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Von Neumann's growth model and the 'classical' tradition

Heinz D. Kurz and Neri Salvadori*

1. Introduction

'It is obvious to what kind of theoretical models the above assumptions correspond' (von Neumann 1945: 2). With this remark John von Neumann (1903–57) concluded the exposition of the premises underlying his famous growth model, which was first published in German in 1937 and then translated into English and published in 1945 (see von Neumann 1937, 1945). What was obvious to him need no longer be obvious to us. However, scrutinizing the contemporary literature on the von Neumann model shows that there exists a clearly dominant view as to the nature and theoretical affiliation of von Neumann's contribution. This dominant view is well expressed by Kenneth Arrow, who, in a contribution to a volume celebrating the 50th anniversary of the publication of the growth model, wrote: 'Though von Neumann makes no reference . . . , it seems very clear that he took Cassel's work as a starting point' (Arrow 1989: 17). This interpretation is shared by the editors of that volume, who maintained that the Cassellian system 'forms the backdrop to the model expounded in his 1937 paper' (Dore, Chakravarty and Goodwin 1989: 2; see also Weintraub 1985: 77). And Lionel McKenzie in his entry 'general equilibrium' in *The New Palgrave* contended that Cassel's model 'was generalized to allow joint production in a special context by von Neumann' (1987: 500). The reference is to Gustav Cassel's *Theoretische Sozialökonomie* published in 1918, which contains a considerably simplified version of Walras's theory (see Cassel 1918).[1] It is known as the 'Walras–Cassel model', a name coined by Robert Dorfman, Paul Anthony Samuelson and Robert Solow (1958: 346).

This paper examines the conventional interpretation of the von Neumann model and confronts it with an alternative interpretation. The idea of writing this paper was born while we were working on a book manuscript dedicated to the theory of production from a von Neumann–Sraffa point of view (see Kurz and Salvadori 1992a and 1992b). Since one concern of the book is with tracing

the historical origins of the concepts used, we studied the literature on the two proximate originators of the approach adopted by us, only to find out that their contributions are frequently regarded as belonging to vastly different or even diametrically opposed traditions in economic thought. Hence our view as to the compatibility of the two approaches was questioned.

Delving deeper into the matter amplified our doubts about the conventional interpretation of the von Neumann model. These doubts concern both the circumstantial evidence put forward in support of a 'Walras–Cassel connection' of von Neumann's growth model and, much more important, the possibility of reconciling characteristic features of the latter with neoclassical (long-period) theory. Since from our point of view the conventional interpretation does not stand up to examination, the question was close at hand whether a different interpretation could be tried which is both plausible and not in conflict with the facts known to us. We think that we can offer elements of such an interpretation, in which the von Neumann model emerges as belonging to the 'classical' tradition of economic thought. It deserves to be stressed that for this interpretation it is of no importance whether von Neumann was familiar with the writings of the classical economists or those working in that tradition; in all probability he was not and did not care whether his analysis was 'classical', 'neoclassical' or else. What matters is the similarity of the structure of the respective approaches. Interestingly, though, von Neumann may well have come across pieces of economic analysis of classical derivation while he was a *Privatdozent* at the University of Berlin from 1927 to 1929. However, since we lack direct evidence in favour of the interpretation put forward here, coming either from von Neumann himself or from the group around him, it would be presumptuous of us to demand more than that our interpretation be heard together with the traditional one. It is up to the reader to decide which of the two, if either, is more convincing.

For the purpose of this paper we shall adopt the following distinction between the 'classical' and the 'neoclassical' approach to the theory of distribution and relative prices in conditions of free competition, i.e. in the absence of substantial barriers to entry or exit. The 'classical' tradition focuses attention on goods that are reproducible. Production is conceived as a circular flow: commodities are produced by means of commodities. The wage rate(s) are assumed to be given from outside the system of production, determined by social conditions. The means of production are divided into scarce and reproducible: scarce means of production, such as land, yield their owners a (differential) rent, whereas reproducible means of production, i.e. capital goods, yield their owners a uniform rate of profits on the value of the capital invested. Hence, there is a fundamental *asymmetry* in the classical theory of distribution.

In contradistinction, in the 'neoclassical' tradition all prices, including the prices of 'factor services', are conceived as indexes of scarcity. Wages, profits

and rents are determined *symmetrically* in terms of supply and demand. This requires that supply and demand are conceived as schedules relating price and quantity, where either the supply or the demand curve or both incorporate some substitutability between factor services or goods such that the two curves intersect. The point of intersection gives the equilibrium price and quantity. In the *long-period* versions of neoclassical analysis, with which we will be exclusively concerned in this paper, the economy is assumed to be in a self-replacing state, which means that the prices of the newly produced means of production are exactly the same as those of the means of production that entered as inputs at the beginning of the production process, and that a uniform rate of profits (or interest) is obtained on the supply price of capital goods.[2]

The structure of the paper is as follows. In the first part, comprising Sections 2–4, the conventional interpretation of the von Neumann model will be scrutinized. Section 2 summarizes that interpretation; Section 3 sketches the von Neumann model; and Section 4 points out the difficulties in the conventional view. In the second part, comprising Sections 5–7, the von Neumann model will be compared to major contributions to the 'classical' tradition preceding von Neumann. In Section 5 central concepts employed by him are traced back to classical authors and authors working in that tradition. Section 6 provides a summary statement of a contribution by Robert Remak, who was a colleague of von Neumann's at the University of Berlin. Section 7 argues that von Neumann's paper can be read as containing, among other things, an implicit answer to the paper by his fellow-mathematician. Section 8 draws some conclusions.

2. On the conventional interpretation of the von Neumann model

The essential reasons given in the literature in support of the 'neoclassical' interpretation are as follows. First, in 1936 von Neumann gave his paper in Karl Menger's famous Mathematical Colloquium at the University of Vienna; the paper was then for the first time published in the proceedings of the colloquium, *Ergebnisse eines mathematischen Kolloquiums* (von Neumann 1937). Since the earlier contributions to the colloquium dedicated to economics dealt with the problem of the existence of an equilibrium solution of the 'Walras–Cassel model', it is concluded that von Neumann was concerned with essentially the same problem, adopting the same (neoclassical) perspective.[3]

While circumstantial evidence of this kind is not without interest, it cannot of course replace a proper demonstration of the 'family resemblance' of the analyses under consideration. Such a demonstration is all the more needed since we know from von Neumann that he had read his paper for the first time in the winter of 1932 at the Mathematical Seminar of Princeton University (cf. von

Neumann 1945: 1), i.e. more than one year before Schlesinger and Wald gave their papers at Menger's colloquium on 19 March, 1934.[4] Such a family resemblance could be shown to exist if in terms of scope, method and content the analyses were similar. According to some authors there is clear evidence that this is the case (see, for example, Weintraub 1985, or Punzo 1989 and 1991).

In terms of *scope*, von Neumann is said to share Cassel's concern with equi-proportionate growth in the production of all commodities (e.g. Weintraub 1985: 77). Cassel presents two models, one of a 'continuous stationary society' (Cassel 1932: 144), the other of an economy growing along a steady-state path. In his first model it is assumed that n commodities are produced by using m primary resources, or factors of production, in given supply, employing a single fixed coefficients technology. This provides the basis for his second model, which is sketched only verbally. He introduces it in the following terms: 'We must now take into consideration the society which is progressing at a uniform rate. In it, the quantities of the factors of production which are available in each period . . . are subject to a uniform increase' (ibid.: 152). The *exogenously* given uniform and constant rate of growth of the various endowments gives also the rate of expansion of the economy as a whole. In Cassel's view this 'generalisation' of the previous model does not cause substantial problems: the original set of equations giving the supply and demand for goods and factors is easily adapted to the new case, 'so that the whole pricing problem is solved' (ibid.: 153).

As regards the *method* used we may distinguish between several aspects. In terms of the notion of equilibrium adopted, Cassel, the Viennese economists and von Neumann are all concerned with long-run competitive equilibria characterized by the absence of extra profits. Yet there appear to exist two even more important aspects which account for the close link seen by many interpreters between the von Neumann model and neoclassical general equilibrium analysis. First, it is pointed out that von Neumann on the one hand and Schlesinger and Wald on the other 'share one essential outlook, that of emphasizing inequalities rather than equalities as the true characterization of economic equilibrium' (Arrow 1989: 18). It is indeed a widespread opinion that the original novelty of the contributions to Menger's seminar consisted in the introduction of complementary slackness conditions, and that von Neumann in his paper simply made use of the same device.[5] Second, interpreting 'method' in the technical sense of the mathematical technique used to prove the existence of an equilibrium, the tool developed by von Neumann, i.e. a generalization of Brouwer's fixed point theorem, soon became *the* basic tool of neoclassical general equilibrium theory.

Finally, it is pointed out that in terms of *content* the Rule of Free Goods is employed by Schlesinger, Wald and von Neumann. This rule is taken to express the neoclassical view that a good that is in excess supply assumes a zero price. In a controversy with Kaldor, Solow claimed that 'the pricing side of von

Neumann's model contained assumptions which took us back to Menger, Walras and the marginal productivity theory' (see Lutz and Hague 1961: 297).

Hence, on all three counts the conventional interpretation appears to be well founded. Moreover, there is some evidence that von Neumann was familiar with writings of major marginalist authors. Kaldor who knew von Neumann from Budapest, their home town, and who was on friendly terms with him, recalls that 'One day he expressed an interest in economics and he asked me whether I could suggest a short book which gives a formal mathematical exposition of prevailing economic theory.' Kaldor suggested Wicksell's *Über Wert, Kapital und Rente* (cf. Wicksell 1893). 'He read it in a very short time and expressed some scepticism of the "marginalist" approach on the grounds that it gives too much emphasis to substitutability and too little to the forces which make for mutually conditioned expansion.' According to Kaldor, von Neumann subsequently had a look at the original Walrasian equations (cf. Walras [1874] 1954). 'He told me afterwards that they provide no genuine solution, since the equations can result in negative prices (or quantities) just as well as positive ones' (Kaldor 1989: viii).

Thus, while the works of Wicksell and Walras appear to have been a source of inspiration to von Neumann, according to Kaldor's recollection he was not only dissatisfied with the fact that no proper existence proof of equilibrium was provided but also with the economic substance of the argument put forward. The following summary statement of von Neumann's model provides the basis for the ensuing critical discussion of the dominant interpretation of that model.

3. The von Neumann growth model

Von Neumann assumes that there are n goods which can be produced by m constant returns to scale production processes. The problem is to establish which processes will actually be used and which not, being 'unprofitable'.[6] Von Neumann takes the real wage rate, consisting of the 'necessities of life', to be given and paid at the beginning of the (uniform) production period. In addition, he assumes 'that all income in excess of necessities of life will be reinvested' (1945: 2). The characteristic features of the model include: (i) 'Goods are produced not only from "natural factors of production", but in the first place from each other. These processes of production may be circular' (ibid. 1); (ii) the processes of production 'can describe the special case where good G_j can be produced only jointly with certain others, viz. its permanent joint products' (ibid.: 2); (iii) both circulating and fixed capital can be dealt with: 'wear and tear of capital goods are to be described by introducing different stages of wear as different goods, using a separate P_i [process i] for each of these' (ibid.: 2). These assumptions are coupled with the Rule of Free Goods: 'if there is excess production of G_j, G_j becomes a free good and its price $[p_j] = 0$' (ibid.: 3).

Von Neumann's approach can be summarized as follows. Let **A** and **B** be the mxn input and output matrices, respectively, where **A** includes the means of subsistence in the support of workers; and let **q** be the m-dimensional vector of activity levels and **p** the n-dimensional price vector. $\alpha = 1 + g$ is the expansion factor, where g is the expansion or growth rate; $\beta = 1 + r$ is the interest factor, where r is the rate of interest (or rate of profits). The model is subject to the following axioms.

$$\mathbf{q}^T\mathbf{B} \geqq \alpha\mathbf{q}^T\mathbf{A}. \tag{1}$$

$$\mathbf{B}\mathbf{p} \leqq \beta\mathbf{A}\mathbf{p}. \tag{2}$$

$$\mathbf{q}^T(\mathbf{B} - \alpha\mathbf{A})\mathbf{p} = 0. \tag{3}$$

$$\mathbf{q}^T(\mathbf{B} - \beta\mathbf{A})\mathbf{p} = 0. \tag{4}$$

$$\mathbf{q} \geq \mathbf{0} \text{ and } \mathbf{p} \geq \mathbf{0}. \tag{5}$$

Axiom (1) implies that α times the inputs for a given period are not larger than the outputs of the previous period. (2) is the no extra profits condition. (3) states the free disposal assumption. (4) implies that processes which incur extra costs will not be operated. Finally, (5) requires that both the intensity and the price vector are semipositive. In order to demonstrate that for any pair of nonnegative matrices **A** and **B** there exist solutions for **q** and **p** and for α, $\alpha \geq 0$, and β, $\beta \geq 0$, von Neumann in addition assumes:

$$\mathbf{A} + \mathbf{B} > \mathbf{0}, \tag{6}$$

which implies that every process requires as an input or produces as an output some positive amount of every good.

On the basis of these givens von Neumann determines (i) which processes will be operated; (ii) at what rate the economic system will grow; (iii) what prices will obtain; (iv) what the rate of interest will be. He is able to demonstrate the existence of a solution and that, of necessity, $\alpha = \beta$, i.e. the growth and the interest factor are equal.

The stimulation to publish an English version of the paper came from Nicholas Kaldor, then chairman of the editorial committee of *The Review of Economic Studies*. Kaldor arranged also for the translation of the paper and was concerned with rendering the mathematically demanding paper attractive to an audience of economists. A first step in the pursuit of this goal appears to have been the adaptation of the paper's title (cf. Kaldor 1989: x), a literal translation of the original German version of which would have been 'On an economic system of equations and a generalization of Brouwer's fixed point theorem'. The second part of the title which reflects von Neumann's assessment that the main achievement of the paper consisted in the generalization of a mathematical

theorem was dropped entirely, and the neutral term 'economic system of equations' was replaced by the not so neutral term 'model of general economic equilibrium'.

The second step consisted in asking David Champernowne, 'the most mathematically-minded economist I knew, to write an explanatory paper *ad usum delphini*, for the use of the semi-numerates, to appear alongside it in the *Review of Economic Studies*' (ibid.: x).[7] In a footnote to the introduction of his paper, Champernowne thanks Nicholas Kaldor for help with economic ideas, and Piero Sraffa and a Mr Crum for 'instruction in subjects discussed in this article' (Champernowne 1945: 10, n.1). Interestingly, in Champernowne's interpretation von Neumann's model emerges as one characterized by essentially 'classical' features. Before we deal with the classical tradition and von Neumann's paper, a critical discussion of the now conventional view will be provided.

4. Some difficulties in the conventional interpretation

It is a characteristic feature of neoclassical theory of whichever variety that it attempts to explain all prices and quantities, including the prices of productive services and the employment levels of these services, in terms of demand and supply. The data or independent variables from which the theory starts are the following. It takes as given

(i) initial endowments of the economy and who owns them;
(ii) preferences of consumers; and
(iii) the set of available techniques.

On the basis of these data the theory tries to find an 'equilibrium' price vector that simultaneously clears all markets for goods and services. In some representations of the theory demand and supply functions, or correspondences, are constructed for each good and each service. The intersection between a demand and the corresponding supply function then gives the equilibrium values of the quantity traded and the price ruling in the respective market.

Those who claim that von Neumann's model can be given a neoclassical interpretation would have to demonstrate that the former starts from the same set of data (i)–(iii) and centres around the same theoretical concepts: 'demand' and 'supply'. Such a demonstration is still lacking, and the following discussion shows why.

In von Neumann's model there are no initial endowments that could constrain productive activity and economic expansion: it is explicitly assumed that primary factors are available in abundance and that there is no historically given endowment of the economy with physical or value capital.[8, 9]

This observation leads to the following one. As mentioned in Section 1, the neoclassical economists explain all distributive variables, including profits, symmetrically in terms of supply and demand in regard to the respective factors of production, including a factor called 'capital'. This *necessitates* that one starts from a given 'quantity of capital', the 'scarcity' of which is seen to be reflected in the level of the rate of profits, or rate of interest.[10] In contradistinction, and this concerns a crucial difference, in the von Neumann model we encounter exactly the same asymmetry in the theory of distribution that is characteristic of classical analysis: the real wage rate is given from outside the system and profits are conceived as a residual magnitude. As Kaldor stressed at the 1958 Corfu conference on the theory of capital, there is no reason to presume 'that von Neumann's model was merely Wicksell, Marshall or the whole neo-classical school in a new disguise' (cf. Lutz and Hague 1961: 296–7).

Finally, it deserves to be mentioned that in the von Neumann model the (long-term) rate of growth is determined *endogenously* rather than exogenously, as in Cassel's neoclassical analysis which takes as given the rates of growth of all primary factors and assumes their continuous full employment. No such assumption is to be found in von Neumann.

In von Neumann's model preferences can at most be said to play a rather concealed role: the only route through which they could exert some influence on the equilibrium solution is via the so-called 'necessities of life' which are taken into account in the (augmented) input matrix **A** (see Section 3 above). If the necessities of life reflect to some extent consumers' choice, as is argued by Samuelson (1989), it might be said that tastes play a role in the determination of relative prices and income distribution. For, with a different vector of wage goods reflecting workers' needs, even with given available methods, the method(s) chosen, the product(s) that have zero prices and the rate of interest may be different (see the numerical example in Steedman 1977: 186–91).

Samuelson is of course right in stressing that a change in the real wage rate may, and generally will, result in a change in the equilibrium solution of a von Neumann model. Yet in von Neumann's analysis the vector of goods constituting the means of subsistence of workers does not depend on relative prices. Hence, while it is perhaps an exaggeration to maintain that the von Neumann model is characterized by 'a complete omission of final demand' (Arrow 1989: 22), it is of course true that 'In contrast to Walras's formulae . . . , no direct marginalistic connection between prices and quantities is assumed' (Menger 1973: 56).[11]

As regards the assumption of a given set of alternative processes of production from which producers can choose, there is no material difference between the neoclassical (with the Walras–Cassel model as a special case) and the von Neumann model.[12] However, as has already been noted, there are important differences in the way in which the latter and the Walras–Cassel model

conceptualize production. While in the Walras–Cassel model production is conceived as the direct transformation of the services of the original factors of production into final goods, in the von Neumann model it is assumed that production takes time and that commodities are produced by means of commodities: the outputs of a process are available one time unit later than the inputs enter it. While the Walras–Cassel model sets aside capital goods, the von Neumann model takes into account both circulating and fixed capital.

Hence salient features of any type of (long-period) neoclassical model, including the Walras–Cassel variant of it, are absent in von Neumann's formulation. We may therefore conclude that the conventional interpretation of the latter is in serious trouble. We have also suggested that there exist some striking parallels between the approach chosen by von Neumann and that of the old classical economists. The following section will scrutinize the relationship between the two in greater detail. In the course of tracing back major concepts used in von Neumann's model in the history of economic thought we shall also take the opportunity to question some received opinions regarding the originality of ideas.

5. The 'classical' tradition

Several authors have emphasized the 'classical' nature of von Neumann's model. The first to point out that characteristic features of it are difficult to reconcile with 'the more traditional [i.e. neoclassical] approach' was David Champernowne (1945). These features include: society is assumed to be stratified in two classes, 'workers' and 'the propertied class'; 'workers spend all their income and capitalists save theirs' (ibid.: 16, n.1); emphasis is on 'the circular nature of the production process' (ibid.: 12); prices 'depend on supply conditions alone and not on the tastes of consumers. This emphasis is important because the orthodox analysis has distributed attention evenly between marginal utility and conditions of supply' (ibid.: 12; similarly: 17); 'the rate of interest is not determined as the supply price of waiting, abstinence or saving', no reference is made 'to marginal products or to the marginal efficiency of capital' or to the (Austrian) concept of the 'period of production' (ibid.: 12). Similarly, in his contribution to the 1958 Corfu conference Kaldor called the von Neumann model 'a variant of the classical approach of Ricardo and Marx' (Kaldor 1961: 181; see also Lutz and Hague 1961: 295); and Michio Morishima stressed that 'Marx's theory contains in itself a way to the von Neumann Revolution' (1973: 3; see also Walsh and Gram 1980, and Goodwin 1986).

In what follows we shall briefly deal with the historical roots of the concept of production as a circular flow; the notion of a uniformly expanding economy; the

Rule of Free Goods as applied to original factors of production and produced goods; and the use of inequalities in the formal analysis of the existence of a cost-minimizing system of production.

5.1. Production as a circular flow and the concept of a uniformly expanding economy

(a) *Profits and growth.* The concept of 'the circular nature of the production process' emphasized by von Neumann can be traced back to the very beginnings of classical political economy.[13] It is present as early as in the works of William Petty and Richard Cantillon and was given a clear two-sectoral expression in the *Tableau Économique* of François Quesnay. The concept of circular flow surfaces in the writings of Adam Smith; it is put into sharp relief in David Ricardo's *Essay on Profits* (cf. Ricardo, *Works* VI) and in the second edition of Robert Torrens's *Essay on the External Corn Trade* (cf. Torrens 1820).[14] In this essay Torrens lays down, 'as a general principle', that the agricultural rate of profit is determined in physical terms and takes the exchange value of manufactured goods relatively to corn to be so adjusted that the same rate of profit obtains in manufacturing (cf. ibid.: 361).[15] And in his *Essay on the Production of Wealth*, published in 1821, he shows that the applicability of that principle is not limited to the case in which there is only one sector which is in the special position of not using the products of other sectors while all the others must use its product as capital. However, the case of uniform input proportions put forward by him to illustrate the argument (cf. Torrens 1821: 372–3) is hardly less special.[16]

Further important contributions based on the concept of production as a circular flow were put forward, among others, by Karl Marx (1956, part III; 1959, part II); Ladislaus von Bortkiewicz (1906–7; 1907) who elaborated on the formalization of Ricardo's theory of value and distribution by Vladimir K. Dmitriev (1974); and the Russian mathematical economist Georg von Charasoff (1910).[17] Von Charasoff built on the foundations laid by his fellow-countrymen in an attempt to reformulate Marx's theory in a way that is logically unassailable. He deserves the credit for discussing prices and the rate of profits on the one hand and quantities and the rate of growth on the other within the framework of a physically fully specified input-output system, and for pointing out the remarkable symmetry of the two sets of variables.[18]

(b) *Anticipating 'duality'.* Von Charasoff develops his main argument within the framework of an interdependent model of (single) production, which exhibits all the properties of the later input-output model. The central concept of his analysis is that of a 'series of production' (*Produktionsreihe*): it consists of a sequence, starting with any (semipositive) net output vector (where net output

is defined exclusive of wage goods), followed by the vector of the means of production and the means of subsistence in the support of workers needed to produce this net output vector, then the vector of the means of production and the means of subsistence needed to produce the previous vector of inputs, and so on. Von Charasoff calls the first input vector 'capital of the first degree' (*Kapital erster Ordnung*), the second 'capital of the second degree' (*Kapital zweiter Ordnung*), etc. This series 'has the remarkable property that each element of it is both the product of the following and the capital of the preceding element; its investigation is indispensable to the study of all the theoretical questions in political economy' (Charasoff 1910: 120).

The series under consideration is obviously closely related to the expanded Leontief inverse. In the case of circular production it is infinite. Tracing it backward, first all commodities that are 'luxury goods' disappear from the picture, next all commodities that are specific means of production needed to produce the luxury goods, then the specific means of production needed in the production of these means of production, etc. On the implicit assumption that none of the commodities mentioned so far enters in its own production,

> it is clear that from a certain finite point onward no further exclusions have to be made, and all the remaining elements of the series of production will always be made up of the selfsame means of production, which in the final instance are indispensable in the production of all the different products and which therefore will be called *basic products* (*Grundprodukte*).

Von Charasoff adds:

> The whole problem of price boils down . . . to the determination of the prices of these basic products. Once they are known, the prices of the means of production used in the production of luxuries and finally also the prices of the latter can be derived.
>
> (ibid.: 120–1)

A further property of the 'series of production' deserves to be stressed: the capital of the second degree is obtained by multiplying the capital of the first degree by the augmented input matrix.

> Yet since the physical composition of a sum of capitals is obviously always a medium between the physical compositions of the summands, it follows that capitals of the second degree deviate from one another to a smaller extent than is the case with capitals of the first degree.
>
> (ibid.: 123)

The farther one goes back in the 'series of production', the more equal the compositions of the capitals become, i.e. capitals of a sufficiently high degree 'may practically be seen as different quantities of one and the same capital: the *original* or *prime capital* (*Urkapital*)'. As Charasoff observes,

> this original type, to which all capitals of lower degree converge, possesses the property of growing in the course of the process of production without any qualitative change, and that the rate of its growth gives the general rate of profits.
>
> (ibid.: 124)

The rate of profits can thus be ascertained in terms of a comparison of two quantities of the same composite commodity: the 'original capital'. Von Charasoff emphasizes: 'The original capital expresses the idea of a surplus-value yielding, growing capital in its purest form, and the rate of its growth appears in fact as the general capitalist profit rate' (ibid.: 112).[19] In the hypothetical case in which all profits are accumulated, the proportions of the different sectors equal the proportions of the original capital. In this case the actual rate of growth equals the rate of profits: the system expands along a von Neumann ray.

These considerations provide the key to a solution of the problem of price. For, if the various capitals can be conceived 'as different amounts of the selfsame capital . . . , then prices must be proportional to the dimensions of these, and the problem of price thus finds its solution in this relationship based on law' (ibid.: 123). The solution to the price problem can therefore be cast in a form, in which 'the notion of labour is almost entirely by-passed' (ibid.: 112). Implicit in this reasoning is the abandonment of the labour theory of value as a basis for the theory of relative prices and the rate of profits: taking the technical conditions of production and the real wage rate as given, prices both of basics and of non-basics and the general profit rate can be determined without having recourse to labour values.

Von Charasoff was perhaps the first author to note clearly what von Neumann more than two decades later was to call 'the remarkable duality (symmetry) of the monetary variables (prices p_j, interest factor β) and the technical variables (intensities of production q_i, coefficient of expansion of the economy α)' (von Neumann 1945: 1).[20]

5.2. The Rule of Free Goods

As we have seen in Section 2, it is widely held that the original novelty of the contributions to Menger's colloquium consisted in the use of inequalities in economic analysis. Whether a productive resource in fixed supply is scarce or not is no longer taken as given from outside, as in previous theory, but is decided endogenously and is thus a part of the solution of the system.[21]

While there can be no doubt that the introduction of complementary slackness conditions represents an important achievement, it is questionable whether the underlying idea is really new. In what follows we shall distinguish between the application of the Rule of Free Goods to 'original' factors of production, in particular different qualities of land on the one hand and (one or several qualities of) labour on the other, and to produced commodities.

The notion that in conditions of free competition the services of certain factors of production, such as some qualities of land, which are in excess supply assume a zero price, was a standard element in classical rent theory from James Anderson to David Ricardo. See, for example, the following statement by Ricardo in which reference is to land available in abundant quantity: 'no rent could be paid for such land, for the reason stated why nothing is given for the use of air and water, or for any of the gifts of nature which exist in boundless quantity' (*Works*: 69; see also Sraffa 1960: 75). At most, one could say that there is old wine in new bottles. What *is* new, is that the applicability of the Rule of Free Goods is defined differently. In classical economics that rule was *not* applied to labour; see, for example, Ricardo's discussion of the labour displacing effects of the introduction of machinery: the presence of unemployed labourers does not drive the wage to zero (cf. *Works* I, ch. 31). In contradistinction, in early contributions to neoclassical general equilibrium theory the rule is taken to be indiscriminately applicable to *all* primary inputs, including labour. Hence, the 'reservation price' for all primary inputs is taken to be zero, whereas in classical economics that for labour is positive.

Interestingly, von Neumann applied the Rule of Free Goods in the same way as the classics. While he assumed 'That the natural factors of production, including labour, can be expanded in unlimited quantities' (1945: 2), this did not make him treat all these factors alike. Rather, he singled out labour as the only factor that is exempt from that rule; all other primary factors, although needed in production, 'disappear' from the scene because they are taken to be non-scarce.[22] Labour is assumed to receive an exogenously given wage bundle which is independent of the degree of employment.[23]

By contrast, von Neumann rather generalized the Rule of Free Goods to products. This is possible because unlike the Viennese economists (and Walras), who assumed single production, he allowed joint production: with single production no produced commodity can be a free good, other than in the ultra-short period. Interestingly, the Rule of Free Goods as applied to products can likewise be traced back to the writings of the classical economists. Adam Smith pointed out that with joint production the proportions in which the products can be produced need not coincide with those in which they are wanted. Hence some products may be overproduced, with the consequence that 'the greater part of them would be thrown away as things of no value' (see Smith, *WN*, I. xi. c. 4; see also Kurz 1986).[24]

These considerations show how misleading it can be to try to infer the economic content of a model from the analytical tools or 'method' used. The way in which von Neumann used the inequality method appears to preclude the possibility of interpreting his model in a straightforward manner as belonging to the neoclassical tradition. At the same time the use he made of that

method does not seem to be in conflict in any simple or obvious way with a classical interpretation of his model.

5.3. The choice of technique problem and the use of inequalities

(a) *The classical approach.* Ever since the inception of systematic economic analysis the problem of the choice of technique has played an important role. Scrutiny shows that the classical economists proceeded in two steps. They first analysed an economy using a *given* system of production. Thus, in the chapter 'On Value' of the *Principles* Ricardo is concerned with investigating the relationship between relative prices and the level of the rate of profits for a given system of production. It is only subsequently that the problem of the choice of technique is addressed.

This latter problem can be divided into two sub-problems: (i) Which methods of production should be chosen from a given set of alternative methods? (ii) Should a newly available method of production be adopted? Problem (i) is investigated, for example, in the second chapter of Ricardo's *Principles*, 'On Rent'. Emphasis is on which kinds of land (or methods of production) will be used in order to produce given outputs. With free competition the choice of technique problem consists in finding, given the real wage rate, a cost-minimizing system of production, including the cultivation of land, for which commodity prices, rents and the rate of profits are non-negative and no process yields extra-profits. Problem (ii) – in modern parlance, whether an invention will become an innovation – is investigated in chapter 31, 'On Machinery'. There Ricardo also provides, albeit in a rudimentary form, an analysis of the transition of the economy from one long-period position to another. Initially the capitalist 'who made the discovery of the machine, or who first usefully applied it, [would make] . . . great profits for a time' (*Works* I: 387), i.e. would pocket 'extra' or 'surplus profits'. Competition would then bring about a fall in prices to costs of production and force other capitalists to adopt the superior method of production. The adjustment process would eventually establish a new long-period position characterized by a new system of production and the associated new levels of the rate of profits, of real wages, and of prices (similarly Smith, *WN*, I. x. b. 43). Ricardo was thus also concerned with investigating the logical generation of a long-period position of the economy.[25]

(b) *Inequalities.* Only a few years after the publication of the third edition of Ricardo's *Principles* (1821) a group around William Whewell at the University of Cambridge applied 'symbolic language . . . to the solution of some problems in Political Economy' (Tozer 1838: 507).[26] This included the treatment of the choice of technique problem in algebraic terms employing *in*equalities.

Whewell in a paper published in 1831 investigates the case where a given amount of commodities can be produced either by direct labour alone, without the assistance of machinery, i.e. what Ricardo called 'unassisted labour', or by labour operating a machine that lasts for only a year and is itself the product of a series of labour inputs. He demonstrates that 'the machine can be employed without loss' if (in Whewell's notation)

$$1 + 1' + 1'' + \&c. < L$$

(Whewell 1831: 20), where the LHS of the inequality gives the direct and indirect amount of labour needed to produce the given output by means of the machine, while the RHS gives the amount of unassisted labour required with the alternative method of production. Thus, Whewell adds, 'when machinery is employed, it has always cost less labour than would obtain the same produce without machinery' (ibid.). John Edward Tozer, whose algebraic formulation is more sophisticated, follows Whewell in using inequalities in the discussion of the choice of technique problem. Summarizing his argument in terms of p and p_1, i.e. the price of produce before and after the introduction of machinery, he writes: 'It may be observed that p_1 cannot be $>$ p; if it were, more than the ordinary profit would arise from employing labour, and the machine would be superseded' (1838: 512).

The classical approach to the problem of the choice of technique in terms of extra-profits and extra-costs was also adopted by Karl Marx. His discussion of the falling tendency of the rate of profits in Volume III of *Capital* starts from the premiss: 'No capitalist ever voluntarily introduces a new method of production, no matter how much more productive it may be, . . . so long as it reduces the rate of profit' (Marx 1959: 264). Yet if no capitalist ever 'voluntarily' does so, how is it then possible that the general rate of profits declines? Marx's answer reads as follows. While a capitalist who first employs a new method of production that allows him to produce at lower costs per unit of output will reap extra profits, competition will eventually lead to the general adoption of the new method and bring about a fall in prices. It is this fall in prices which, according to Marx, is the proximate reason why the general rate of profits is bound to fall in consequence of the gradual replacement of an old method of production by a new one.

Marx's analysis is of particular interest since it was the focus of a criticism elaborated by Ladislaus von Bortkiewicz in the final part of his tripartite treatise 'Wertrechnung und Preisrechnung im Marxschen System' (von Bortkiewicz 1906–7), in which another formalization of the choice of technique problem in terms of inequalities is provided. Since we are not aware of any evidence showing that von Bortkiewicz was familiar with the writings of Whewell or Tozer, we may credit him with the independent introduction of a new tool in economic analysis. Compared with the discussions of his precursors, von Bortkiewicz's is economically more interesting.

Von Bortkiewicz accuses Marx of having committed an elementary error by not taking into account that the price changes 'affect the product in the same measure as the capitalist's advances' (1906–7: III, 458). He then demonstrates in terms of some simple models of production that the introduction and generalization of a new method of production can never reduce the rate of profits, given the real wage rate, and will raise it if the new method contributes directly or indirectly to a cheapening of wage goods (cf. ibid.: 454–68).[27] The comparison of two methods by means of which a commodity can be produced is carried out on the premiss 'that prices (and thus also the price expression of the commodity bundle constituting the real wage) are still the old ones' (ibid.: 457). The criterion adopted is whether a method incurs extra costs or yields extra profits: if it incurs extra costs it will not be adopted; if it yields extra profits it will be introduced and will gradually replace the old method.

Hence, there is a striking parallel between the analyses of the choice of technique problem of early authors working in the classical tradition and von Neumann, which is expressed also formally in the use of inequalities. Moreover, taking together the contributions of von Bortkiewicz and Charasoff, we have, *in nuce*, a combination of some of the constituent elements of the von Neumann model. What is missing are the assumptions of (i) joint production, and (ii) the Rule of Free Goods, which however, as we have seen, are not extraneous to the classical approach. Hence von Neumann's approach can be said to have been anticipated in all important material aspects by authors whose contributions can be strictly located within the classical tradition. It goes without saying that this characterization is not meant to play down the importance of von Neumann's contribution. After all it was he who provided a comprehensive and general formulation of what other authors were able to put forward only partially and with respect to special cases, and it was he who was able to prove the existence of a solution.

Next we turn to Robert Remak, a colleague of John von Neumann's while a *Privatdozent* at the Berlin Institute of Mathematics. Interestingly, in contributions to the history of general equilibrium analysis in which von Neumann's model generally features prominently, Robert Remak is hardly given any attention at all.[28] This neglect is particularly harmful since a potentially important link to the von Neumann model is lost.

6. Remak on 'superposed price systems'

Robert Remak was a student of Georg Frobenius and H.A. Schwarz. In 1929 he acquired the *venia legendi* in mathematics at the University of Berlin and was a *Privatdozent* there until 1933.[29] John von Neumann had become a *Privatdozent* at the same university in 1927; he held the position until 1929 (see Ulam 1958).

According to the information gathered by Wittmann from some of Remak's former friends and colleagues, Remak was in all probability stimulated by a group of economists around von Bortkiewicz to study the problem of the conditions under which positive solutions of systems of linear equations obtain (cf. Wittmann 1967: 401). His 1929 paper was a result of these studies (see Remak 1929). Unfortunately, of Remak's paper only the greater part of the third section dealing with the existence problem of price equilibrium is available in English (cf. Baumol and Goldfeld 1968: 271–7). Hence the motivation of his paper and its economic reasoning are largely unknown in the English speaking world. In what follows we shall briefly summarize the main argument.[30]

6.1 Methodological issues

Remak begins his paper with a definition of what he means by an exact science which bears a close resemblance to Leontief's 'naturalistic' point of view (cf. Leontief 1928): an exact science regards as 'exactly correct' only what can be ascertained by physical observation, counting or calculation (1929: 703). He then applies this definition to 'economics', which he tends to equate with Marshallian demand and supply analysis;[31] his concern is particularly with the demand side. He argues:

> All existing approaches in theoretical economics always start from these [demand] functions, which characterize the buyer's behaviour at different prices. However, since this behaviour can be neither experimentally nor theoretically ascertained quantitatively, there is no way to get from these theories to practical calculations. We will therefore take into consideration approaches which result in quantitative calculations that can also be carried out practically.
>
> (ibid.: 711–12; similarly Leontief 1928: 622)[32]

The alternative Remak suggests are what he calls 'superposed price systems' (*superponierte Preissysteme*): 'A superposed price system has nothing to do with values. It only satisfies the condition that each price covers the prices of the things required in production, and the consumption of the producer on the assumption that it is both just and feasible' (ibid.: 712).[33] Its calculation requires obviously a detailed knowledge of the socio-technical relations of production, i.e. the methods of production in use and the needs and wants of producers (ibid.: 712–13).

For most of the paper, and particularly in its third part which formalizes the argument, Remak assumes (implicitly) a stationary economy. Yet he makes it clear that this is but a first step towards an analysis of a dynamic economic system, i.e. one evolving over time: while a stationary economy can be represented by a single point in what Remak calls the 'economic phase space' made up of a finite number of economic coordinates, a developing economy

involves 'a moving point which in the phase space describes a curve' (ibid.: 717).

6.2. 'Superposed prices'

Remak then constructs 'superposed prices' for an economic system in which there are as many single-product processes of production as there are products, and each process or product is represented by a different 'person'.[34] It would not affect the logic of the argument, if the term 'person' were to be replaced by the term 'industry' or 'activity' (see also Wittmann 1967: 404). The amounts of the different commodities acquired by a person over 'a certain period of time, e.g. a year', in exchange for its own product, are of course the amounts needed as means of production to produce this product, given the technical conditions of production, and the amounts of consumption goods in support of the person (and its family), given the levels of sustenance. With an appropriate choice of units, the resulting system of 'superposed prices' can be written (using matrix notation)

$$\mathbf{p} = \mathbf{Ap}, \tag{7}$$

where $\mathbf{A}$ is the augmented matrix of inputs (means of production and consumption) per unit of output, and $\mathbf{p}$ is the vector of exchange ratios. Remak then discusses system (7) and arrives at the conclusion that there exists a solution to it which is semipositive and unique except for a scale factor.[35]

6.3. Socialism vs. capitalism

Model (7) refers to a kind of ideal economy with independent producers, no wage labour and hence no profits; it thus bears a close resemblance to Marx's concept of 'simple commodity production'. However, it could also be interpreted as reflecting a socialist economic system.[36] Although Remak does not refer to Marx nor to any socialist author, it is clear that his paper is intended to contribute to the then politically heated debate on socialism vs. capitalism. As Remak stresses in the introductory section of his paper:

> The question of whether or not an exact economics is possible is not of a purely theoretical interest, but is of fundamental practical importance. The socialist doctrine maintains the possibility of another, a better economic order which utilizes the given technical possibilities much more effectively to the benefit of the population. Diametrically opposed to this is the capitalist economic doctrine, which claims that through the free play of forces, which includes monopolies and other phenomena, the economic optimum will already be realised, and that any other regulation of economic life, by preventing this free play, would entail a smaller produce. . . . *The main task of an exact economics would consist in deciding between these two views by means of exact instruments of calculation.*
>
> (1929: 704; emphasis added)

In Remak's view there are two problems to be solved here. The first concerns the question whether an appropriate price system for a socialist economy can be found. Without being able to demonstrate that a system of ' "reasonable" prices' actually exists, the socialist alternative would be deprived of its rational basis: 'These prices . . . represent a "necessary" condition in the mathematical sense for an efficient economy exempt from unemployment and crises to exist' (1933: 840). Remak takes pride in having shown with his concept of 'superposed prices' that such a solution in fact exists and how it can be determined. Towards the end of his article he also expresses the conviction that the technical problem of numerically solving large systems of linear equations can be expected to be overcome soon, given the progress made in the development of electric calculating machines (cf. ibid.: 735).

The second and much more difficult problem concerns the comparative assessment of the economic efficiency of capitalism and socialism, respectively. Remak does not pretend to be possessed of a definite answer to this intricate question. He indicates, however, the direction in which an answer should be sought. In his view the problem boils down to the question of whether the modern capitalist economy is 'extremal', that is, whether it fully uses its productive potential or forgoes production possibilities. In view of unemployment and idle plant and equipment Remak sees reason to conjecture that it fails on this account (ibid.: 706, 721–2). How can this failure be explained?

6.4. On the 'non-extremality' of capitalism

Although Remak's discussion is occasionally rather cloudy, two closely connected causes are singled out as responsible for the malfunctioning of the capitalist economy: first, the role money plays in the system, and second, the distribution of income and thus purchasing power between capital owners and workers. Scrutiny shows that Remak advocates some kind of under-consumption-cum-miscalculation explanation of effective demand failures. In one place he writes:

> Today wages are reckoned as a part of the commodity; the latter on its way to completion is subject to several high percent mark ups, so that the worker eventually buys only a fractional part of his own daily work. It does not follow, however, that he gives the remaining part of his work to capitalism, since it is clearly conceivable that a wrong method of calculation gives rise to a lack of sales and thus prevents the realisation of a technically feasible additional production.
>
> (ibid.: 733–4)[37]

Remak does not provide a formalization of his view of the determination of prices in a capitalist economy. The price system he appears to have in mind can, however, easily be constructed following the hints he gives. There are two kinds

of mark ups: a general mark up for the economy as a whole, i.e. the rate of interest (ibid.: 713), and a mark up specific to an industry (or a firm). On the same technological premises as those underlying the construction of system (7) (single production, no choice of technique, etc.) the system of prices would now be given by

$$\mathbf{p}_c = (\mathbf{I} + \mathbf{M})[(1 + r)\mathbf{C}\mathbf{p}_c], \tag{8}$$

where $\mathbf{I}$ is the identity matrix, $\mathbf{M}$ is the diagonal matrix of the sectoral mark ups $m_i \geq 0, i = 1, 2, \ldots, n$, r is the rate of interest, $\mathbf{p}_c$ is the vector of 'capitalist' prices, and $\mathbf{C}$ is the matrix of material inputs, $\mathbf{N}$, plus wage goods per unit of output, i.e.

$$\mathbf{C} = \mathbf{N} + \mathbf{l}\mathbf{w}^T, \tag{9}$$

$\mathbf{l}$ being the vector of labour inputs and $\mathbf{w}^T = (w_1, w_2, \ldots, w_n)$ the real wage bundle per unit of labour.[38] System (8) is sketched only verbally by Remak; no discussion of its mathematical properties is provided.

As we have seen, in Remak's opinion there are reasons to suppose that the problem of underutilization of productive resources in modern capitalism is closely related to the general levels and the structure of the m_i's and the level of r. The question is close at hand whether a transition from price system (8) to the system of 'reasonable' prices (7) would remedy the idleness of labour and capital. Remak's answer is cautiously in the affirmative. The investigation of system (7) is taken to serve the purpose of finding out whether 'an economy which is perceived to be both just and efficient ('*zweckmäßig*') can be brought about by appropriate directions regulating the formation of prices of all commodities' (ibid.: 724). In his second article, which was written under the impact of the Great Depression, Remak concludes that it can be surmised 'that the system of "reasonable" prices would allow merchants to apply only much lower mark ups than the usual ones, which would lead in effect to putting a severe curb on profits' (1933: 841).

7. Von Neumann and Remak

Wittmann (1967: 407–8) points out that Remak gave his paper at a meeting of the Berlin Mathematical Society and that his ideas were discussed at the Institute of Mathematics in Berlin. He also conjectures that von Neumann was familiar with Remak's ideas. According to Wittmann's sources most of Remak's colleagues 'derided' the conclusions of his paper.

It is possible that von Neumann was among those colleagues who took a critical position towards Remak's contribution. We may even consider the possibility that von Neumann's paper contains, *inter alia*, an implicit answer to his colleague. Since we do not know of any statement to this effect by von

Neumann himself, the only evidence on which such an interpretation could possibly rest has to derive from a careful textual comparison of the papers of the two authors. Such a comparison leads in fact to some remarkable observations.

Both authors are concerned with the efficiency, or lack thereof, of what von Neumann calls 'the normal price mechanism' of a capitalist economy (von Neumann 1945: 1). While Remak contended that the way prices are formed in a capitalist economy is partly responsible for the fact that the system is statically (and dynamically) non 'extremal', i.e. inefficient, a main result of von Neumann's paper reads: 'the normal price mechanism brings about . . . the technically most efficient intensities of production' (1945: 1).[39, 40] The other factor mentioned by Remak as potentially detrimental to efficiency, money, is also touched upon by von Neumann. The passage just quoted is followed by the adjunct: 'This seems not unreasonable since we have eliminated monetary complications' (ibid.: 1).

In Remak's paper scarce natural resources, such as land, play no significant role. He rather focuses attention on systems of production that are in a self-replacing state and in which there are at most three types of income: wages, interest and profits. By implication, none of the natural resources utilized is scarce and therefore yields its owner a rent. In accordance with the capitalism vs. socialism debate Remak is interested in, emphasis is on the conflict between workers and capital owners over the distribution of the product. Interestingly, the total neglect of the problem of scarcity is also a characteristic feature of von Neumann's model. If his concern had been with generalizing the 'Walras–Cassel model', as is maintained by the conventional interpretation, this neglect would be totally incomprehensible, whereas it can easily be understood if one of his implicit aims was refuting Remak's view.

Just like Remak, von Neumann adopts a circular notion of production and considers the means of subsistence an integral part of the advances at the beginning of the uniform period of production. However, in every respect von Neumann's model is more general than Remak's. Repeatedly one gets the impression that where Remak drops an idea or poses a question that is beyond the scope of his own model, von Neumann offers a conceptualization and provides an answer. While Remak emphasizes that what is at stake is the question of the dynamic (in)efficiency of an economy, but then restricts his discussion essentially to the case of a stationary system, von Neumann adopts a dynamic framework of the analysis, albeit limited to the case of steady-state growth. While Remak is aware of the fact that an important aspect of the efficiency issue is how the problem of the choice of technique is decided, von Neumann tackles the problem head-on. While Remak notes incidentally that production and consumption activities may generate 'waste' which has to be disposed of,[41] von Neumann starts directly from the assumption of general joint production coupled with the assumption of free disposal of all superfluous

products. While Remak discusses markup pricing without, however, addressing the problem of the mutual consistency of the markups, including the rate of interest, the given real wage rate(s) and the given technical conditions of production, von Neumann demonstrates that the rate of interest, i.e. the general markup across all processes of production, is uniquely determined by the technical alternatives, given the real wage rate(s).

Circumstantial evidence and a detailed textual comparison seem to support the conjecture that von Neumann's model contained, among other things, an answer to his mathematical colleague. Compared with the widespread opinion that von Neumann's model was meant to provide a solution to a problem posed by Cassel, that of uniform growth, and not dealt with by the Viennese mathematical economists, this interpretation appears to us to be more plausible. Indeed, in our view there are too many elements in the analyses of von Neumann and the Viennese that are difficult to reconcile (see, in particular, Section 4 above), while we are not aware of any aspect contradicting our interpretation. It goes without saying that we cannot prove that we are right: *se non è vero, è ben trovato.*

8. Conclusion

This paper has shown that the conventional interpretation of von Neumann's growth model is difficult to sustain. Most important, in von Neumann there is no endowment of the economy with a given (physical or value) 'quantity of capital' that constrains productive capacity and provides the basis, in terms of its relative 'scarcity', for a determination of the rate of interest. It is a characteristic feature of the von Neumann model that the distributive variables, the wage rate and the rate of interest, are not determined in the conventional, symmetric way in terms of the demand for and supply of the respective factors of production, labour and 'capital'. Moreover, whereas in the growth model of Cassel the (long-term) rate of growth of the system is given from outside by a 'natural' rate of growth, assuming the full employment of all primary factors, in von Neumann the rate of growth is endogenously determined and full employment of labour (or natural resources) is not assumed.

While the structure of the von Neumann model is difficult to reconcile with the neoclassical point of view, it is fully compatible with the classical one. This concerns in particular the asymmetric treatment of the wage rate, the independent variable, and the rate of interest, the dependent one. It is shown that von Neumann's approach has been anticipated in all relevant aspects by authors whose contributions can be strictly located within the classical tradition. These aspects concern: (i) the concept of production as a circular

flow; (ii) the notion of a uniformly expanding economy in which the rate of expansion is endogenously determined, i.e. a 'quasi-stationary system'; (iii) the concept of duality of the relationship between relative quantities and the rate of growth on the one hand and that between relative prices and the rate of interest (rate of profits) on the other; (iv) the use of inequalities in the discussion of the problem of the choice of technique; (v) and the way the Rule of Free Goods is applied to primary factors of production and to products, respectively. The authors referred to include, among others, Smith, Ricardo, Torrens, Whewell, von Bortkiewicz and Charasoff.

Next it is argued that von Neumann's model may be interpreted as containing, *inter alia*, an answer to the ideas laid out in a paper by his fellow-mathematician Robert Remak. Both circumstantial evidence and, more important, a careful textual comparison of Remak's paper on 'superposed price systems' and von Neumann's analysis support this interpretation. In contradistinction to Cassel and the Viennese economists Schlesinger and Wald, and in accordance with Remak, von Neumann set aside scarce natural resources and adopted a circular flow concept of production which differs from the neoclassical concept of a one-way avenue that leads from primary factors of production to consumption goods. It is argued that von Neumann was particularly concerned with refuting Remak's opinion that the 'normal price mechanism' in a capitalist economy is inefficient. It is concluded that there are too many elements in the analyses of von Neumann and the Viennese economists that are difficult to reconcile, while there appears to be none contradicting the interpretation put forward in this paper.

H.D.K., University of Graz
N.S., University of Pisa

Notes

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1 Cassel's book was published in English as *The Theory of Social Economy* in 1923; a revised translation of the fifth German edition was published in 1932 (see Cassel 1932).

2 It would of course be quite inappropriate, indeed pointless, to compare the von Neumann model, which is long period, with any short-period neoclassical model. In the latter the endowment of the economy with 'capital' is specified in terms of an arbitrarily given vector of heterogeneous capital goods. Therefore, in these models, flukes apart, an equilibrium is characterized by differential rates of return on the supply prices of the various capital goods.

3 In reading about Menger's colloquium one occasionally gets the impression that it was concerned with little else than the above problem. However, in the period of its existence 1932–7 only two people other than von Neumann, the banker and economist Karl Schlesinger and the mathematician Abraham Wald, read altogether four papers at the seminar, three of which were also published in the *Ergebnisse* (see Schlesinger 1935, and Wald 1935 and 1936). Another paper by Wald could not be published, first 'owing to a lack of space' (*Ergebnisse*, 8: 84) and then because of the colloquium's untimely termination due to the pending *Anschluß* of Austria to Hitler Germany in 1938. Wald, who fled Europe with the arrival of the Nazis, seems to have lost the paper on his way to the United States; on the history of Wald's paper, see Chipman (1965: 720, footnote 18). In what follows we shall refer to Schlesinger and Wald as the 'Viennese economists'.

4 As Karl Menger recalled: 'Wald's paper on the equations concerning production greatly interested von Neumann, as he told me when passing through Vienna soon after its publication. It reminded him of equations he had formulated and solved in 1932 and now offered to present in our Colloquium' (Menger 1973: 55). See also Hicks (1960: 676, fn. 1) and the story told by Jacob Marschak to Axel Leijonhufvud and Earlene Craver, as reported by Weintraub (1985: 74, n.) and Arrow (1989: 25). Although there is some uncertainty as to the year in which the event at the Kaiser Wilhelm Institut in Berlin took place, Marschak's story provides further evidence that von Neumann had developed his ideas several years before he gave his talk at Menger's colloquium. Weintraub comments on this: 'This story . . . suggests that the genesis of von Neumann's *Ergebnisse* paper was quite specific and roughly contemporary with von Neumann's [1928] paper on game theory. The min–max idea, the duality ideas, and the strategy of proof to be used later for the fixed-point theorem are found in each paper. The papers appear, then, to be naturally related not only by content, but also by place of origin' (ibid.). With regard to the last observation it would appear to be natural to pay special attention to the Berlin scientific community around the time when von Neumann was there as a lecturer and researcher. However, von Neumann's 'Berlin connection' is not dealt with by Weintraub. In his book neither von Neumann's fellow-mathematician Robert Remak nor Ladislaus von Bortkiewicz, Berlin's eminent Professor of Statistics and Political Economy, are mentioned. On the possible implications of this omission, see below.

5 Although it is not clearly stated, this seems to be the implication of the following passage in Arrow (1989: 23): 'Von Neumann makes no reference to the papers of Schlesinger and Wald, though he is publishing in the same journal two years later. He does state that the paper had been delivered to the Princeton Mathematical Club

in 1932, so that it may be taken to be independent of Wald and Schlesinger.' Then follows the remarkable adjunct: 'Wald must have been very self-effacing; he was one of the editors of the volume of the *Ergebnisse* in which von Neumann's paper appeared.'

6 Brody (1989: 141) has put forward the interesting conjecture that the new tools employed by von Neumann, i.e. the use of inequalities rather than equations and the adoption of max–min criteria for the existence of equilibrium, may have come to his attention while studying chemistry in Berlin under W. Ostwald. Ostwald had translated J.W. Gibb's 'On the Equilibrium of Heterogeneous Substances' (1875–8), who had used these tools to describe chemical processes. This interpretation may throw light on the parallel drawn by von Neumann between the function ϕ (X, Y) in his analysis and that of 'thermodynamic potentials in phenomenological thermodynamics' (von Neumann 1945: 1).

7 It is interesting to note that in the title of Champernowne's paper (see Champernowne 1945) the title of the English version of von Neumann's paper is referred to incompletely: the adjective 'general' is left out.

8 It is true, though, that both in von Neumann and in those long-period versions of neoclassical theory that start from a given endowment of the economy with value capital, the *proportions* in which the different capital goods are needed are fully adjusted to the data, or independent variables, of the respective approaches. Hence, these proportions are taken to be a part of the solution of the system rather than a given (as in neoclassical short-period analysis). However, in contradistinction to neoclassical long-period models, in von Neumann the aggregate value or 'quantity' of the capital stock is not among the data of the problem.

9 This is one of the reasons why Koopmans considered von Neumann's paper 'not very good economics' (Koopmans 1974). The assumption of a given initial endowment of the economy with capital goods was only subsequently appended to von Neumann's growth model, e.g. in Dorfman, Samuelson and Solow (1958). This, together with the assumption of a given terminal endowment with capital goods, has led to the development of 'turnpike theorems'. Another reason for this harsh judgement was the treatment of the consumption of workers, which, in Champernowne's interpretation, reduced 'the role of the worker-consumer to that of a farm animal' (Champernowne 1945: 12).

10 Since 'capital' is set aside in the formulations of the Viennese economists, it is not surprising that the concept of the rate of interest (or rate of profits) makes no appearance.

11 In another place Arrow writes: 'Why von Neumann discarded the whole apparatus of demand functions, we cannot know' (1989: 25). See, however, Kaldor's recollection quoted in Section 2 of this paper.

12 It should be noted, though, that the Viennese economists, following Cassel's basic model, assumed that there is only one fixed-coefficients method of production for each commodity, i.e. there is no choice of technique.

13 For a brief account of the classical concept of production, see Kurz and Salvadori (1992a: ch. 1).

14 On Torrens's contribution see also Schefold (1981: section 4) and de Vivo (1985, 1986).

15 Torrens acknowledges his indebtedness to Ricardo's 'original and profound inquiry into the laws by which the rate of profits is determined' (ibid.: xix).

16 Torrens also indicates that if the entire 'surplus' or 'profit' were to be accumulated, then the rate of expansion of the economy would be equal to the rate of profits. Hence Torrens may be said to have anticipated, in embryonic form, what Champernowne

(1945: 10) in his interpretation of the von Neumann model called a 'quasi-stationary state'.

17 Dmitriev published his essay on Ricardo in Russian in 1898. This essay together with two others, one on Cournot's theory of competition, the other on marginal utility theory, was reprinted in 1904. A French translation of the three essays was edited by A. Zauberman in 1968 (see Dmitriev 1968), an English translation by D.M. Nuti in 1974 (see Dmitriev 1974). (According to Nuti (cf. Dmitriev 1974: 30), the only copy of Dmitriev's 1904 book available in the West was in the possession of Piero Sraffa.) Both von Bortkiewicz and Charasoff published in German. Charasoff who was born in Tiflis in 1877 obtained a doctorate in mathematics in 1901 at the University of Heidelberg.

18 Charasoff's contributions have only recently been rediscovered by Egidi and Gilibert (1984); see also Duffner and Huth (1988). For a summary statement of his main argument, see Kurz (1989: 44–6).

19 The family resemblance with Sraffa's notion of the 'Standard system' in which the rate of profits 'appears as a ratio between quantities of commodities irrespective of their prices' (Sraffa 1960: 22) is close at hand.

20 As is well known, the concept of production as a circular flow figures prominently also in Leontief's 1928 Ph.D. thesis written under the supervision of von Bortkiewicz at the University of Berlin (see Leontief 1928) and in his subsequent formulation of input-output analysis. Due to a lack of space we cannot enter into a proper discussion of his works; see, however, the brief remarks on Leontief in Section 6.

21 Prior to the Viennese economists the Danish economist F. Zeuthen (1933) had argued that Cassel's resource constraints ought to be written as inequalities. In a review article published in Swedish only one year after Cassel's *Theoretische Sozialökonomie* Knut Wicksell had already pointed out that the Cassellian system may possess no solution or may have solutions where some factor prices are zero because there is an excess supply of the respective factors (cf. Wicksell 1934: appendix 1, p. 228). (This reference may help to answer a query by Baumol and Goldfeld 1968: 268, n.)

22 Assuming that natural resources are non-scarce is of course not the same thing as assuming that there are no natural resources at all. Von Neumann's model is frequently misinterpreted in the latter sense. In this context it deserves to be noted that von Neumann does not define goods in the same way as Debreu (1959: 32): he does not consider a particular plot of land in a particular location as a special good. However, with the system growing forever, the point will surely come where some natural resource(s) will become scarce. Surprisingly, von Neumann does not seem to have seen this point. As Professor Samuelson has pointed out to us in private correspondence, 'More by inadvertance than conscious intention, v.N. failed to emphasize the *basic classical* notion of land resources as unproducible or diminishable.' The total neglect of the problem of scarce primary resources such as land distinguishes his analysis in fact both from the analyses of the classical and the neoclassical economists. For a possible explanation of this neglect, see Section 7 below.

23 'At most, one could say that a "Rule of Zero 'Excess' Wages" is applied because labour is less than fully employed' (Steedman 1987: 419). The interpretation given by Dore of von Neumann's use (or rather non-use) of the Rule of Free Goods is difficult to sustain: according to Dore (1989: 83) in the von Neumann model 'Cassel's "principle of scarcity" . . . is given an extreme binary interpretation whereby a resource has either a positive economic value if it is fully utilized, or its value is zero. . . . Unless every single man and woman is fully employed, the social value of labour is

zero; this is indeed extreme. Why did von Neumann resort to this formulation?' The answer to this question is: he did not.

24 Thus, Varri's contention (1982: 10–11) that the Rule of Free Goods is 'completely extraneous' to the theory of value of 'classical derivation' does not stand up to examination – unless, of course, Adam Smith is declared non-classical.

25 Therefore, it is seriously misleading to characterise the classical approach as one which is exclusively concerned with 'a fixed economic universe' and thus 'cannot account for the generation of an equilibrium because it refers to an empirically unique observed economy' (Punzo 1991: 15).

26 On Whewell and the group of mathematical economists, see Campanelli (1982) and Henderson (1985).

27 This finding anticipates the essence of the Okishio theorem (see Okishio 1963).

28 For example, there is no reference to Remak's contribution in Weintraub (1985), Punzo (1989, 1991) or Dore, Chakravarty, and Goodwin (1989). See, however, Gilibert (1991: 396) who deserves the credit for having drawn attention to the importance of Remak's paper in his attempt to reconstruct the history of mathematical economics at the beginning of this century.

29 Remak died in the concentration camp at Auschwitz.

30 It is interesting to note that the papers by Remak (1929) and Leontief (1928) have several elements in common. These include: (i) the general methodological position adopted; (ii) the concept of price put forward; and (iii) the description of the economic process in terms of what Sraffa (1960: 3) was to call 'the methods of production and productive consumption'.

31 Marshall's *Principles of Economics* is the only book referred to in the entire paper (cf. ibid.: 709, fn.). Therefore, the foundation of the view conveyed by Baumol and Goldfeld (1968: 267) that Remak aimed at pointing out 'a serious gap in Walras' argument' is unclear.

32 See also Kaldor's recollection (cf. Section 2 above) of the reservations expressed by von Neumann with regard to the marginalist theory of demand.

33 In an addendum to his paper published in 1933, Remak stresses: 'A price does not emerge from supply and demand, it is rather a number which has to satisfy certain conditions. The price of a commodity must cover the prices of the expenses contained in it including the cost of living, which may be taken to be known, of the people participating in its production. This leads to the superposed price systems' (1933: 840). Remak also talks of' "reasonable" prices' (*'vernünftige' Preise*). See also Leontief (1928: 598) who stresses that the concept of value adopted by him has nothing to do with any intrinsic property of goods as judged by the consumer; it rather refers to the 'exchange relation' deduced from the 'relations of production'.

34 The somewhat unfortunate phrasing of the problem by Remak may have been the source of the misconception that his concern was with a pure exchange economy; for this interpretation, see Gale (1960: 290) and Newman (1962: 60).

35 It should be mentioned that Remak does not make use of the mathematical tools provided by Perron and his own former teacher Frobenius.

36 The view that system (7) is open to alternative interpretations is especially emphasized by Remak in his second paper (1933: 840).

37 While most of Remak's argument refers to an economy with a given productive capacity, he touches also upon the dynamic features of a capitalist economy. In his view there is the danger that the innovative potential of such an economy will not be fully exploited: 'Today's economy allows increases in value in consequence of technological change in favour of capital only. These increases can, however, be utilized only partially, since the producer will not find the buyers of all the goods he

could produce if it were not for the limited sales possibilities' (1929: 708; see also p. 722). As seen by Remak, the modern capitalist economy is neither statically nor dynamically efficient, or 'extremal'.

38 In the case in which there is a single uniform mark up, m, throughout the economy, equation (i) would simplify to

$$\mathbf{p}_c^* = (1 + R)\mathbf{C}\mathbf{p}_c^*, \tag{10}$$

where $(1 + R) = (1 + m)(1 + r)$ and $\mathbf{p}_c^*$ is the corresponding price vector.

39 In the German original von Neumann uses the expression: 'die rein technisch *zweckmäßigste* Verteilung der Produktionsintensitäten'. He thus uses the same terminology as Remak. More important, the conception of efficiency adopted by the two authors appears to be the same.

40 Interestingly, Champernowne in his commentary on the von Neumann model remarks on the above passage: 'This may immediately suggest an argument in favour of free enterprise in the real world' (Champernowne 1945: 16).

41 Remak even mentions the possibility of 'negative prices' in this context (1929: 726) and points out that the negativity of the price of a substance that has to be removed corresponds with the positivity of the price of the respective disposal service.

References

Arrow, K.J. (1989). Von Neumann and the existence theorem for general equilibrium. In Dore, Chakravarty and Goodwin 1989: 15–28.

Baumol, W.J. and Goldfeld, S.M. (eds) (1968). *Precursors in Mathematical Economics*. LSE Series of Reprints of Scarce Works on Political Economy, No. 19. London: London School of Economics.

Bortkiewicz, L.v. (1906–7). Wertrechnung und Preisrechnung im Marxschen System. *Archiv für Sozialwissenschaft und Sozialpolitik*, 23 (1906): 1–50, 25 (1907): 10–51 and 445–88; quoted as 1906–7 I, II and III.

—— (1907). Zur Berichtigung der grundlegenden theoretischen Konstruktion von Marx im 3. Band des 'Kapital'. *Jahrbücher für Nationalökonomie und Statistik*, 34: 319–35.

Brody, A. (1989). Economics and thermodynamics. In Dore, Chakravarty and Goodwin 1989: 141–8.

Campanelli, G. (1982). W. Whewell's contribution to economic analysis: the first mathematical formulation of fixed capital in Ricardo's system. *The Manchester School*, 50: 248–65.

Cassel, G. (1918). *Theoretische Sozialökonomie*. Leipzig: A. Deichert.

—— (1932). *The Theory of Social Economy*, revised English translation of the 5th German edition of Cassel (1918) by L. Barron. New York 1932: Harcourt Brace. Reprinted (1967) New York: Augustus M. Kelley Publishers.

Champernowne, D.G. (1945). A note on J.v. Neumann's article on 'A model of economic equilibrium'. *Review of Economic Studies*, 13: 10–18.

Charasoff, G.v. (1910). *Das System des Marxismus: Darstellung und Kritik*. Berlin: H. Bondy.

Chipman, J.S. (1965). A survey of the theory of international trade, Part 2. *Econometrica*, 33: 685–760.

Debreu, G. (1959). *Theory of Value*. New York: Wiley.

De Vivo, G. (1985). Robert Torrens and Ricardo's 'corn-ratio' theory of profits. *Cambridge Journal of Economics*, 9: 89–92.

—— (1986). Torrens on value and distribution. *Contributions to Political Economy*, 5: 23–36.

Dmitriev, V.K. (1968). *Essais Economiques. Esquisse de synthèse organique de la théorie de la valeur-travail et de la théorie de l'utilité marginale*, ed. with an introduction by A. Zauberman. Paris: Editions du Centre national de la récherche scientifique.

—— (1974). *Economic Essays on Value, Competition and Utility*, ed. with an introduction by D.M. Nuti. Cambridge: Cambridge University Press.

Dore, M. (1989). The legacy of von Neumann. In Dore, Chakravarty and Goodwin 1989: 82–99.

Dore, M., Chakravarty, S. and Goodwin, R. (eds) (1989). *John von Neumann and Modern Economics*. Oxford: Clarendon.

Dorfman, R., Samuelson, P.A. and Solow, R.M. (1958). *Linear Programming and Economic Analysis*. New York, Toronto and London: McGraw-Hill.

Duffner, H. and Huth, T. (1988). Georg Charasoff's theory of value, capital and prices of production. University of Bremen, mimeo.

Egidi, M. and Gilibert, G. (1984). La teoria oggettiva dei prezzi. *Economia Politica*, 1: 43–61. In English: The objective theory of prices. *Political Economy. Studies in the Surplus Approach*, 5 (1989): 59–74.

Ergebnisse eines mathematischen Kolloquiums (1931–37). Ed. K. Menger in collaboration with K. Gödel and G. Nöbeling, 6 (1935), 7 (1936) and 8 (1937). Leipzig and Vienna: Franz Deuticke.

Gale, D. (1960). *The Theory of Linear Economic Models*. New York: McGraw-Hill.

Gilibert, G. (1991). La scuola russo-tedesca di economia matematica e la dottrina del flusso circolare. In G. Beccatini (ed.), *Le scuole economiche*. Turin: Utet, pp. 387–402.

Goodwin, R.M. (1986). Swinging along the turnpike with von Neumann and Sraffa. *Cambridge Journal of Economics*, 10: 203–10.

Henderson, J.P. (1985). The Whewell group of mathematical economists. *The Manchester School*, 53: 404–31.

Hicks, J.R. (1960). Linear theory. *Economic Journal*, 70: 671–709.

Kaldor, N. (1961). Capital accumulation and economic growth. In Lutz and Hague 1961: 177–222.

—— (1989). John von Neumann: a personal recollection. Foreword in Dore, Chakravarty and Goodwin 1989: vii–xi.

Koopmans, T.C. (1974). Contribution to 'General discussion on past and future of the von Neumann Model'. In Los and Los (eds), *Mathematical Models of Economic Growth*. Amsterdam: North-Holland, pp. 3–4.

Kurz, H.D. (1986). Classical and early neoclassical economists on joint production. *Metroeconomica*, 38: 1–37. Reprinted in Salvadori and Steedman 1990.

—— (1989). Die deutsche theoretische Nationalökonomie zu Beginn des 20. Jahrhunderts zwischen Klassik und Neoklassik. In B. Schefold (ed.), *Studien zur Entwicklung der ökonomischen Theorie*, vol. VIII. Berlin: Duncker & Humblot, pp. 11–61.

Kurz, H.D. and Salvadori, N. (1992a). Theory of production I. *Dynamis*, Quaderno 1/92, Working papers of IDSE (Istituto di ricerca sulla Dinamica dei Sistemi Economici). Milan, pp. 238.

Kurz, H.D. and Salvadori, N. (1992b). Theory of production II. *Dynamis*, Quaderno 2/92, Working papers of IDSE (Istituto di ricerca sulla Dinamica dei Sistemi Economici). Milan, pp. 192.

Leontief, W. (1928). Die Wirtschaft als Kreislauf. *Archiv für Sozialwissenschaft und Sozialpolitik*, 60: 577–623. An English translation of parts of this paper was published,

with an introduction by P.A. Samuelson, as: The economy as a circular flow. In *Structural Change and Economic Dynamics*, 2 (1991): 177–212.

Lutz, F.A. and Hague, D.C. (eds) (1961). *The Theory of Capital*. London: Macmillan.

Marx, K. (1956). *Capital*, vol. II. Moscow: Progress Publishers. English translation of *Das Kapital*, vol. II, ed. F. Engels. Hamburg (1885): Meissner.

—— (1959). *Capital*, vol. III. Moscow: Progress Publishers. English translation of *Das Kapital*, vol. III, ed. F. Engels. Hamburg (1894): Meissner.

McKenzie, L.W. (1987). General equilibrium. In J. Eatwell, M. Milgate and P. Newman (eds), *The New Palgrave. A Dictionary of Economics*, vol. 2. London and Basingstoke: Macmillan, pp. 498–512.

Menger, K. (1973). Austrian marginalism and mathematical economics. In J.R. Hicks and W. Weber (eds), *Carl Menger and the Austrian School of Economics*. Oxford: Clarendon Press, pp. 38–60.

Morishima, M. (1973). *Marx's Economics. A Dual Theory of Value and Growth*. Cambridge: Cambridge University Press.

Neumann, J.v. (1928). Zur Theorie der Gesellschaftsspiele. *Mathematische Annalen*, 100: 295–320.

—— (1937). Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes. *Ergebnisse eines mathematischen Kolloquiums*, 8: 73–83.

—— (1945). A model of general economic equilibrium. *Review of Economic Studies*, 13: 1–9. English translation of von Neumann (1937).

Newman, P. (1962). Production of commodities by means of commodities. *Schweizerische Zeitschrift für Volkswirtschaft und Statistik*, 98: 58–75.

Okishio, N. (1963). A mathematical note on Marxian theorems. *Weltwirtschaftliches Archiv*, 91: 287–99.

Punzo, L.F. (1989). Von Neumann and Karl Menger's mathematical colloquium. In Dore, Chakravarty, and Goodwin 1989: 29–65.

—— (1991). The school of mathematical formalism and the Viennese circle of mathematical economists. *Journal of the History of Economic Thought*, 13: 1–18.

Remak, R. (1929). Kann die Volkswirtschaftslehre eine exakte Wissenschaft werden? *Jahrbücher für Nationalökonomie und Statistik*, 131: 703–35.

—— (1933). Können superponierte Preissysteme praktisch berechnet werden? *Jahrbücher für Nationalökonomie und Statistik*, 138: 839–42.

Ricardo, D. (1951 ssq.). *The Works and Correspondence of David Ricardo*, ed. P. Sraffa in collaboration with M.H. Dobb. Cambridge: Cambridge University Press, vols. I and VI. In the text referred to as *Works*, volume number and page number.

Samuelson, P.A. (1989). A revisionist view of von Neumann's growth model. In Dore, Chakravarty and Goodwin 1989: 100–22.

Schefold, B. (1981). Nachfrage und Zufuhr in der klassischen Ökonomie. In F. Neumark (ed.), *Studien zur Entwicklung der ökonomischen Theorie*, vol. I. Berlin: Duncker & Humblot, pp. 53–91.

Schlesinger, K. (1935). Über die Produktionsgleichungen der ökonomischen Wertlehre. *Ergebnisse eines mathematischen Kolloquiums*, 6: 10–11.

Smith, A. (1976). *An Inquiry into the Nature and Causes of the Wealth of Nations*. vol. II of *The Glasgow Edition of the Works and Correspondence of Adam Smith*, ed. R.H. Campbell, A.S. Skinner and W.B. Todd. Oxford: Oxford University Press. In the text quoted as *WN*, book number, chapter number, section number, paragraph number.

Sraffa, P. (1960). *Production of Commodities by Means of Commodities. Prelude to a Critique of Economic Theory*. Cambridge: Cambridge University Press.

Steedman, I. (1977). *Marx after Sraffa*. London: N.L.B.

—— (1987). Free goods. In J. Eatwell, M. Milgate, and P. Newman (eds), *The New Palgrave. A Dictionary of Economics*, vol. 2. London and Basingstoke: Macmillan, pp. 419–20.

Torrens, R. (1820). *An Essay on the Influence of the External Corn Trade upon the Production and Distribution of National Wealth*, 2nd edn. London: Hatchard.

—— (1821). *An Essay on the Production of Wealth*. London: Longman, Hurst, Rees, Orme, and Brown. Reprint (1965) ed. J. Dorfman, New York: Augustus M. Kelley.

Tozer, J.E. (1838). Mathematical investigation of the effect of machinery on the wealth of a community in which it is employed, and on the fund for the payment of wages. *Transactions of the Cambridge Philosophical Society*, vol. VI, part III. Cambridge: Pitt Press, pp. 507–22. Reprinted (1968) New York: Augustus M. Kelley.

Ulam, S. (1958). John von Neumann, 1903–1957. *Bulletin of the American Economic Society*, 64: 1–49.

Varri, P. (ed.) (1982). *I prodotti congiunti*. Milan: Vita e Pensiero.

Wald, A. (1935). Über die eindeutige positive Lösbarkeit der neuen Produktionsgleichungen. *Ergebnisse eines mathematischen Kolloquiums*, 6: 12–18. Comments by Schams and Menger, pp. 18–20.

—— (1936). Über die Produktionsgleichungen der ökonomischen Wertlehre (II. Mitteilung). *Ergebnisse eines mathematischen Kolloquiums*, 7: 1–6. Comment by Gödel, p. 6.

Walras, L. (1954). *Elements of Pure Economics*. London: George Allen & Unwin. English translation of the definitive edition of Walras's *Elements d'économie politique pure* (first published 1874) by W. Jaffé.

Walsh, V. and Gram, H. (1980). *Classical and Neoclassical Theories of General Equilibrium: Historical Origins and Mathematical Structure*. New York and Oxford: Oxford University Press.

Weintraub, E.R. (1985). *General Equilibrium Analysis. Studies in Appraisal*. Cambridge: Cambridge University Press.

Whewell, W. (1831). Mathematical exposition of some of the leading doctrines in Mr. Ricardo's 'Principles of political economy and taxation'. *Transactions of the Cambridge Philosophical Society*, pp. 1–44. Reprinted in Whewell 1971.

—— (1971). *Mathematical Exposition of Some Doctrines of Political Economy*. Reprint of four papers published by Whewell in 1829, 1831 and 1850. New York: Augustus M. Kelley.

Wicksell, K. (1893). *Über Wert, Kapital und Rente nach den neueren nationalökonomischen Theorien*. Jena: Fischer. English translation (1954) *Value, Capital and Rent*. London: George Allen & Unwin.

—— (1934). *Lectures on Political Economy*, vol. I: General Theory. Transl. from Swedish by E. Classen and ed., with introduction, by Lionel Robbins. London: George Routledge & Sons.

Wittmann, W. (1967). Die extremale Wirtschaft. Robert Remak – ein Vorläufer der Aktivitätsanalyse. *Jahrbücher für Nationalökonomie und Statistik*, 180: 397–409.

Zeuthen, F. (1933). Das Prinzip der Knappheit, technische Kombination und ökonomische Qualität. *Zeitschrift für Nationalökonomie*, 4: 1–24.

Abstract

The paper shows that the conventional neoclassical interpretation of von Neumann's growth model cannot be sustained. Other than in models in the tradition of Walras

and Cassel, in his model there is no given 'capital' endowment that constrains productive capacity and provides the basis, in terms of its relative 'scarcity', for a determination of the interest rate. Von Neumann's model is rather fully compatible with, and has been anticipated in all relevant aspects by, authors whose contributions can be strictly located within, the classical tradition. This concerns in particular the asymmetric treatment of the distributive variables. Finally it is argued that von Neumann's model may be interpreted as containing, *inter alia*, an answer to the ideas put forward by his fellow-mathematician Robert Remak. Both circumstantial evidence and a careful textual comparison of Remak's paper on 'superposed price systems' and von Neumann's analysis support this interpretation.