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SOME OBSERVATIONS ON INPUT-OUTPUT ANALYSIS

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1. THE SCOPE OF INPUT-OUTPUT ANALYSIS

The analysis of inter-industry relations, usually referred to as input-output analysis, serves the purpose of establishing the quantitative relations between various branches of production which must be maintained in order to assure a smooth flow of production in the national economy. It studies the conditions of mutual consistency of the outputs of the various branches of the national economy which result from the fact that the output of one branch is the source of input in other branches.

The idea that certain proportions must be maintained between the outputs of various branches of the national economy is at the basis of the equilibrium analysis of classical political economy and neo-classical economics. The proportions referred to are, however, conceived by classical and neo-classical economic theory basically in 'horizontal' terms, i.e., as proportions between final products designed to satisfy the wants of consumers. Under conditions of competitive capitalism, of free mobility of capital, the tendency of the rate of profit towards a 'normal' level in each branch of the national economy leads towards an equilibrium of output of the various branches. In equilibrium, output is adjusted to the demand for the various products. In a planned economy, it is believed, proper planning should assure the establishment of equilibrium proportions.

While this idea of 'horizontal' equilibrium proportions undoubtedly points to an important aspect of the relations between the output of the various branches of the national economy, it overlooks the need of maintaining another kind of proportions, determined not by conditions of consumers' demand, but by conditions of technological relations associated with the fact that the output of certain products serves

—entirely or in part—as input in the process of producing other products. We may call this the problem of 'vertical' proportions.

This problem of 'vertical' proportions is the subject matter of input-output analysis. The problem was first posed by Quesnay in his famous 'Tableau Economique'. Its insight was lost by classical and neo-classical economic theory. A systematic treatment as well as the fundamental solution of the problem was given by Marx in his schemes of reproduction of capital contained in volume II of Das Kapital. Outside of Marxist political economy the problem was scarcely seen; neo-classical economics confining itself to the study of equilibrium conditions of the 'horizontal' type.

However, in business cycle theory of bourgecis economists the problem of 'vertical relations' between investment goods and consumers' goods was bound to reappear, for it is this type of relation which is at the bottom of the phenomenon of crises and depressions. Consequently, it plays an important role in Keynesian theory. The 'vertical' character of the relations involved causes that 'disproportionalities' in this field are not automatically solved by the process of competition through capital moving from less profitable to more profitable branches of the economy. It also explains why smooth economic development is not automatically assured under conditions of capitalism, even independently of the handicaps resulting from the specific features of monopoly capitalism.

The importance of a study of the 'vertical' relations between various branches of the economy, *i.e.*, of input-output analysis, is not limited to conditions of a capitalist economy. As was already pointed out by Marx, since input-output relations are based on technological conditions of production, proper proportions in this field must be maintained in any economic system. A study of such relations is therefore necessary for purposes of socialist economic planning as well as for the understanding of the working-mechanism of capitalist economy. Under conditions of socialism input-output analysis is a necessary tool of ascertaining the internal consistency of national economic plans.

In the socialist countries input-output analysis takes the form of various 'statistical balances' which serve as tools of national economic planning. These balances are conceived as concretisations of the general idea underlying the reproduction schemes of Marx. In the USA Professor Leontief has developed a type of input-output analysis which, too, can be conceived as a concretisation of Marx's idea of input-output relations taking place in the process of reproduction of the national product. Professor Leontief's analysis takes explicitly into account the technological relations between output and input. Though applied first to the economy of the USA, this analysis like all input-output analyses is also applicable to a socialist economy. Indeed, it seems to me, that this analysis achieves its full justification only if applied as a tool of economic planning. Its technique, though first applied to a capitalist economy, points beyond the historical limitations of capitalism and can come fully into its own only under conditions of planned economy.

2. The marxian schemes

Marx's analysis of reproduction is based on two devices. First, the value of the total national product during a period of time (e.g. a year) is considered as being composed of three parts—the value of the means of production used-up during this period (to be denoted by c—in Marx's terminology the constant capital used up), the value of the labour power directly engaged in production (to be denoted by v—in Marx's terminology the variable capital, i.e., the revolving wage fund), the surplus generated (to be denoted by s). Thus:

Total national product $= c+v+s$.

Here, c is the replacement of the means of production used-up, $v+s$ is the total value added (or national income).

Secondly, the national economy is divided into two departments: one producing means of production, the other producing consumers' goods. Using the subscripts 1 and 2 to indicate the two departments, respectively, we shall write:

$$\begin{aligned}\text{total output of means of production} &= c_1+v_1+s_1\\ \text{total output of consumers' goods} &= c_2+v_2+s_2\\ \text{total national product} &= c+v+s\end{aligned}$$

where $$c = c_1+c_2, \quad v = v_1+v_2, \quad s = s_1+s_2.$$

In a stationary economy (Marx's simple reproduction) :

$$\begin{aligned}\text{total demand for means of production} &= c_1+c_2\\ \text{total demand for consumers' goods} &= v_1+v_2+s_1+s_2\end{aligned}$$

The total demand for means of production is equal to the joint replacement requirement of both departments, the total demand for consumers' goods is equal to the joint wage fund and surplus of both departments.

Putting equal demand and output of means of production, we obtain

$$c_1+c_2 = c_1+v_1+s_1 \qquad \ldots \quad (2.1)$$

which simplifies to $$c_2 = v_1+s_1. \qquad \ldots \quad (2.2)$$

The same result is obtained from putting equal total demand and output of consumers' goods.

That is $$v_1+v_2+s_1+s_2 = c_2+v_2+s_2. \qquad \ldots \quad (2.3)$$

This is so, because the total national product $c+v+s$ is being given . Equation (2.3) can be deduced from equation (2.1).

Equation (2.2) indicates an input-output relation between the two departments of the national economy. Indeed, let us write,

$$\begin{array}{ccc} c_1 & + & \boxed{v_1+s_1} \\ \boxed{c_2} & + & v_2+s_2 \end{array} \qquad \ldots \quad (2.4)$$

Department 1 produces means of production. Part of its output equal in value to c_1 is retained within the department for replacement of the means of production used up. The remainder (in the rectangle) equal in value to v_1+s_1 is transmitted to department 2 in exchange for consumers' goods. Department 2 produces consumers goods. Part of its output equal in value to v_2+s_2 is retained within the department for consumption. The remainder in the rectangle equal in value to c_2 is transmitted to department 1 in exchange for the means of production needed for replacement of those which were used-up. In order that production goes on smoothly, the output of the two departments must be co-ordinated in such a way that a balanced exchange takes place between the two departments, i.e., $c_2 = v_1+s_1$. The above table (2.4) thus indicates the input output relations between the two departments: equation (2.2) gives the condition of proper balance between the two departments.

In an expanding economy (Marx's expanded reproduction) not all the surplus is consumed; part of it is accumulated to increase the amount of means of production and to employ more labour power. We shall express this by writing,

$$s = \bar{s}+s_c+s_v$$

where $\bar{s}$ is the part of the surplus consumed, s_c the part of the surplus used to increase the amount of means of production, s_v the part of the surplus used to employ more labour power.

Dividing as before, the economy into two departments, we have,

total output of means of production $= c_1+v_1+\bar{s}_1+s_{1c}+s_{1v}$

total output of consumers' goods $= c_2+v_2+\bar{s}_2+s_{2c}+s_{2v}$

total national product $= c+v+\bar{s}+s_c+s_v$.

Furthermore;

total demand for means of production $= c_1+c_2+s_{1c}+s_{2c}$

total demand for consumers' goods $= v_1+v_2+s_{1v}+s_{2v}+\bar{s}_1+\bar{s}_2$.

The total demand for means of production is equal to the joint replacement and expansion requirement of both departments. The total demand of consumers' goods is equal to the joint wage fund, the joint expansion of the wage fund and the joint surplus consumed in both departments.

Equality of demand and output of means of production implies

$$c_1+s_{1c}+c_2+s_{2c} = c_1+v_1+\bar{s}_1+s_{1c}+s_{1v} \quad \dots \quad (2.5)$$

which leads to

$$c_2+s_{2c} = v_1+\bar{s}_1+s_{1v}. \quad \dots \quad (2.6)$$

The same result can be obtained from the condition of equality of demand and output of consumer's goods.

Equation (2.6) indicates the input-output relation between the two departments in an expanding economy. It can be presented by means of the following table :

$$\begin{array}{c|c}
c_1+s_{1c} & + \quad \boxed{v_1+\bar{s}_1+s_{1v}} \\
\hline
\boxed{c_2+s_{2c}} & + \quad v_2+\bar{s}_2+s_{2v}
\end{array} \qquad \dots \quad (2.7)$$

In department 1 part of the product equal in value to c_1+s_{1c} is retained within the department for replacement of the means of production used up and for expansion of the amount of means of production in the department. The remainder (contained in the rectangle) is transmitted to department 2 in exchange for consumers' goods. In department 2 part of the product equal in value to $v_2+\bar{s}_2+s_{2v}$ is retained for consumption. The remainder (contained in the rectangle) is transmitted to department 1 in exchange for means of production for replacement of the means of production used-up and for expansion of the amount of means of production in the department. The proper balance between the two departments is thus expressed by equation (2.6).

3. Input-output relations in a multi—sector model

Professor Leontief's input-output tables are designed to study the relations between a larger number of sectors of the national economy. Let the economy be divided into n production sectors denoted by the indices $1, 2, \dots, n$. Denote by X_i the total or gross output of the i-th sector by X_{ij} the quantity of the product of the i-th sector transmitted to the j-th sector where it is used as input. Further denote by x_i the net output of the i-th sector, viz., that part of the gross output X_i which is not allocated to another sectors to be used there as input. The net output x_i can be consumed, exported, or accumulated for the purpose of investment.

We have thus,

$$X_i = \sum_{j=1}^{n} X_{ij}+x_i \qquad (i = 1, 2, \dots, n). \qquad \dots \quad (3.1)$$

It is convenient to represent the input-output relations between the sectors of the economy in the form of a table as follows:

X_1	X_{11}	X_{12}	…	…	…	…	X_{1n}	x_1
X_2	X_{21}	X_{22}	…	…	…	…	X_{2n}	x_2
: :	: :	: :	: :	: :	: :	: :	: :	: :
X_n	X_{n1}	X_{n2}	…	…	…	…	X_{nn}	x_n

… (3.2)

The items in the square matrix in the center of the table represent the input-output relations, or the 'interflows' between the various branches of the national economy (also called 'intersector deliveries'.) The column on the right hand side represents the net outputs and the column on the left hand side the gross outputs of the various products. The rows are subject to the balance relation indicated by equation (3.1).

Since the process of production requires not only the use of means of production but also the application of direct labour, we may supplement the above input-output table by introducing the amounts of labour force employed in production. Let us denote the total labour force available in the national economy by X_0; the labour force employed in producing the output of the i-th sector of the economy by X_{0i} and, finally, by x_0 the labour force not employed productively. The latter may be either unemployed (labour reserve) or employed in non-productive occupations, i.e., in occupations which do not produce material goods (e.g., personal services). With regard to the allocation of the total labour force the following equation holds :

$$X_0 = \sum_{i=1}^{n} X_{i0} + x_0. \qquad \dots \quad (3.3)$$

Introducing the allocation of the labour force into the input-output table, we obtain the following table

$$\begin{array}{c|cccccccc|c}
X_0 & X_{01} & X_{02} & \cdots & \cdots & \cdots & \cdots & X_{0n} & & x_0 \\
\hline
X_1 & X_{11} & X_{12} & \cdots & \cdots & \cdots & \cdots & X_{1n} & & x_1 \\
X_2 & X_{21} & X_{22} & \cdots & \cdots & \cdots & \cdots & X_{2n} & & x_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
X_n & X_{n1} & X_{n2} & \cdots & \cdots & \cdots & \cdots & X_{nn} & & x_n \\
\hline
 & Y_1 & Y_2 & \cdots & \cdots & \cdots & \cdots & Y_n & &
\end{array} \qquad \dots \quad (3.4)$$

The items in the square matrix in the center of the table are 'interflows' for 'inter-sector deliveries'. The upper row in the center represents the allocation of the labour force to the various branches of the economy. Similarly as before, the column at the right represents the remainder of the labour force not allocated productively (x_0), and the net outputs of the various products (x_i; $i = 1, \dots, n$). The column on the left hand side represents the total labour force X_0 and the gross outputs X_i ($i = 1, 2, \dots, n$.) of the various branches.

The entries in table may be expressed either in physical units or in value units. In the latter case, the table is sometimes called a 'transaction table' rather than our input-output table. Whatever the units, the rows of the table can always be summed, for each row is expressed in the same units (e.g., man-hours, tons, gallons, yards, pieces). Thus the equations (3.1) and (3.2) hold under all circumstances. We may call them the 'allocation equations'.

The columns, however, can be summed only if the entries of the table are expressed in value units (e.g., rupees) i.e., if the table is a transaction table, otherwise the items of a column would be non-homogeneous. We shall write these sums in the following form.

$$Y_j = X_{0j} + \sum_{i=1}^{n} X_{ij} \qquad (j = 1, 2, \ldots, n). \qquad \ldots \quad (3.5)$$

Obviously, Y_j is the cost of the output of the j-th branch, X_{0j} being the cost of the labour force employed and $\Sigma\ X_{ij}$ the cost of the means of production used-up in producing the output. We may call the equations (3.5) the 'cost equations'. The costs of producing the output of the various branches of the economy are indicated in the row at the bottom of table (3.4).

The excess of the value of the output of a branch of the national economy over the cost of producing the output is the surplus produced in this branch. Denoting the surplus produced in the j-th branch by S_j, we have

$$S_j = X_j - Y_j \qquad \ldots \quad (3.6)$$

and in view of (3.5),

$$X_j = X_{0j} + \sum_{i=1}^{n} X_{ij} + S_j \qquad (j = 1, \ldots, n). \qquad \ldots \quad (3.7)$$

This is the relation which in a multi-sector model corresponds to the Marxian decomposition of the value of the output of a branch of the national economy into $c_j + v_j + s_j$ $(j = 1, 2)$. Here $\Sigma\ X_{ij}$ stands for c_j and X_{0j} stands for v_j in the Marxian notation. The value added in the sector is $X_{0j} + S_j$.

Introducing the surplus produced in the various branches of the economy into the transaction table and taking account of the relation (3.7) we obtain the following transaction table:

$$\begin{array}{|c|cccc|c|}
\hline
X_0 & X_{01} & X_{02} & \ldots\ldots\ldots\ldots & X_{0n} & x_0 \\
\hline
X_1 & X_{11} & X_{12} & \ldots\ldots\ldots\ldots & X_{1n} & x_1 \\
\cdot\cdot & \cdot\cdot & \cdot\cdot & \ldots\ldots\ldots\ldots & \cdot\cdot & \cdot\cdot \\
X_n & X_{n1} & X_{n2} & \ldots\ldots\ldots\ldots & X_{nn} & x_n \\
\hline
 & S_1 & S_2 & \ldots\ldots\ldots\ldots & S_n & \\
\cline{2-5}
 & X_1 & X_2 & \ldots\ldots\ldots\ldots & X_n & \\
\hline
\end{array} \qquad \ldots \quad (3.8)$$

From table (3.8) it is apparent that the gross output of a branch, say X_i, can be obtained either by summation of the entries of a row or by summation of the entries of a column. Consequently, we have

$$\sum_{j=1}^{n} X_{ij} + x_i = X_{0i} + \sum_{j=1}^{n} X_{ji} + S_i \qquad (i = 1, \ldots, n). \qquad \ldots \quad (3.9)$$

This results directly from the equations (3.1) and (3.7). On both sides of equation (3.9) X_{ii} is appearing under the summation sign : it is the part of the output retained in the sector for replacement. Eliminating X_{ii} from the equation, we obtain

$$\sum_{j\neq i} X_{ij}+x_i = X_{0i}+\sum_{j\neq i} X_{ji}+S_i \qquad (i = 1, \ldots, n). \qquad \ldots \quad (3.10)$$

This equation states that (measured in value units) the outflow from the sector to other sectors—plus the net output is equal to the inflow from other sectors plus the value added in the sector.

Equation (3.10) is the analogue, in a multisector model, of the Marxian equations (3.2) and (3.6) of the previous section which hold in a two-sector model. The mentioned Marxian equations are obtained—just like equation (3.10)—by putting equal the value of the output of the sector and the total allocation of the sector's output and by eliminating on both sides the part of the output retained in the sector.

In order to see the exact analogy of equation (3.10) and the equations of the Marxian two-sector model, let us transform equation (3.10) in the following way. Suppose that the net output x_i is partly reinvested in the sector and partly consumed or allocated to other sectors; the corresponding parts will be indicated by x'_i and x'' respectively. Thus we have

$$x_i = x'_i+x''_i \qquad (i = 1, \ldots, n). \qquad \ldots \quad (3.11)$$

Further, suppose that the surplus produced in the sector is used partly for consumption, partly for employment of additional labour force in the sector, and partly for addition to the means of production used in the sector. Denote these quantities by S_i S_{i0} and x'_i respectively. Thus

$$S_i = \bar{S}_i+ S_{i0}+x'_i. \qquad \ldots \quad (3.12)$$

Substituting (3.11) and (3.12) into equation (3.10) and eliminating x'_i on both sides, the equation reduces to

$$\sum_{j\neq i} X_{ij}+x''_i = \sum_{j\neq i} X_{ji}+X_{0i}+S_{i0}+\bar{S}_i \qquad (i = 1, \ldots, n). \qquad \ldots \quad (3.13)$$

In this form not only the quantities X_{ii} retained in the sector for replacement but also the quantity retained in the sector for expansion is eliminated. Equation (3.13) states that the net outflow to other sectors and to consumption is equal to the inflow from other sectors and to the part of the value added not retained in the sector. This is the exact counterpart—in a multisector model — to the Marxian equation (3.6) in the previous section.

If the number of sectors is reduced to two, equation (3.13) becomes identical with equation (3.2) of the preceding section. In this case (3.13) reduces to

$$X_{12}+x''_1 = X_{21}+X_{01}+S_{10}+\bar{S}_1. \qquad \ldots \quad (3.14)$$

The corresponding transaction table takes the form:

X_0	X_{01}	X_{02}	$x'_{01} + x'_{02} + x''_0$
X_1	X_{11}	X_{12}	$x'_1 + x''_1$
X_2	X_{21}	X_{22}	$x'_2 + x''_2$
	$\bar{S}_1$ S_{10} x'_1	$\bar{S}_2$ S_{20} x'_2	

... (3.15)

Sector 1 produces means of production, sector 2 produces consumers' goods. As consumer's goods are not a means of production, $X_{21} = 0$, and as means of production are not consumed, x''_1 are the means of production allocated to sector 2 for expansion. Using the notation of the preceding section, we shall write:

$$X_{01} = v_1 \; ; \quad X_{02} = v_2$$
$$X_{11} = c_1 \; ; \quad X_{12} = c_2 \; ; \quad X_{21} = 0$$
$$x''_2 \; = s_{2c} \; ; \quad S_{10} = s_{1v}$$

Thus equation (3.15) takes the form

$$c_2 + s_{2c} = v_1 + s_{1v} + \bar{s}_1$$

which is identical with equation (2.6) of the preceding section. In a stationary economy, $s_{2c} = s_{1v} = 0$, and the equation reduces to $c_2 = v_1 + s_1$, i.e., to equation (2.2) of the preceding section.

It should also be noticed that of the equations (3.10) or (3.13) (which are equivalent to (3.10)), only $n-1$ are independent. From the transaction table (3.8) it is apparent that

$$\sum_i \left(\sum_j X_{ij} + x_i\right) \equiv \sum_i \left(X_{0i} + \sum_j X_{ji} + S_i\right) \equiv \sum_i X_i \qquad \ldots \quad (3.16)$$

This implies directly that one of the equations (3.10) can be deduced from the remaining $n-1$. This corresponds to the property of the Marxian two sector model where only one relation like equation (2.6) or (2.2) of the preceding section holds between the two sectors.

Eliminating the double sums on both sides of the identity (3.16), we obtain

$$\sum_i x_i = \sum_i X_{0i} + \sum_i S_i \qquad \ldots \quad (3.17)$$

which indicates that the net product of the national economy, or national income is equal to the total value added during the period under consideration.

4. Technological relations and value relations

In order to study the effect of the technological conditions of production upon input-output relations we have to distinguish sharply between input-output tables expressed in physical units and transaction tables which are expressed in value units. For this purpose we shall use a separate notation.

2

The physical output of the i-th sector will be denoted by Q_i, the physical net output by q_i and the physical interflow from the i-th to the j-th sector by q_{ij} $(i, j = 1, \ldots, n)$. The physical total labour force (measured, for instance, in properly weighted man-hours) will be denoted by Q_0, the physical labour power employed in the i-th sector by q_{0i} and the remainder not employed productively by q_o. The physical input-output table can thus be written in the form

$$\begin{array}{|c|cccc|c|}
\hline
Q_0 & q_{01} & q_{02} & \cdots\cdots & q_{0n} & q_0 \\
\hline
Q_1 & q_{11} & q_{12} & \cdots\cdots & q_{1n} & q_1 \\
Q_2 & q_{21} & q_{22} & \cdots\cdots & q_{2n} & q_2 \\
:\,: & :\,: & :\,: & :\,: & :\,: & :\,: \\
Q_n & q_{n1} & q_{n2} & \cdots\cdots & q_{nn} & q_n \\
\hline
\end{array} \qquad \ldots \quad (4.1)$$

The rows of the table are subject to the allocation balance

$$Q_i = \sum_j q_{ij} + q_i \qquad (i = 0,\ 1, 2, \ldots, n). \qquad \ldots \quad (4.2)$$

The technological conditions of production can be described by the technical coefficients, called also coefficients of production:

$$a_{ij} = q_{ij}/Q_j \qquad (i = 0, 1, \ldots, n; j = 1, \ldots, n). \qquad \ldots \quad (4.3)$$

The coefficient a_{oj} indicates the labour power employed in producing a unit of output of the j-th sector, the remaining coefficients a_{ij} indicate the amount of output of the i-th sector needed to produce a unit of output of the j-th sector.

In the socialist countries the values of these coefficients are generally available in form of the 'technical norms' used in planning and administration of production. These norms indicate the amounts of labour power, raw materials etc., which are allowed to be used per unit of output. In the absence of such 'technical norms' in the industries the technical coefficients can be obtained approximately from statistical input-output tables, according to formula (4.3). This method was employed by Professor Leontief.

Introducing the technical coefficients (4.3), the allocation equations (4.2) become

$$Q_i = \sum_j a_{ij}\, Q_j + q_i \qquad (i = 0, 1, \ldots, n).$$

It is convenient to separate the first equation relating to labour power from the remaining ones. We have then

$$Q_0 = \sum_j a_{0j} Q_j + q_0 \qquad \ldots \quad (4.4)$$

and the remaining equation can be written in the form

$$(1 - a_{ii}) Q_i - \sum_{j \neq i} a_{ij}\, Q_j = q_i \qquad (i = 1, \ldots, n). \qquad \ldots \quad (4.5)$$

Thus the equations (4.5) can be solved separately from equation (4.1). The matrix of the coefficients of these equations

$$\begin{pmatrix} 1-a_{11}, & -a_{12} & \dots\dots\dots & \dots\dots\dots & -a_{1n} \\ \dots & & \dots\dots\dots & \dots\dots\dots & \dots \\ -a_{n1}, & -a_{n2}, & \dots\dots\dots & \dots\dots\dots & 1-a_{nn} \end{pmatrix} \qquad \dots \quad (4.6)$$

is called the 'technical matrix'. It describes the technological conditions of production.[1]

In the system (4.5) there are n equations and $2n$ variables, i.e., the gross outputs $Q_1, \dots, Q_n$ and the net outputs, $q_1, \dots, q_n$. If the technical matrix is non-singular as we shall assume to be the case, there are thus n degrees of freedom. We can fix in the national economic plan the net outputs $q_1, \dots, q_n$ and the gross outputs $Q_1, \dots, Q_n$ are then uniquely determined by the equations (4.5). Or, instead, we can fix in the plan the gross outputs and the net outputs available which will result uniquely from the equations. Or, finally, we can fix in the plan a number of gross outputs and of net outputs, together n in number — and the remaining n gross and net outputs are determined by the equations.

If the technical matrix happens to be singular, the number of degrees of freedom is increased according to the order of nullity of the matrix. Thus if the rank of the matrix is m $(m < n)$, the order of nullity is $n-m$ and the number of degrees of freedom is $n + n-m$. Thus we must fix in the plan $2n-m$ variables, the remaining m variables being then obtained from the equations (4.5).

Having the gross outputs $Q_1, \dots, Q_n$ either from the equations (4.5) or directly from the plan, we can substitute them into equation (4.4). This gives us the total labour force employed $\sum_{j=1}^{n} a_{0j}Q_j$, and taking the total labour force Q_0 as a datum, we can calculate q_0 i.e., the labour force remaining outside productive employment.

To show the relation between the transaction table and the physical input-output table (1), we must take explicitly account of prices. Denote by p_0 the remuneration of a unit of labour force, and by $p_1, p_2, \dots, p_n$ the prices of the products of the various sectors. Further p_0' denotes the earning of the labour force not employed in production. We have then

$$\begin{aligned} X_i &= p_i Q_i, \ x_i = p_i q_i \qquad \dots \quad (4.7) \\ x_0 &= p_0' q_0, \\ X_{ij} &= p_i q_{ij}. \end{aligned}$$

[1] It should be noticed that this technical matrix differs from the matrix used by Professor Leontief in so far that in Professor Leontief's matrix the coefficeints a_{ii} in the diagonal are absent; his diagonal consists only of unities. This is due to the fact that he does not take into account the fact that part of the output is retained in the sector as means of production, e.g., part of the output of agriculture is retained as seed and as fodder for breeding of animals, part of the coal is retained in the coal mines on fuel etc. If the number of sectors in the model is small, the sectors being accordingly large, this omission may be serious.

We shall also denote by Π_i the surplus per unit of gross physical output of the sector, i.e.,

$$S_i = \Pi_i Q_i \qquad (i = 1, \ldots, n). \qquad \ldots \quad (4.8)$$

Introducing these relations into the transaction table (4.8) of the preceding section we obtain the following form of the transaction table:

$p_0\Sigma q_{0j}+p'_0q_0$	p_0q_{01},	p_0q_{02},	............,	p_0q_0	$p'_0\,q_0$
p_1Q_1	p_1q_{11}	p_1q_{12},	............,	p_1q_{1n}	p_1q_1
p_2Q_2	p_2q_{21}	p_2q_{22},	............,	p_2q_{2n}	p_2q_2
: :	: :	: :	: : : : : :	: :	: :
p_nQ_n	p_nq_{n1}	p_nq_{n2},	............,	p_nq_{nn}	p_nq_n
	Π_1Q_1	Π_2Q_2	............,	Π_nQ_n	
	p_1Q_1	p_2Q_2	............,	p_nQ_n	

… (4.9)

Summing the columns we obtain the equations

$$p_0q_{0i}+\sum_j p_jq_{ji}+\Pi_iQ_i = p_iQ_i$$

which are identical with equations (3.7) in the preceding section. Taking account of the technical coefficients (a_{ij}), these equations can be written.

$$a_{0i}p_0+\sum_j a_{ji}\,p_j+\Pi_i = p_i$$

or, more conveniently,

$$(1-a_{ii})p_i-\sum_{j\neq i} a_{ji}p_i-a_{0i}p_0 = \Pi_i \qquad \ldots \quad (4.10)$$

The matrix of the coefficients is

$$\begin{pmatrix} 1-a_{11}, & -a_{21}, & \ldots\ldots & \ldots\ldots & -a_{n1}, & -a_{01} \\ \cdot\,\cdot & \cdot\,\cdot & \cdot\,\cdot & \cdot\,\cdot & \cdot\,\cdot & \cdot\,\cdot \\ -a_{1n}, & -a_{2n}, & \ldots\ldots & \ldots\ldots, & 1-a_{nn}, & -a_{0n} \end{pmatrix} \qquad \ldots \quad (4.11)$$

There are n equations and $2n+1$ variables i.e., n prices $p_1, \ldots, p_n$ the wage rate p_0 and n per-unit surpluses, $\Pi_1, \ldots, \Pi_n$. If the matrix is of rank n, there are thus $n+1$ degrees of freedom. We can fix, for instance, the wage rate p_0 and the per unit surpluses $\Pi_1, \ldots, \Pi_n$, the n prices are then uniquely determined. Or, instead, we can fix the n prices mentioned and the wage rate, the per unit surpluses are then uniquely determined, or any other combination of $n+1$ variables can be fixed, the n remaining ones resulting from the equations.

If the rank of the matrix is less than n, the number of degrees of freedom increases correspondingly. The important point to be noticed is that these relations between prices of products, wage rate and per unit surpluses are entirely determined by the technological conditions of production as represented by the technical matrix of the coefficients of equations (4.10). The $n \times n$ submatrix containing the first n columns is simply the transpose of the technical matrix (4.6).

Now we can show the relation between the physical input-output relations and the input-output relations in value terms as expressed in a transaction table. The rows of the transaction table (4.9) are subject to the allocation balance

$$p_i Q_i = \sum_j p_i q_{ij} + p_i q_i$$

or, introducing the technical coefficients according to (4.3)

$$p_i Q_i = \sum_j p_i a_{ij} Q_j + p_i q_i$$

This can also be written in the form

$$p_i Q_i = \sum_j a'_{ij} \, p_j Q_j + p_i q_i \qquad \ldots \quad (4.12)$$

where

$$a'_{ij} = (p_i/p_j) a_{ij} \qquad (i, j = 1, \ldots, n). \qquad \ldots \quad (4.13)$$

In view of (4.7), the equations (4.12) can be written in the form

$$X_i = \sum_j a'_{ij} X_j + x_i$$

or

$$(1 - a'_{ii}) X_i + \sum_{j \neq i} a'_{ij} X_j = x_i \qquad (i = 1, \ldots, n). \qquad \ldots \quad (4.14)$$

These equations establish the relations between the value of the net outputs $x_1, \ldots, x_n$, and the value of the gross outputs of the various sectors.

The matrix of the coefficients of these equations is

$$\begin{pmatrix} 1-a'_{11}, & -a'_{12}, & \ldots & \ldots & \ldots & -a'_{1n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -a'_{n1}, & -a'_{n2}, & \ldots & \ldots & \ldots & 1-a'_{nn} \end{pmatrix} \qquad \ldots \quad (4.15)$$

i.e., analogous to the matrix (4.6), only that the coefficients a'_{ij} appear instead of the coefficients a_{ij}.

The coefficients a'_{ij} can be written in the form

$$a'_{ij} = X_{ij}/X_j \qquad (i, j = 1, \ldots, n). \qquad \ldots \quad (4.16)$$

They indicate the value of the input of the product of the i-th sector ($i = 1, \ldots, n$) required to produce a unit of value of output of the j-th sector. We shall call these coefficients the 'input coefficients'.

In addition, input coefficients of the type

$$a_{oj} = X'_{oj}/X_j \qquad \dots \quad (4.17)$$

can be introduced which indicate the value of direct labour power needed to produce a unit of value of product of the j-th sector. With the aid of these coefficients the value of the total labour force employed in production can be calculated, i.e.,

$$X_0 - x_0 = \sum_j a'_{oj} X_j. \qquad \dots \quad (4.18)$$

The input coefficients derive their significance from their simple behaviour with regard to aggregation of two or several sectors into one single sector. For instance, let us aggregate the j-th sector and the k-th sector and denote the new sector thus obtained as the l-th sector.

The value of the gross output of the new sector is then

$$X_l = X_j + X_k \qquad \dots \quad (4.19)$$

and the value of the part of the product of the i-th sector allocated as input to the new sector is

$$X_{il} = X_{ij} + X_{ik} \qquad \dots \quad (4.20)$$

The new input coefficient is, consequently,

$$a'_{il} = X_{il}/X_1 = \frac{X_{ij} + X_{ik}}{X_j + X_k} .$$

In view of the definition (4.16), this is equal to

$$a'_{i1} = \frac{a'_{ij} X_j + a'_{ik} X_k}{X_j + X_k} \qquad \dots \quad (4.21)$$

i.e., the new input coefficient is the weighted mean of the input coefficients before aggregation.

The input coefficients can be given a simple interpretation on the basis of the Marxian theory of value. If the prices of the products express the amount of socially necessary labour required to produce a physical unit of output, the input coefficients indicate the quantity of social labour engaged in one sector necessary to produce in another sector a unit of value (i.e., an amount representing a unit of social labour.) This quantity is entirely determined by the technological conditions of production. The transaction table indicates the allocation of the social labour among the various sectors of the national economy and shows the interflow of social labour between the various sectors of the economy. Aggregation of sectors can be performed by mere summation and the input coefficients are transformed under aggregation by simple averaging.

The Marxian theory, however, points out that in a capitalist economy prices do not exactly reflect the amount of social labour necessary to produce a unit of output. Systematic deviations arise between the 'prices of production', i.e., equilibrium prices under competitive capitalism, and the values of products measured in labour. These deviations are the result of the technologically determined differences in ratios of capital goods and direct labour employed on one hand, and the equalisation of the rates of profit by competition on the other hand. Monopoly produces further systematic deviations. Consequently, transaction tables of a capitalist economy give only an approximate picture of allocation of social labour. In a socialist economy transaction tables give a picture of the allocation of social labour to the extent that prices express the amount of social labour required in production. Therefore, in a socialist economy, a proper system of prices reflecting the amounts of social labour required in production is a necessary instrument of effective accounting of the allocation of society's labour force among the various branches of national economy.

5. Consumption and investment

The net output of any sector of the national economy may be consumed, exported or accumulated for future use. Accumulated output may be designed for future consumption or allocated to increase the quantity of means of production, i.e., invested in the process of production. In the first case we shall consider it as another form of consumption; the last mentioned use will be called productive investment. The part of the net output exported can be considered as destined for consumption or productive investment in proportion as the goods imported in return consist of consumers' goods or means of production. Thus the total net output of a sector may be divided up into a part consumed and a part utilized for productive investment.

Consider the net physical output q_i of the i-th sector and denote the part consumed by $q_i^{(1)}$ and the part invested productively by $q_i^{(2)}$. Then

$$q_i = q_i^{(1)} + q_i^{(2)}. \qquad \dots \quad (5.1)$$

Further

$$k_i = q_i^{(1)}/Q_i\,; \quad \alpha_i = q_i^{(2)}/Q_i \qquad \dots \quad (5.2)$$

Thus, k_i is the proportion of the gross output Q_i of the sector i consumed, and α_i is the proportion of the gross output Q_i used for productive investment. We shall call them the 'rate of consumption' and 'rate of investment', respectively.

Obviously,

$$q_i = (k_i + \alpha_i)\, Q_i \qquad \dots \quad (5.3)$$

The allocation equations (4.5) of the preceding section can then be written as homogeneous equations of the form

$$(1 - a_{ii} - k_i - \alpha_i) Q_i - \sum_{j \neq i} a_{ij}\, Q_j = 0 \qquad (i = 1, \dots, n). \qquad \dots \quad (5.4)$$

In order that these have a non-trivial solution it is necessary that

$$\begin{vmatrix} 1-a_{11}-k_1-\alpha_1, & -a_{12}, & \ldots\ldots & \ldots\ldots & -a_{1n} \\ \ldots & \ldots & \ldots\ldots & \ldots\ldots & \ldots \\ -a_{n1}, & -a_{n2}, & \ldots\ldots & \ldots\ldots & 1-a_{nn}-k_n-\alpha_n \end{vmatrix} = 0 \quad \ldots \quad (5.5)$$

i.e., the rates of consumption and rates of investment of the various sectors cannot be fixed independently of each other. Their mutual relations depend on the rank of the matrix of (5.5).

This may be conveniently illustrated by the example of a two sector model. Taking the sectors 1 and 2, the determinantal equation (5.4) becomes

$$(1-a_{11}-k_1-\alpha_1)(1-a_{22}-k_2-\alpha_2) = a_{12}a_{21} \quad \ldots \quad (5.6)$$

or,

$$\frac{1-a_{11}-k_1-\alpha_1}{a_{12}} = \frac{a_{21}}{1-a_{22}-k_2-\alpha_2}. \quad \ldots \quad (5.7)$$

This means that the fractions of the gross output of each sector going to the other sector for current use in production, i.e., $1-a_{ii}-k_i-\alpha_i$ is proportional to the technical co-efficients relating the two sectors to each other. It is seen from (5.6) that if the rates of consumption are kept constant, the rate of investment of one sector can be increased only at the expense of reducing the rate of investment of the other sector. A similar relation holds for the rates of consumption of the two sectors, if the rates of investment are kept constant.

Now suppose that sector 1 produces means of production and sector 2 produces consumers' goods. Means of production are needed to produce consumers' goods but themselves are not consumed; consequently, $a_{12} > 0$ and $k_1 = 0$. Consuers' goods are only usable for consumption; they are neither needed currently to produce means of production nor are they investable in production. Consequently, $a_{21} = 0$ and $\alpha_2 = 0$. Thus the equation (5.6) turns into

$$(1-a_{11}-\alpha_1)(1-a_{22}-k_2) = 0$$

As consumers' goods are not invested, their total net output is consumed, i.e., $1-a_{22}-k_2 = 0$. Consequently, $1-a_{11}-\alpha_1$ is arbitrary and the rate of investment α_1, can be arbitrarily fixed.

In a communist economy distribution of the national product is divorced from the input of labour and follows the principle, 'to each according to his need'. Under such circumstances, the rates of consumption can be set by policy provided their mutual relations resulting from (5.5) are observed. These relations are entirely expressed in physical terms and no value relations are involved; they depend entirely on the technical coefficients.

In a socialist economy distribution of the national product is based on the remuneration for labour performed. Under capitalism it depends also on property in means of production which permits certain classes to appropriate the surplus generated in production. Therefore, in a socialist economy the rates of consumption are related to the remuneration of the labour force both in productive and non-productive employment. In a capitalist economy they depend also on the use property owners make of the surplus they appropriate.

In order to determine the rates of consumption, it is best to start from a transaction table. We have seen in section 3, equation (3.17), that the net product of the national economy is equal to the total value added in production, i.e.,

$$\sum_i x_i = \sum_i X_{0i} + \sum_i S_i.$$

Introducing the rates of consumption and of investment, we can write this in the form

$$\sum_i k_i X_i = \sum_i X_{0i} + \sum_i S_i - \sum_i \alpha_i X_i. \qquad \ldots \quad (5.8)$$

The left hand side of this equation represents the part of the total value of the net product of the economy (national income) devoted to consumption.

Let W_i be the fraction of the part of the national income devoted to consumption spent for the product of the i-th sector ($i = 1, \ldots, n$). We consider these fractions to be 'behavioural data' and shall call them 'consumption parameters'. Then

$$k_i X_i = W_i \left(\sum_j X_{0j} + \sum_j S_j - \sum_j \alpha_i X_j\right), \qquad (i = 1, \ldots, n; \quad \sum W_i = 1). \quad \ldots \quad (5.9)$$

(The subscripts in the summation signs on the right hand side are denoted by j in order to avoid confusion with the subscript i on the left hand side).

Introducing input coefficients and writing

$$S_j = \Pi'_j X_j \qquad (j = 1, \ldots, n) \qquad \ldots \quad (5.10)$$

we can write

$$k_i X_i = W_i \left(\sum_j a'_{0j} X_j + \sum_j \Pi'_j X_j - \sum_j \alpha_j X_j\right) \qquad (i = 1, \ldots, n). \qquad \ldots \quad (5.11)$$

Substituting this in the allocation equations (4.14) of the preceding section which indicate the allocation balances in the rows of the transaction table, we obtain

$$[1 - a'_{ii} - \alpha_i - W_i(a'_{0i} + \Pi'_i - \alpha_i)]X_i - \sum_{j \neq i} [a'_{ij} + W_i(a'_{ij} + \Pi'_j - \alpha_j)]X_j = 0, \quad \ldots \quad (5.12)$$

$$(i = 1, \ldots, n).$$

In order that these equations have a non-trivial solution we must have the determinant

$$\begin{vmatrix} 1-a'_{11}-\alpha_1-W_1(a'_{01}+\Pi'_1-\alpha_1), & \ldots\ldots & -a'_{1n}-W_1(a'_{0n}+\Pi'_n-\alpha_n) \\ \ldots & \ldots & \ldots \\ -a'_{n1}-W_n(a'_{01}+\Pi'_1-\alpha_1), & \ldots\ldots & 1-a'_{nn}-\alpha_n-W_n(a'_{0n}+\Pi'_n-\alpha_n) \end{vmatrix} = 0. \quad \ldots \quad (5.13)$$

This condition establishes the relations which must be maintained between the rates of investment $\alpha_1, \ldots, \alpha_n$ when the rates of consumption are determined by the 'demand equations' (5.1).

The expressions

$$a'_{0j}+\Pi'_j-\alpha_j \qquad (j = 1, \ldots, n) \qquad \ldots \quad (5.14)$$

which occur in the determinant (5.5) indicate the part of the value added per unit of output value of the sector which is devoted to consumption. By multiplying these expressions by W_i we get the fraction of it which goes into consumption of the product of the i-th sector.

For illustration let us consider a two sector model. The determinantal equation can then be written in the form

$$\frac{1-a'_{11}-\alpha_1-W_1(a'_{01}+\Pi'_1-\alpha_1)}{a'_{12}+W_1(a'_{02}+\Pi'_2-\alpha_2)} = \frac{a'_{21}+W_2(a'_{01}+\Pi'_1-\alpha_1)}{1-a'_{22}-\alpha_2-W_2(a'_{02}+\Pi'_2-\alpha_2)} \quad \ldots \quad (5.15)$$

This equation indicates that the fraction of the value of gross output of each sector remaining after deduction of the part retained in the sector for replacement (a'_{ii}), and for consumption $W_i(a'_{0i}+\Pi'_i-\alpha_i)$ and of the part devoted to investment (α_i) is proportional to the total demand (per unit of value of its output) of the other sector for the product of the first. The latter is equal to the sum of the input coefficient—a'_{ij} and the output of the other sector required for consumption, i.e., $W_i(a'_0+\Pi'_j-\alpha_j)$.

Transforming the input coefficients into technical coefficients according to formula (4.13) of the preceding section and observing that

$$\Pi'_j = \frac{\Pi_j}{p_j}, \qquad (j = 1, \ldots, n) \qquad \ldots \quad (5.16)$$

we can write the determinantal equation (5.13) in the abbreviated form

$$\left| \delta_{ij}-\frac{p_i}{p_j}a_{ij}-W_i\left(\frac{p_0}{p_j}a_{0j}+\frac{\Pi_j}{p_j}-\alpha_j\right)\right| = 0 \qquad \ldots \quad (5.17$$

where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$. This equation contains the wage rate p_0, the product prices $p_1, \ldots, p_n$ and the per-unit surpluses $\Pi_1, \ldots \Pi_n$. These quantities cannot be eliminated from the equation.

Thus when the rates of consumption are determined by 'demand equations' like (5.11) linking them to the national income, the relation between the rates of investment in the various sectors of the national economy cannot be expressed in purely physical and technological terms. They have to be expressed in value terms and are found according to (5.13) to depend on the input coefficients, the rates of surplus $\Pi'_1 \ldots, \Pi'_n$ and the consumption parameters $W_1 \ldots, W_n$ of the various sectors.

As in the light of the Marxian theory of value the input coefficients can be interpreted as indicating technological conditions of production, the relations between the rates of investment are found to depend, in addition to the technological conditions of production, on behaviourial parameters relating consumption of the various products to national income and on the per-unit surpluses in the various sectors. The latter can be considered as 'sociological parameters'. In a capitalist economy they are equal to the proportion of the value of each sector's output appropriated by the owners of means of production. In a socialist economy the surpluses are set by considerations of social policy, providing the resources for productive investment and for society's collective consumption.

6. Investment and economic growth

The part of the net outputs of the various sectors invested in production is added to the means of production available in the next period. This makes possible in the next period an increase in the output of the various sectors of the national economy. The investment done in one period adds to the amount of means of production in operation in the next period. In consequence, a larger output is obtained in the next period. The outputs of successive period (years, for instance) are linked up in a chain through the investments undertaken in each period. Thus, productive investment generates a process of growth of output.

Let $Q_i(t)$ be the gross physical output of the i-th sector of the economy during the time period indicated by t, e.g., the year 1955, and let α_i be the rate of investment of the i-th sector as defined by (5.2) in the preceding section. The quantity of the output of the sector invested is thus $\alpha_i Q_i(t)$. By this amount increases the stock of product of the i-th sector available in the economy as means of production.

This increment is partly retained in the sector and partly allocated to other sectors. Denote the increment allocated to the j-th sector by $\Delta q_{ij}(t)$, $(i, j = 1, \ldots, n.)$ The index t indicates the period during which the allocation takes place.

We have
$$\alpha_i Q_i(t) = \sum_j \Delta q_{ij}(t). \qquad \ldots \quad (6.1)$$

However, not all the increment allocated is used-up by the various sectors during a single unit period of time. For instance, if it consists of machines or other durable equipment it will last for several units of time (years) and only a fraction of it is used up during a unit period of time. Let the durability of the part of the output of the i-th sector allocated to the j-th sector as additional means of production be T_{ij} units of time. T_{ij} is taken as a parameter given by the technological conditions of production and may be called the 'turnover period' of the particular type of productive equipment. The reciprocal of the turnover period, i.e., $1/T_{ij}$ is the rate of used up per unit of time, it is also called 'rate of replacement' or 'rate of amortisation'.

In order to produce a unit of physical output of the product of the j-th sector during a unit period of time the quantity a_{ij} of the product of the i-th sector must be used up during that period of time; a_{ij} is the technical coefficient. Thus to increase in the next period the output of the j-th sector by an additional unit, the quantity of output of the i-th sector $a_{ij} \cdot T_{ij}$ must be allocated to the j-th sector. Then exactly a_{ij} of output of the i-th sector will be used-up in the next unit period in the sector and this will produce one unit of output.

The quantities

$$b_{ij} = a_{ij}T_{ij} \qquad (i, j = 1, \ldots, n) \qquad \ldots \quad (6.2)$$

may be called the 'investment coefficients'. The investment coefficients indicate the quantity of output of one sector which must be invested in the other sector in order to increase by one unit the other sector's output in the next unit period.

The investment coefficients as well as their reciprocals reflect technological conditions of production; given the technical coefficients, the investment coefficients are proportional to the turnover periods of the various types of means of production.

Write $Q_j(t)$ for the physical gross output of the j-th sector in the unit period under consideration and $Q_j(t+1)$ for the physical gross output of this sector in the next unit period. An increment of output of the j-th sector equal to $Q_j(t+1)-Q_j(t)$ requires the investment in the sector of the following quantity of the output of i-th sector.

$$\Delta q_{ij} = b_{ij}[Q_j\,(t+1)-Q_j\,(t)] \qquad (i, j = 1, \ldots, n). \qquad \ldots \quad (6.3)$$

In view of (6.1), we have

$$\alpha_i Q_i\,(t) = \sum_j b_{ij}[Q_j\,(t+1)-Q_j\,(t)] \qquad (i = 1, \ldots n). \qquad \ldots \quad (6.4)$$

These equations express the relations between the allocation of the part of the net product of each sector devoted to investment in the various sectors of the economy and the increments of output obtained in the various sectors in the next unit period.

If the amounts of product of the various sectors invested during the unit period t, i.e., $\alpha_i Q_i(t)$ are given ($i = 1, \ldots, n$), the increments of output in the next unit period can be calculated from the equations (6.4).

Denote by
$$B = \begin{pmatrix} b_{11}, & b_{12}, \ldots\ldots\ldots\ldots, & b_{1n} \\ b_{21}, & b_{22}, \ldots\ldots\ldots\ldots, & b_{2n} \\ \cdots & \cdots \quad \cdots \quad \cdots & \cdots \\ b_{n1}, & b_{n2}, \ldots\ldots\ldots\ldots, & b_{nn} \end{pmatrix} \qquad \ldots \quad (6.5)$$

the matrix of the investment coefficients. The increments of output in the various sectors are then

$$Q_j(t+1)-Q_j(t) = \frac{1}{|B|}\sum_i |B_{ij}|\,\alpha_i Q_i(t) \qquad \ldots \quad (6.6)$$

where $|B|$ is the determinant of the matrix B and $|B_{ij}|$ is the co-factor of the element.

It is convenient to write

$$B_{ij} = \frac{|B_{ij}|}{|B|} \qquad \ldots \quad (6.7)$$

and express (6.6) in the form

$$Q_j(t+1)-Q_j(t) = \sum_i B_{ij}\, \alpha_i\, Q_i\,(t) \qquad (j = 1, \ldots, n). \qquad \ldots \quad (6.8)$$

The coefficients B_{ij} indicate the increment of output obtained in the j-th sector from an additional unit of the i-th sectors' product invested in the j-th sector. They may be called 'intersector output-investment ratios'. The matrix of the coefficients B_{ij}, is the inverse of the matrix B.

The increments of output in the various sectors depend on the investment coefficients and on the amounts of product of the various sectors invested. The investment coefficients, in turn, depend on the technical coefficients and turnover periods. By virtue of (6.2) the matrix of investment coefficients can be presented as follows:

$$B = \begin{pmatrix} a_{11}T_{11}, & a_{12}T_{12}, & \ldots\ldots & a_{1n}T_{1n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1}T_{n1}, & a_{n2}T_{n2} & \ldots\ldots & a_{nn}T_{nn} \end{pmatrix} \qquad \ldots \quad (6.9)$$

In this way the investments done in one unit period lead to an increase of output in the next period. If the rates of investment remain constant, the investments in the successive unit periods are,

$$\alpha_i Q_i(t+1),\ \alpha_i Q_i(t+2),\quad \ldots\ldots\ldots, \qquad (i = 1, \ldots, n).$$

The investments of the first unit period t are the initial 'shock' which sets in motion the process of economic growth. The investments in the successive unit periods carry the process forward from one stage to another.

The course of the process of economic growth can be deduced from the equation (6.4) or, for that matter, also from the equivalent equations (6.8). These are linear difference equations with constant coefficients. The characteristic equation of the system (6.4) is

$$0 = \begin{vmatrix} \alpha_1+b_{11}(1-\lambda), & b_{12}(1-\lambda), & \ldots, & b_{1n}(1-\lambda) \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ b_{n1}(1-\lambda), & b_{n2}(1-\lambda), & \ldots & \alpha_{nn}+b_{nn}(1-\lambda) \end{vmatrix} \qquad \ldots \quad (6.10)$$

The solution of the difference equations indicating the gross output in the unit period t_s can be written in the form

$$Q_j(t_s) = \Sigma\, C_k h_{jk}\, \lambda_k^{t_s} \qquad (j = 1, \ldots, n) \qquad \ldots \quad (6.11)$$

where the λ_k are the roots of the characteristic equation, the C_k are constants determined by the outputs $Q_j(t_s)$ in the initial unit period t_s, the h_{jk} are constants determined by the matrix of the coefficients of equation (6.4), i.e., by the matrix

$$\begin{pmatrix} \alpha_1+b_{11}, & b_{12}, & \ldots\ldots\ldots, & b_{1n} \\ \ldots & \ldots & \ldots \quad \ldots & \ldots \\ b_{n1}, & b_{n2}, & \ldots\ldots\ldots, & \alpha_n+b_{nn} \end{pmatrix} \qquad \ldots \quad (6.12)$$

Thus the constants C_k reflect the initial situation of the national economy while the constants h_{jk} depend on the technological structure of the economy as expressed by the technical coefficients and the turnover periods as well as on the rates of investment.[1]

This analysis can be generalized by considering the rates of investment as variable in time, i.e., considering functions $\alpha\,(t)$ instead of constants $\alpha_i (i = 1, \ldots, n)$. In a similar way, changes in technical coefficients and turnover periods can be investigated. Instead of the constant investment coefficients, we would have to consider functions of time $b_{ij}(t)$, where $i, j = 1, \ldots, n$. The difference equations (6.5) become then,

$$\alpha_i(t)\, Q_i(t) = \sum_j b_{ij}(t)[Q_j(t+1) - Q_j(t)] \qquad \ldots \quad (6.13)$$

Since the coefficients in these equations are not constant, the equations require more complicated methods of treatment.

The increments in output from one unit period to the next one can, however, be easily computed. They are, in analogy with (6.8),

$$Q_j(t+1) - Q_j(t) = \sum_i B_{ij} \alpha_i(t)\, Q_i(t), \qquad \ldots \quad (6.14)$$

the matrix of the coefficients B_{ij} being now the inverse of the matrix

$$B(t) = \begin{pmatrix} b_{11}(t), & b_{12}(t), & \ldots\ldots\ldots, & b_{1n}(t) \\ : : & : : & : : \quad : : & : : \\ b_{n1}(t), & b_{n2}(t), & \ldots\ldots\ldots, & b_{nn}(t) \end{pmatrix} \qquad \ldots \quad (6.15)$$

The relations between investment and the process of growth of output are here presented entirely in physical terms. They are found to depend solely on the techno-

[1] In the above, the roots λ_k are assumed to be all distinct. In case of a multiple root the corresponding h_{jk} on the right hand side of (6.11) is not a constant but a polynomial of degree one less than the multiplicity of the root. The coefficients of this polynomial are determined by the technological structure of the economy expressed by the matrix and the rates of investment. The coefficients C_k remain determined by the initial situation.

logical structure of the economy and on the rates of investment chosen. The process of economic growth, however, can also be presented in value terms.

In such a case, the technological investment coefficients b_{ij} are replaced by a set of coefficients,

$$b'_{ij} = \frac{\Delta X_{ij}}{X_j(t+1)-X_j(t)} \qquad (i, j = 1, \ldots, n), \qquad \ldots \quad (6.16)$$

indicating the value of the output of the i-th sector which must be invested in the j-th sector in order to obtain in the latter a unit increment of output value. These coefficients may be called 'investment-outlay coefficients' or simply, 'outlay coefficients.'[1]

In view of the relations (4.7) in section 4, the outlay coefficients are related to the investment coefficients as follows:

$$b'_{ij} = \frac{p_i}{p_j} . b_{ij}. \qquad \ldots \quad (6.17)$$

Taking into account (6.2), they can also be written in the form:

$$b'_i = a'_{ij}\ T_{ij} = \frac{p_i}{p_j}\ . a_{ij}\ T_{ij}. \qquad \ldots \quad (6.18)$$

Using the relations (4.7) of section 4 the difference equations (6.4) expressing the relations between investments in the various sectors of the economy and the increments of output obtained can be written in the value form:

$$\alpha_i X_i(t) = \sum_j b'_{ij}[X_j(t+1)-X_j(t)], \qquad \ldots \quad (6.19)$$

and the solutions of these equations are obtained by means of their characteristic equation which is

$$0 = \begin{vmatrix} \alpha_1+b'_{11}(1-\lambda), & \ldots & \ldots & b'_{1n}(1-\lambda) \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ b_{n1}(1-\lambda), & \ldots & \ldots, & \alpha_n+b'_{nn}(1-\lambda) \end{vmatrix} \qquad \ldots \quad (6.20)$$

The process of growth of the value of the output of the various sectors of the economy is thus determined—given the values of the initial outputs $X_1(t_0), \ldots, X_n(t_0)$ by the outlay coefficients b_{ij} and the rates of investment α_{ij}.

*Usually the term 'capital-coefficients' is used to denote the outlay coefficients. For reasons exposed by the Marxian theory the term 'capital' is not appropriate in a socialist economy because it covers up the fundamental difference between the role of capital as value of means of production used to appropriate by their owners the surplus produced in the national economy and the role of means of production as an instrument in the physical process of production. We, therefore, prefer to use the term 'outlay coefficients', meaning by 'outlay' the money value of the physical investments.

The outlay coefficients behave under aggregation of two or several sectors into one sector in a similar way like the input coefficients. The outlay coefficients of the new sector resulting from aggregation are the weighted means of the outlay coefficients of the sectors aggregated.

Indeed, denote by the subscript l the sector resulting from aggregation of the j-th sector and the k-th sector. The outlay coefficients of the new sector are then

$$b_{il} = \frac{\Delta X_{il}}{X_l(t+1)-X_l(t)}.$$

Since

$$\begin{cases} \Delta X_{il} = \Delta X_{ij} + \Delta X_{ik} \\ X_l(t) = X_j(t) + X_k(t) \\ X_l(t+1) = X_j(t+1) + X_k(t+1) \end{cases} \qquad \ldots \quad (6.21)$$

we obtain, taking into account the definition (6.16),

$$b'_{il} = \frac{b'_{ij}\,[X_j(t+1)-X_j(t)]+b'_{ik}\,[X_k(t+1)-X_k(t)]}{[X_j(t+1)-X_j\,(t)]+[X_k(t+1)-X_k(t)]} \qquad \ldots \quad (6.22)$$

The merit of presentation of the process of growth of output resulting from investment in value terms consists in the possibility it gives to aggregate sectors. But it must be pointed out that the outlay coefficients do not reflect only the technological structure of the economy. As seen from (6.17), they depend also on the relative prices of the products. The result of their averaging under aggregation also depends on the relative prices of the products of the sectors aggregated.

However, on the basis of the Marxian theory of value, the outlay coefficients may, under appropriate circumstances, be interpreted as indicating the quantity of social labour employed in the sector of the economy which must be 'stored up' in order to increase the output of another by an amount representing one unit of social labour. Under such interpretation, which requires that prices reflect the amounts of social labour necessary to produce a physical unit of product, the outlay coefficients too represent the technological structure of the economy.

The way in which the growth of output set in motion by investment depends entirely on the technological structure of the economy is further elucidated by the fact that the investment coefficients are, according to (6.2), products of the technical coefficients and the turnover periods, or that the outlay coefficients, according to (6.8) are the products of the input-coefficients and the turnover periods.[1] Thus the technological conditions determining the growth of output resulting from investment

*The fact that the investment coefficient are not independent of the technical coefficients but are derived from them by multiplication by the turnover periods seems to have been pointed out first by David Hawkins, "Some conditions of macroeconomic stability," *Econometrica* 1948, p. 313. Usually they are wrongly taken as independent data, like for instance by Professor Leontief in, *Studies in the Structure of the American Economy,* Oxford University Press, New York 1953, p. 56.

consist entirely of two factors. One are the technical coefficients indicating current input-output relations during a unit period. The other are the turnover periods which simply indicate the durability of the various means of production and, consequently the rate of use-up of the means of production in a single unit period of time.

This disposes definitely of any mystical notions about the 'productivity' of a mythical entity 'capital' conceived as a separate factor of production distinguished from the physical means of production. Such metaphysical entity is proved to be non-existent.

In a capitalist economy 'capital' consits of private property rights to means of production which permit the owners of the means of production to appropriate the surplus produced in the national economy. 'Capital' is the power to appropriate surplus. This power, under capitalism, is measured by the money value of the means of production and hired labour power a person (or corporation) can command. In a socialist economy such property rights are absent. There exist simply physical means of production and certain technological conditions expressed by the technical coefficients and turnover periods. From these technological conditions there result certain consequences concerning the quantity of social labour which must be 'stored up' in order to achieve a planned increase in output. Thus there is no need in a socialist economy for any concept of 'capital'. Such concept would only obscure the technological character of the conditions of the process of economic growth.

7. Effects of investment on national income and employment

The equations (6.19) of the preceding section can be transformed in a shape analogous to equation (6.8), i.e., in a shape which presents the increment of the value of output of a sector of the national economy as a linear combination of the investments undertaken in the various sectors. For greater generality it is convenient to consider the rates of investment, α_i, as variable in time, i.e., $\alpha_i(t)$. We obtain then,

$$X_j(t+1) - X_j(t) = \sum_i B'_{ij}\, \alpha_i(t)\, X_i(t) \qquad (j = 1, \ldots, n). \qquad \ldots \quad (7.1)$$

The coefficients B'_{ij} are the elements of a matrix $(B_{ij})^{-1}$ which is the inverse of the matrix of the outlay coefficients

$$B' \equiv \begin{pmatrix} b'_{11}, & b'_{12}, & \ldots & \ldots, & b'_{1n} \\ :: & :: & :: & :: & :: \\ b'_{n1}, & b'_{n2}, & \ldots & \ldots, & b'_{nn} \end{pmatrix} \qquad \ldots \quad (7.2)$$

This means that,

$$B'_{ij} = \frac{|B'_{ij}|}{|B'|}\,; \qquad (i, j = 1, \ldots, n,) \qquad \ldots \quad (7.3)$$

where $|B'|$ is the determinant of B and $|B'_{ij}|$ is the co-factor of the element b'_{ij}.

The coefficients B'_{ij} may be called 'intersector output-outlay ratios'. They indicate the increment of the output (measured in value) of the j-th sector resulting from a unit increase of investment outlay in the i-th sector.

Summing the equation (7.1) over all sectors of the national economy, we obtain

$$\sum_j [X_j\ (t+1)-X_j\ (t)] = \sum_j \sum_i B'_{ij}\ \alpha_i\ (t)\ X_i\ (t)$$

or, writing

$$\beta_i = \sum_j B'_{ij} \qquad (i = 1, \ldots, n). \qquad \ldots \quad (7.4)$$

$$\sum_j (X_j\ (t+1)-X_j\ (t)) = \sum_i \beta_i\ \alpha_i\ (t) X_i\ (t). \qquad .. \quad (7.5)$$

The left hand side of equation (7.5) is the increment, from one unit period to the next, of gross national product. The coefficients β_i on the right hand side indicate the effect of a unit increase in investment outlay in the various sectors of the economy on national gross product. They can be called simply 'output-outlay ratios' of the various sectors.

A further simplification of equation (7.5) can be achieved by expressing the investment outlays in the various sectors as a fraction of the total investment outlay in the national economy. Denote by $\alpha(t)$ the overall rate of investment in the national economy during the unit period t. The total investment outlay during the unit period is

$$\alpha(t)\ \sum_i X_i(t).$$

Denoting further by $\mu_i\ (t)$ the proportion of the total investment outlay which is undertaken in the i-th sector of the economy, we have

$$\alpha_i\ (t)\ X_i\ (t) = \mu_i\ (t)\ \alpha(t) \sum_i X_i(t)\ ; \qquad (\sum_i \mu_i\ (t) = 1). \qquad \ldots \quad (7.6)$$

Substituting the relation (7.6) into equation (7.5) and observing that

$$\sum_i X_i\ (t) = \sum_j X_j\ (t),$$

we arrive at

$$\sum_j (X_j\ (t+1)-X_j\ (t)) = \alpha(t) \sum_j X_j\ (t) \sum_j \beta_i\ \mu_i\ (t),$$

which also can be written as

$$\frac{\sum_j (X_j(t+1)-X_j\ (t))}{\sum_j X_j\ (t)} = \alpha(t) \sum_j \beta_i\ \mu_i\ (t). \qquad \ldots \quad (7.7)$$

The left hand side of (7.7) is the rate of increase of gross national product and will be denoted by $r(t)$. In order to simplify the right hand side we shall put

$$\beta(t) = \sum_i \beta_i\ \mu_i\ (t) \qquad \ldots \quad (7.8)$$

Since $\sum_i \mu_i(t) = 1$, β can be interpreted as the average output-outlay ratio of the national economy. Equation (7.7) can thus be expressed in the simple form

$$r(t) = \alpha(t)\,\beta(t). \qquad \ldots \quad (7.9)$$

Thus the rate of increase of gross national product is the product of the overall rate of investment and of the average output-outlay ratio.

Now we can calculate the effect of a given investment programme upon gross national income after a number of unit periods of time. Let $\sum_j X_j(t_0)$ be the gross national product in the initial unit period t_0, and let the investment programme be given by the overall rates of investment $\alpha(t_0), \ldots, \alpha(t_n)$ and the fractions $\mu_i(t_0), \ldots \mu_i(t_n)$ of the total investment outlay allocated to the various sectors of the economy, $(i = 1, \ldots, n)$. We obtain, then, the average output-outlay ratios, $\beta(t_0), \ldots \beta(t_n)$. The gross national product in unit period $t_s (t_s > t_0)$ is,

$$\sum_j X_j(t_s) = \prod_{t=t_0}^{t_s} [1+\alpha(t)\beta(t)] \sum_j X_j(t_0). \qquad \ldots \quad (7.10)$$

If the overall rate of investment $\alpha(t)$ and the allocation fractions $\mu_i(t)$ are the same during each unit period, say α and μ_i, this reduces to

$$\sum_j X_j(t_s) = (1+\alpha\,\beta)^{t_s - t_0} \cdot \sum_j X_j(t_0). \qquad \ldots \quad (7.11)$$

National income is the value of the total net output of the national economy. The value of the net output of the i-th sector in unit period, t is according to the allocation equation (4.12) or (4.14)

$$x_i(t) = X_i(t) - \sum_j a'_{ij} X_j(t), \qquad \ldots \quad (7.12)$$

where the a'_{ij} are input coefficients. National income in unit period t thus is

$$\sum_i x_i(t) = \sum_i X_i(t) - \sum_j X_j(t) \sum_i a'_{ij}.$$

Remembering that $\sum_i X_i(t) = \sum_j X_j(t)$

$$\sum_i x_i(t) = \sum_j x_j(t)$$

we obtain

$$\sum_j x_j(t) = (1 - \sum_{i,j} a'_{ij}) \sum_j X_j(t). \qquad \ldots \quad (7.13)$$

Thus the national income in any unit period differs from the gross national product of that period by a constant factor, $(1 - \sum_{i,j} a'_{ij})$. The double sum in this factor expresses the fraction of national product which is allocated for replacement of the products used up in the process of production during the unit period (i.e. for replacement). The factor itself indicates the fraction of gross national products which constitutes net product, i.e., national income.

Since national income differs from gross national product by a constant multiplier, the rate of increase of national income is necessarily equal to the rate of increase of gross national product. Consequently, the relation (7.9) holds for national income as well as gross national product.

Furthermore, we find that national income in unit period t_s is related to national income in the initial unit period t_0 $(t_s > t_0)$ by formulae analogous to (7.10) and (7.11), namely,

$$\sum_j x_j(t_s) = \prod_{t=t_0}^{t_s} [1+\alpha(t)\,\beta\,(t)] \sum_j x_j(t_0), \qquad \ldots \quad (7.14)$$

and, in the case when $\alpha(t) = \text{const}$ and $\beta(t) = \text{const}$

$$\sum_j x_j(t_s) = (1+\alpha\beta)^{t_s-t_0}. \sum_j x_j(t_0). \qquad \ldots \quad (7.15)$$

The total employment generated by the gross national product is calculated as follows. Denote, as in section 4 by a'_{0j} the input coefficient indicating the value of direct labour force needed to produce a unit of value of product in the j-th sector. We shall call them for convenience 'employment coefficients'. The total employment (in value units) corresponding to gross national product in unit period t is, according to the balance equation (4.1)

$$\sum_j a'_{0j} X_j(t).$$

Consequently, the increment of total employment from one unit period to the next is $\sum_j a'_{0j}\,[\underset{j}{X}\,(t+1) - \underset{j}{X}\,(t)]$.

Taking into account equation (7.1), we find

$$\sum_j a'_{0j}\,[\underset{j}{X}\,(t+1) - \underset{j}{X}\,(t)] = \sum_j a'_{0j} \sum_i B'_{ij}\,\alpha_i(t) X_i(t),$$

or, in view (7.6),

$$\sum_j a'_{0j}\,[X_j(t+1) - X_j(t)] = \sum_j a'_{0j} \sum_i B'_{ij}\,\mu_i(t)\,\alpha\,(t) \sum_i X_i(t). \qquad \ldots \quad (7.16)$$

This expression can be simplified as follows. Write

$$\gamma_i = \sum_j a'_{0j} B'_{ij} \qquad (i = 1, \ldots, n), \qquad \ldots \quad (7.17)$$

γ_i is the additional amount of employment (in value units) created in the national economy by a unit increase in investment outlay in the i-th sector of the economy. We may call it the 'employment outlay ratio' of the i-th sector. Then we obtain

$$\frac{\sum_j a'_{0j}\,[X_j(t+1) - X_j(t)]}{\sum_j X_j(t)} = \alpha(t) \sum_i \gamma_i\,\mu_i(t),$$

or, by introducing the average employment-outlay ratio of the national economy

$$\gamma(t) = \sum_i \gamma_i \mu_i (t), \qquad \ldots \quad (7.18)$$

$$\frac{\sum_j a'_{0j} [X_j(t+1) - X_j(t)}{\sum_j X_j(t)} = \alpha (t) \gamma (t). \qquad \ldots \quad (7.19)$$

The left hand side of (7.19) indicates the increment of total employment from one unit period to the next in relation to the value of the gross national product in the initial unit period. Let us write,

$$a'_0(t) = \frac{\sum_j a'_{0j} X_j (t)}{\sum_j X_j(t)}, \qquad \ldots \quad (7.20)$$

i.e., the average employment coefficient of the national economy. Substituting this into (7.19) we obtain the rate of increase of total employment from one unit period to the next;

$$\frac{\sum_j a'_{0j} X_j (t+1) - X_j(t)}{\sum_j a'_{0j} X_j (t)} = \frac{\alpha(t) \gamma(t)}{a'_0(t)},$$

or, denoting the left hand side by $\rho(t)$,

$$\rho(t) = \frac{\alpha (t) \gamma (t)}{a'_0(t)}. \qquad \ldots \quad (7.21)$$

Thus we find that the rate of increase of total employment is the product of the rate of investment and the average employment-outlay ratio divided by the average employment coefficient of the national economy.

The total employment in unit period t is related to the total employment in the initial unit period $t_0 (t_s > t_0)$ by the formula

$$\sum_j a'_{0j} X_j (t_s) = \prod_{t=t_0}^{t_s} \left[1 + \frac{\alpha (t) \gamma(t)}{a'_0(t)} \right] \sum_j a'_{0j} X_j(t_0). \qquad \ldots \quad (7.22)$$

Comparing (7.21) with (7.9), we can establish a relation between the rate of increase of employment and the rate of increase of national income (or, which is the same, of gross national product.) Denote by $\nu(t)$ the ratio of these two rates, i.e.,

$$\nu(t) = \frac{\rho(t)}{r(t)}; \qquad \ldots \quad (7.23)$$

we have

$$\nu(t) = \frac{1}{a'(t)} \cdot \frac{\gamma(t)}{\beta(t)}; \qquad \ldots \quad (7.24)$$

i.e., this ratio is proportional to the ratio of the average employment-outlay ratio and the average output-outlay ratio.

Total employment grows faster, equal or slower than national income according as to whether

$$\frac{\gamma(t)}{a_0'(t)} \begin{array}{c} > \\ = \\ < \end{array} \beta(t). \qquad \ldots \quad (7.25)$$

However, $\gamma(t)$ and $\beta(t)$ are averages depending on the allocation of the total investment outlay among the various sectors of the national economy. Remembering (7.8) and (7.18) we have

$$\nu(t) = \frac{1}{a_0'(t)} \frac{\sum_i \gamma_i \, \mu_i(t)}{\sum \beta_i \, \mu_i \,(t)} \qquad \ldots \quad (2.26)$$

Since the coefficients γ_i and β_i are determined by technological conditions and $a_0'(t)$ is determined by the employment coefficients a_{oj}' and by the way the national product is composed of outputs of the various sectors, $\nu(t)$ can be influenced only by a proper choice of the allocation of investment fractions $\mu_i(t)$.

In order to obtain the greatest rate of increase of national income (or of gross-national output) the allocation fractions $\mu_i(t)$ have to be chosen so as to maximize the average overall output-outlay ratio $\beta(t)$. In order to achieve this, investment outlays must be allocated to the sectors with the highest overall out-lay ratios, β_i.

In order to obtain the greatest possible rate of increase of total employment the allocation fraction $\mu_i(t)$ have to be chosen so as to maximize the average employment outlay ratio $\gamma(t)$. This requires that the investment outlays be allocated to the sectors with the highest overall employment outlay ratio γ_i.

These considerations refer to the rate of increase of national income or of total employment in a given unit period t. If the goal of the policy is to obtain the greatest possible increase of total employment after a longer period of time an additional factor has to be brought into consideration. From (7.21) we see that the rate of increase in total employment is proportional to $\alpha(t)$ i.e., the rate of investment in the unit period. The rate of investment, however, may depend on the national income, because an increase in national income makes it possible to have a greater rate of investment.

Consequently, it may be possible to obtain in the long run a greater increase in total employment by allocating investment outlays not in a way which produces immediately the greatest rate of growth of total employment but in a way which produces the greatest rate of increase of national income. The slower rate of increase of employment in the beginning period is then over-compensated by a more rapid rate of increase of employment in the later period due to an increased rate of investment.

For instance, let

$$\alpha(t) = cI(t), \qquad \ldots \quad (7.27)$$

where $I(t) = \Sigma\, x_j(t)$ is the national income in unit period t and c is a factor of proportionality $(0 < c < 1)$. Then,

$$\rho(t) = \frac{cI(t)\gamma(t)}{a'_0(t)}. \qquad \ldots \quad (7.28)$$

Taking into account relation (7.14), we find that in any given unit period t_k $(t_k < t_0)$ the rate of increase of total employment is

$$\rho(t_k) = c\ \frac{\gamma(t_k)}{a'_o(t_k)}\ I(t_0) \prod_{t=t_o}^{t_k} (1+r(t)), \qquad \ldots \quad (7.29)$$

where $I(t_0)$ is the national income in the initial unit period, t_0.

Thus the rate of increase of total employment in any given unit period is proportional to the increase of national income which took place between the initial unit period and the unit period under consideration.

In expression (7.29) $\gamma\,(t_k)$ depends on the values of the investment allocation fractions $\mu_i(t_k)$ $(i = 1, \ldots, n)$ in unit period t_k whereas $r(t)$ depends on the values of the allocation of investment fractions $\mu_i(t)$ in all the unit periods from t_0 to t_k. This can be seen immediately from the formulae (7.8), (7.9), and (7.18). A change of the values of the allocation (of investment) fractions in each period from t_0 to t_k thus produces a change in the rate of increase of total employment in unit period t_k equal to

$$d\,\rho\,(t_k) = \frac{c}{a'_o(t_k)}\, I(t_0) \left[\prod_{t=t_o}^{t_k} (1+r(t))d\,\gamma\,(t_k) + \gamma(t_k) d \prod_{t=t_o}^{t_k} (1+r(t)) \right]. \qquad \ldots \quad (7.30)$$

The change is positive zero or negative according to the sign of the expression in braces on the right hand side, i.e., according as to whether

$$\frac{d \prod\limits_{t=t_o}^{t_k} (1+r(t))}{\prod\limits_{t=t_o}^{t_k} (1+r(t))} \begin{matrix} > \\ = \\ < \end{matrix} -\frac{d\gamma(t_k)}{\gamma(t_k)}. \qquad \ldots \quad (7.31)$$

The left hand side of (7.31) can be written in the form

$$d \log \prod_{t=t_o}^{t_k} (1+r(t)) = \sum_{t=t_o}^{t_k} \frac{dr(t)}{1+r(t)}.$$

Hence, the expression (7.31) becomes

$$\sum_{t=t_o}^{t_k} \frac{d\,r(t)}{1+r(t)} \begin{matrix} > \\ = \\ < \end{matrix} -\frac{d\gamma(t_k)}{\gamma(t_k)}. \qquad \ldots \quad (7.32)$$

Let us start with values of the allocation of investment fractions which in each unit period from t_0 to t_k maximize the average employment-outlay ratio $\gamma(t)$. Then change these fractions so as to maximize $r(t)$. In each unit period $dr(t) > 0$ and $d\gamma(t_k) < 0$ (except in the trivial case when $\gamma(t) = \beta(t)$ in each unit period, in which case $dr(t) = 0 = d\gamma(t)$). Thus the left hand side of (7.32) increases monotonously with the value of t_k. By choosing t_k large enough it is possible to make the left hand side in (7.32) greater than the right hand side, i.e., to achieve a greater rate of increase of total employment than would be the case if the investment allocation fractions were chosen so as to maximize in each unit period the immediate effect on total employment.

Total employment in the unit period $t_s(t_s \geqslant t_k \geqslant t_0)$ is according to (7.22)

$$\sum_j a'_{oj} X_j(t_s) = \prod_{t_k=t_o}^{t_s} [1+\rho(t_k)] \sum_j a'_{oj} X_j(t_0). \qquad \dots \quad (7.33)$$

Taking logarithms, we find

$$d \log \sum_j a'_{oj} X_j(t_s) = \sum_{t_k=t_o}^{t_s} \frac{d\rho(t_k)}{1+\rho(t_k)} + \text{constant}. \qquad \dots \quad (7.34)$$

As we have seen, a change of the allocation of investment fractions designed to maximize $r(t)$ in each unit period leads to $d\rho(t_k) > 0$ from a certain unit period on wards. Beginning with that unit period the right hand side of (7.34) increases monotonously, with the value of t_s. By choosing t_s large enough it is possible to make (7.34) positive, i.e., to make total employment larger than would be the case if the rate of increase of national income were not maximised in each unit period.

Denote by t_c the critical value of t_s at which the expression starts becoming positive. Over planning periods which are shorter than $t_c - t_0$ the greatest possible total employment is obtained by allocating investment outlays among the various sectors of the national economy so as to maximise in each unit period $\gamma(t)$ by directing them always to the sectors with greatest employment-outlay ratios. Over planning periods exceeding $t_c - t_0$ the greatest possible total employment is obtained by maximising in each unit period $r(t)$, i.e., by allocating investment outlays always to the sectors with the greatest output-outlay ratios.

More complicated conditions for allocation of investment outlays among the various sectors of the national economy are obtained when the principal goal of the policy i.e., greatest possible increase of national income or of total employment during a period of time, is subject to additional conditions imposed like, for instance, a certain predetermined rate of growth of consumption. Such problems can be solved on the basis of the relations established in this chapter by means of the techniques of linear programming.

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