorer than the poorest Europeans in the maximin state (prior to discovery of America). What then? Must Europe share the wealth? Perhaps the answer is that a country (or coalition?) using 'maximin' to distribute wealth among its members need not surrender any of what it can produce "alone," without the new entrant, just as in the two-person case the more productive Crusoe is not called upon to suffer upon making contact with the less productive Friday. Subject to that constraint, the gains from trade are to be regulated to help the less well-off to the maximum. If that is "correct Rawls," then, if America is sufficiently small, she would be allowed to trade at Europe's relative prices and garner the entire gains from trade – provided this free-trade arrangement did not make the Indians better off than the worst-off Europeans (in which event they should expect the tax man).

Rawls's book has had the effect of smoking out a vast array of opponents not just to maximin, but also to utilitarianism (new and old) and to the very concept of a social welfare function. Among these are exponents of perfectionism (along the lines of Nietzsche and deJouvenel), neo-conservatism (the new rage that cites Burke and deTocqueville), objectivism (Rand and Brandon) and 'marxism'. The reader interested in a critique of the concept of a social welfare function (in general, not just in

Rawls) should consult Nozick's dazzling book in defense of the minimalist state, and, as a counter to it, Varian's review of its treatment of distribution.

1.6. *'Fairness': From Foley to Varian*

It is fortunate that Rawls abandoned the term "fairness" that he earlier used to designate his conception of justice (he had planned once to title his book *Justice as Fairness*). Now that term is generally used to denote an idea that Varian credits to Foley (1967) and which has since been studied extensively by Vind, Schmeidler, Yaari, Pazner, Kolm, Feldman, Kirman, and Varian.[8]

The idea is that if endowments of every good and bad (all claims and obligations) can be divided equally among the members of society, and if the resulting trade and production achieves a Pareto-optimal allocation, then the associated distribution(s) of consumptions, satisfactions, and what-not (individual "utilities" if you like) can be declared "fair." The argument is that because no one will envy another person's endowment (in the sense of wishing he could exchange endowments with that person) there seems to be no inequity about starting conditions, and because there exists no Pareto-superior allocation no one could secure a better outcome save at the expense of someone else's outcome.

Consider the Edgeworthian exchange economy (no production requiring labor and no capital accumulation). Is it really fair, Robert Cooter has asked (in conversation), that a Pygmy, of Pygmy appetite, should receive the same initial allotments of goods as a giant receives – if all goods are foods? If some goods are foods? One can see the laudable objective of the approach: to obviate having to obtain information on differences in tastes and handicaps, which would be expensive to collect and unreliable for any reasonable outlay toward its collection. But is it right to ignore the problem simply because it is problematic?

A similar conceptual problem cannot be be avoided in a production economy. Is an equitable allocation one that equalizes the endowment of leisure, hence working-time, or instead one that equalizes before-trade "debts" to produce, hence work-time in efficiency units? We can imagine (just barely) that everyone receives tickets to equal shares of the various

[8]In this section I have relied largely on "secondary" sources, which fact I am told is apparent.

commodities produced and everyone has an initial obligation to put in equal effort (as measured in man-hours honestly worked, in every type of job) which he may then buy his way out of by giving up some consumption tickets, according to the market price of his effort (his wage). If tastes are perchance identical, there exists a fair allocation: it is the one where the after-trade consumptions and efforts are identical and are at their respective efficient levels (given the tastes). But if tastes are not identical, there will not generally exist an allocation (attainable by trade from some identical endowment) which is Pareto-optimal (so it is then unclear which of these Pareto-inferior allocations to declare just).

Pazner and Schmeidler, who first saw the existence problem, suggest this way out of it: Suppose that every Pareto-optimal allocation can be supported by a perfect market. Associated with each is the market value of the consumptions and leisure consumed by each person. The authors call fair – let us say (following Varian) income-fair – the allocation that equalizes the above-mentioned market valuation of "implicit income." Varian convinces us of the existence of such an income-fair allocation by letting everyone be endowed with the rights to $1/n$ of everyone's total time and showing that, by perfect-market trade in these rights, there is available a Pareto-optimal allocation.

Another way out is suggested by Varian: If an allocation is such that each person prefers his consumption-output endowment bundle to that of every other person, then the allocation is wealth-equitable; and if the allocation is Pareto-optimal, it is also wealth-fair. The market allocation after trade from such an endowment can be shown to be efficient and hence wealth-fair.

That way out seems far out. I may not complain that I "envy" Sinatra's consumption-output bundle unless and until I can imitate his singing? Or earn his income singing (or doing) something else? Income-fairness has a better claim to our ethical attention. But its present formulation seems to suffer from its other-worldliness: How could we achieve such an allocation with existing institutions or plausible new ones? (It seems to me that Rawls and Nozick are right that justice is to be solved in "policy space," to be characterized in terms of the choice among potential institutions in view of their allocative consequences.) Wouldn't all or most candidates to the claim of "just institution" produce an allocation that would incite someone's envy of somebody else? What then?

2. *Consequences in policy models: Public finance*

The cash value of an idea, James said, lies in its operational consequences. Evidently an idea is being expressed by the statement "State x is socially preferred to state y if ...." But of course a single statement has no implications unless combined with one or more statements. When we want to know what Rawls's difference principle "means," our first instinct is to add the sentence, "The set of feasible states is like the set of possible divisions of a fixed pie among n persons," and look at the resulting implications. When social choice involves production, however, we cannot afford to stop at fixed pies. Rawls argues that the acceptability of an ethical criterion of distribution depends upon such hypothetical "testing" in a series of hypothetical policy analyses. Our own Koopmans in 1965 reached the conclusion that some ethical principles are simply inoperable or meaningless in some policy-choice problems.

I am not such a purist as to believe that all current-day work in theoretical welfare economics is just so much spadework in moral philosophy. If the research results obtained to date are not to claim substantive interest for current policy, because one is unsure that one's candidate for an ethical criterion will survive all future tests, then *when* is one going to accept an ethical criterion (however provisionally) and advocate the use or implementation of its results in one's policy-choice models? Nevertheless I shall be content here to examine a few research results purely from the standpoint: How well does Rawls's 'maximin' do (so far) as against its utilitarian rival? I will confine the testing ground to public-finance models of production economies. I begin with lump-sum tax models.

2.1. *Lump-sum tax models – atemporal and intertemporal*

The replies of (classical and neo) utilitarians to Rawls have often asserted that all the anomalies that offend our moral intuition are on the side of 'maximin' and that the latter performs well only when utilitarianism performs as well or better. Sen's objections to a linear B function of the individual Us has already been recounted: his equity axiom would give more consumption to the comparatively inefficient consuming machines, contrary to Bentham. That can be fixed up by making W sufficiently concave; the corresponding S can still display the utilitarians' cherished additivity.

Mirrlees (1970) had a surprise, though, for utilitarianism, even when so generalized (or weakened). Consider n producer-consumers with differing efficiencies as producers and identical utility functions. Maximizing minimum utility will yield the highest equal-utility allocation. In that state, the more efficient will enjoy less leisure but more consumption (to compensate) than the less efficient. By contrast, the allocation maximizing an additive social welfare function, $w(U(c^1, l^1) + w(U(c^2, l^2)) + \cdots$, will not. Mirrlees pointed out that if U is additive (and why not, with just two normal goods) and w is Bentham-linear (the sum of the Us), then marginal utilities of consumption are equalized while marginal disutilites of effort are made greater for the more efficient. Hence all consume equally while the more efficient have to work harder and thus receive less utility. This seems ethically bizarre since, as Rawls would say, no one deserved the misfortune of being highly productive. So score one for 'maximin' over utilitarianism – though Rawls might feel that there is no problem of economic justice in the problem posed since, taking the model literally, a worker's marginal productivity does not depend on, nor gain from, the cooperation of other workers.

The notion of lump-sum taxes is customarily applied in matters of intergenerational choice where a "generation" can lump-sum tax itself via reduced transfer payments and public services (without incurring substitution effects upon its incentives to work and save). The differential lump-sum taxing of individuals within a generation according to their different estimated earning potentials is more hazardous than the contingency-taxation of individuals based on observed ex post incomes; that may be why most societies prefer to rely primarily on the latter. But generations do not all co-exist (most earlier ones are dead), so there is little or no possibility for a generation to claim tax credits from its successors if its productivity turned out to be smaller than expected.

Inter-generation capital accumulation is the best-known testing ground of utilitarianism and 'maximin.' Ramsey's (1928) results were apparently regarded (not altogether unreasonably) as a great success for utilitarianism. But Ramsey shied away from the perplexing problems raised by exogenously rising population. Ought we to maximize the time-integral of the rate of *aggregate* utility $U(t)$ or the rate of *average* utility $u(t)$ over the interval (O, T)? If the former and if the population is going to grow rapidly, the sheer numbers of future people will place a huge burden on the early population; and as the time horizon T is increased indefinitely, the initial consumption rate will shrink to zero (or to "subsistence"). With little explanation, and over the objections of Arrow and Kurz (1970), most analysts opted for the latter alternative – "average utilitarianism." In Vickrey's terms, being in one *generation* or another is equiprobable, not one *person* or another. But why? It is a characteristic problem for the "equal ignorance" method.

Neither brand of utilitarianism, average or aggregate, is generally equalitarian with respect to individuals' consumptions. Under *average* utilitarianism, per capita consumption and utility grow monotonically if the initial marginal product of capital exceeds the growth rate of population (under geometric population growth). Under *aggregate* utilitarianism, consumption and utility per head grow if the marginal product of capital is merely greater than zero initially. (In the latter problem, the horizon must be finite as Koopmans showed.)

Now if that is what people want, fine. Within a family, one generation may feel like sacrificing for the next generation if the terms of trade are attractive. But there seems to be something implausible about a conception of justice under which it is an obligation of a generation in a society to make a collective sacrifice in favor of the next generation when, without that sacrifice, the members of the next generation would be as well off as the members of the present generation.

What are the implications of the 'maximin' criterion in Ramsey-type accumulation models? If each person cares only about his own consumption and leisure, that criterion leads to equalization of utilities within and across generations. In a no-finite-horizon model by Solow, this allocation is accompanied by substitution of additional capital for a dwindling natural resource. In a no-horizon model by Phelps and Riley (1976), where generations overlap, from a certain region of initial states the present generation will increase capital and compensate itself with a helicopter drop of additional government debt to be bought by the next generation.

To some these findings are much too rigidly anti-growth: while capital per worker is not generally frozen, individual (lifetime) *utility* is frozen, and at a level determined by the happenstance of initial conditions. But there is nothing in the maximin criterion *per se* to prevent a generation from giving the next generation a better start if they want to. Calvo (1975a) and Phelps and Riley show how utilitarian sentiments (with or without time discounting) can be combined with the maximin criterion to yield a state-region within which there is rising individual utility from generation to generation.[9]

[9] What happens if the children won't do for the grandchildren what the parents would like them to, as in the Phelps/Pollak story? Dasgupta (1971) posits that only the Nash game equilibria are eligible to be just, so if (as it turns out) there is only one it is just. The same

Finally, a brief look at optimal accumulation of "risky" capital where, as in Phelps (1962), generations do not overlap and next period's capital stock is given by $k_{t+1} = r_t(k_t - c_t)$. Here the gross rate of return, r_t, is an independently distributed random variable, and c_t denotes current consumption which is to be chosen in view of k_t and the known distribution. In the (true or pseudo) utilitarian approach we calculate, for each feasible sequence of distributions of consumption, the mathematical expectation of the one-period utilities of each of the successive generations, all of whom have the same preferences regarding risk. An optimal plan is one that maximizes the sum of these expected utilities. (Fishburn (1969) has studied the meaning of this.) In an infinite-horizon model, the optimal plan would be some stationary policy function, $c(k)$. After existence and uniqueness, the principal result is that an increase in the variance of r will increase the optimum volume of saving at each k if and only if a decrease of the expected value of r would do the same. (See Mirrlees (1974) and the references there. For a continuous-time treatment, see Merton (1975) and the references there.)

As a maximin approach, Calvo (1975b) suggests that we look for the infimum of the sequence of one-period expected utilities associated with a sequence of one-period distributions of consumption. A feasible sequence of distributions is called maximin-optimal if no other feasible sequence presents a larger infimum. Then it is shown that an optimal plan exists, characterized by a stationary policy of the type $c(k)$, and that this policy is uniquely determined at least within some intermediate range of k. Without uncertainty, of course, the maximin policy would call for consuming (certain) income. Calvo shows that when the variance of r is positive, the optimal consumption will not be larger; and in the familiar examples of isoelastic utility functions, consumption will be strictly smaller than under certainty. The reason seems to be that if the present generation is to equate its utility to the expected utility of the next generation (who will then do the same), then some extra saving must be

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approach is taken by Prescott and Kydland when they assume that the 1980 Federal Reserve Board will do what it wants to do, not follow the contingency rules the choice (and assurance) of which would be best for us, even if we are the disaster case. Yes, there exists no institutional mechanism to force future generations to do the right thing – there is only their dedication to the postulated conception of justice. (I should mention that Rawls's own treatment of capital accumulation over generations, which is not maximin, has to my mind a game-equilibrium flavor à la Phelps-Pollak – we are to locate a constant saving-income ratio whose expectation will turn out to be confirmed. It would be understandable therefore for Dasgupta to believe that he is explicating Rawls, and perhaps he is.)

done to compensate for risk aversion. (It is hard to say whether Rawls intended for anxious people to have larger portfolios.)

2.2. *Graduated and proportionate taxation*

The problem of optimally graduated taxation in an economy of heterogeneous workers was first studied by Mirrlees (1971). In the Mirrleesian economy, if a person of efficiency w chooses to work l_w hours, his income before tax is wl_w and aggregate consumption is the sum (or integral) of these hours worked expressed in efficiency units. It is desired to optimize the impersonal tax function $t(wl_w)$ subject to the above relation and to the constraint that individuals may adjust their hours worked and corresponding consumptions as they prefer in view of their after-tax wage rates and any lump-sum transfer payment that is equal for all.

Mirrlees sought to maximize an additive utilitarian integral of the individual utilities, as did Sheshinski who specialized to the class of linear tax schedules. Positive taxation was shown to be optimal (the first little bit of taxing having negligible dead-weight cost) and hence, to balance the budget, the government must optimally award a positive lump-sum transfer. Thus we have a utilitarian theory for the "negative income tax." It should have been added, however, that if there were an exogenously given public expenditure (of the resource-absorbing kind) to claim some of the tax revenue, the optimum transfer might have turned out zero. The real surprise among the results was Mirrlees's finding that the optimal marginal tax rate, $t'(\cdot)$, did not show any tendency to rise (monotonically or otherwise) with incomes.

A maximin treatment of the Mirrleesian economy was given by Phelps (1973). It was shown that, at least if the worst off do not or cannot work, the treasury is to maximize tax revenue. So unless the desired public expenditure is the largest feasible, the optimal transfer is assuredly positive. The surprise here was that the marginal tax rate on additional income earned by the highest earner is optimally zero. Since he wouldn't hurt anyone by working more, it is spiteful and inefficient to discourage him from doing so. And since the government could collect a fraction of the inframarginal units of his extra income, everyone could thereby gain (via the higher transfers thus possible.) Therefore, whatever the ups and downs taken by the marginal tax rate, it must go to zero finally as personal income goes to its maximum (under the optimum tax function). Sadka, working independently, showed that this optimality property was not dependent upon the maximin criterion since it follows from Pareto efficiency. But *where* the marginal rate should hit zero depends upon the criterion.

Let us turn now to the optimal mix of proportionate taxes. The now standard approach to this difficult subject was introduced by Diamond and Mirrlees in a one-period setting. Their focus was on the conditions that the tax mix must satisfy generally if it is specified only that the economy is Bergson-optimal, maximizing a quasi-concave welfare function $W(U_1, U_2, \ldots)$. Since our interest here is in the contrasting implications of utilitarianism and maximin I shall not cite their results here.

The problem of the optimal mix of proportionate taxes on labor and property income has been tackled in a series of papers that employ the maximin criterion by Ordover and Phelps. Certain conditions are developed under which one or the other or both kinds of income should bear positive (linear) tax rates. It does not appear that the maximin criterion entails necessarily Draconian taxation of either labor or capital, though in particular cases a substantial wealth tax might be maximin-optimal. It is not known what the implications of intertemporal utilitarianism are in this regard. If utilitarianism calls for greater capital accumulation, as it will under some circumstances, that will entail a smaller lump-sum transfer payment to the current generation coupled, it may be conjectured, with somewhat lighter taxation on either work or saving (or perhaps both). In any case, it seems that the road from Bentham to Rawls is continuous with regard to the tax-and-transfer structure which is optimal for the present generation.

2.3. *Public expenditures and subsidies*

Should a slow learner receive less or more public expenditure toward his education and training than a fast learner? The question reminds one of the classical question of the efficient and the inefficient "utility-producing" individuals acknowledged by Sidgwick and "answered" by Sen with his Weak Equity Axiom.

Arrow (1971) showed that if one's social welfare function is additive (à la Fleming) in the two individuals' educational achievements, then the slow learner should receive less of a given total expenditure as the two achievements tend to perfect substitutes in the welfare function, while he should receive more as the two achievements tend to perfect complements. (See also the subsequent paper by Green and Sheshinski.) It is quite unlikely that this analysis is intended to outline optimal education policy so much as to analyze welfare functions. It is obvious that a realistic model of optimal education outlays would have to include system effects such as the effect both of achievement levels upon the subsequent earning powers of each type of person and upon the transfer payments available to each via the resulting tax revenues.

A rather important practical question is the optimal subsidy to the birth and raising of children, hence optimal population growth. The utilitarian approach (there are variants, of course) was inaugurated by Meade and later taken up again by Dasgupta (1969). Given capital formation, a generation is to add to its numbers up to the point where one more person scoring positive utility would add to the sum of utilities by just enough to cover the losses of utilities from the resulting reduction of consumption per head. When capital formation is simultaneously optimized, the solution is to jump (or proceed as quickly as possible) to that (Golden Rule) *stationary* state where consumption per head is maximized. Why wouldn't exponentially declining population and capital be utilitarian-superior? Evidently because we could not then fit in (even over infinite time) the infinity of potential fetuses that could be accomodated with stationary (or rising) population.

The question that still nags is whether such a solution is even a rough approximation to the utilitarian solution when there is a finite time horizon and variable costs and pleasures to having children. And, in that problem, why are we to weigh the foregone utilities of the potential persons not assigned a berth against the marginal utility of greater consumption by those predeterminedly born or subsequently chosen to be born? While ordinary utilitarianism goes pretty far in attaching proportional weight to the relative numbers of "haves" (and "have-nots"), super-utilitarianism goes farther than almost anyone is prepared to go in attaching proportional weight to the relative numbers of "ares" (and "are-nots"). Since the potential fetus, whom we certainly would wish life and joy if it would not cost us anything, will not know its missed opportunity if unborn, we seem to have a case here that cries out for a lexicographical treatment.

A maximin model of optimal population growth has been developed by Calvo (1975a) and, independently (in a paper I have not yet seen), by Samuelson (1975). Putting aside the pleasure of having children, the model finds the 'utility' of having children in the fact that they will work the capital to produce the output on which our old-age consumption must depend. Under constant returns to scale, the current generation can do better for itself by engaging in some (net) multiplication of the population small enough that future generations can each pass the buck by engaging in net multiplication on their part. If child-rearing is costly, the maximin-optimal plan will require subsidies to that activity and (lump-sum) taxes to balance the government budget. (Calvo has since noticed that the same propositions emerge if we impose merely a kind of efficiency condition on population change – hence they would apply if our criterion were "average utilitarianism," i.e. the sum over generations of utility per head). It is probably unnecessary to remark that the substance of the maximin implications would be altered if one were to introduce scarce land and other natural resources.

3. *Conclusions*

By way of a last word, I should repeat that these optimal-policy models are not to be read for their policy conclusions – the social-choice trade-offs being modeled are too simple for that – but rather for their early-warnings of conceptual difficulties in the applications of their optimality criteria. For that purpose the recent exercises have been illuminating, though maybe not decisive in the contest between maximin and generalized utilitarianism ("average" and "aggregate"). Further, with the analysis of policies by the use of explicit criteria (beyond the Pareto principle of rejecting dominated states) we are back on the track followed up to the Thirties, before the derailment by the "new welfare economics." It is my feeling this is the right track, the only track. Of course, we may never agree altogether on the choice of a welfare criterion; no doubt new criteria will come and go. But is there any other realm of discourse? So raise your glass: the Old Welfare Economics is dead. Long live the Old Welfare Economics.

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