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A Note on J. v. Neumann's Article on "A Model of Economic Equilibrium"[1]

SCOPE OF THE PAPER

The supreme merit of this paper lies in the elegance of the mathematical solution of a highly generalised problem in theoretical economics. But the paper is of considerable interest to economists as well as to mathematicians, because it deals simultaneously with questions on several fields of economics, which until this paper was first read, (in 1932) had seldom been considered together as parts of one problem. For example, in this short paper the author considers which goods will be free goods, and the determination of the prices of goods which are not free : at the same time he examines which productive processes and scales of production will be optimum and which will be unprofitable : he also examines the degree in which each optimum process will be used and the relative amounts of different goods that will be produced. At the same time he demonstrates the mechanism which determines the rate of interest and the rate of expansion of the whole economy.

Approaching these questions as a mathematician, Dr. Neumann places emphasis on rather different aspects of the problem than would an economist. Whereas he takes great care to give an absolutely rigorous mathematical argument and to state his assumptions completely and without ambiguity, he develops his points with the minimum of descriptive explanation. The paper is logically complete and admirably concise. In contrast to the convention among mathematical writers of reducing explanations to a minimum and stating assumptions as concisely as possible, economists more usually provide illustrative examples and repetitions of their argument to ease the reader's task of comprehension. Those accustomed to these less austere conventions may therefore be interested to read the following discursive commentary which develops some of the points of economic interest in Dr. Neumann's classic article.

By adopting extremely artificial assumptions, the author rendered his problem soluble and concentrated attention on some very interesting properties of the actual economic system. But at the same time this process of abstraction inevitably made many of his conclusions inapplicable to the real world : others could be applied only after considerable modifications. It is interesting to enquire how far the properties in his simplified model do correspond to similar phenomena in the real world.

THE APPROACH TO THE PROBLEM

Prof. v. Neumann's method is the familiar one of examining the conditions of equilibrium of his simplified model of the economic world. The first point is to get clear what is meant by equilibrium. The definition of equilibrium is very similar to that of the economist's stationary state : but in v. Neumann's article equilibrium differs from a stationary state's equilibrium in the vital respect that a uniform expansion of the whole system is allowed under equilibrium. Such a state of equilibrium may be called a quasi-stationary state, although v. Neumann does not in fact use this term.

From the point of view of the mathematician, the most important result of the article is the proof that under the simplified conditions there assumed, it will be possible for the model to have any equilibrium position at all, in the sense in which equilibrium

[1] This note is the outcome of conversations with Mr. N. Kaldor, to whom many of the ideas in it are due. I am also indebted to Mr. P. Sraffa of Cambridge and to Mr. Crum of New College, Oxford, for instruction in subjects discussed in this article.

is there defined. This may seem rather surprising, as one is rather apt to investigate conditions of equilibrium without bothering first of all to find out whether any equilibrium position need actually exist at all : one is liable to assume that some equilibrium position is possible. The fact that it is necessary to prove the existence of an equilibrium position before finding out the properties of such a position may be illustrated by the following consideration. Although v. Neumann defines equilibrium to be that of a quasi-stationary state, which may be expanding, he might perhaps have chosen to define equilibrium to be that of a stationary state without expansion or contraction. It so happens that the simplifying assumptions made about his economic model make it impossible in general for it to settle down to a state of stationary equilibrium : if, therefore, he had assumed that such a stationary equilibrium position was possible and had investigated the various conditions which the system must satisfy when in equilibrium, he would evidently have arrived at ridiculous results. For similar reasons, it was therefore necessary for him to prove that there was at least one possible position of quasi-stationary state equilibrium, before it was of any use investigating the properties which the system must have in such an equilibrium position.

Although, to the mathematician, the most interesting part of this paper will be that which proves the existence of at least one equilibrium position ; for the economist the most interesting part is that which analyses the properties of the system when it is in equilibrium. Fortunately, once the existence of an equilibrium position has been demonstrated, the arguments demonstrating the nature of this equilibrium are of quite an elementary nature, and it is possible to translate the rigorous mathematics of this part of the article into a somewhat looser form of words more readily digestible by those who are unused to thinking in terms of symbols.

Before turning to these arguments, it is useful to examine the manner in which v. Neumann approaches the economic problem. As we have seen, he is concerned not with short period problems but with the properties of the economic system when it has settled down to an equilibrium position which may be described as a quasi-stationary state. In such a state, all prices remain constant, the production of all goods remains in the same proportion although a uniform geometric rate of growth is allowed to the whole system. Thus if in any period the output of one particular good doubles, so then does the output of every other good double in that period, and the population and quantity of each kind of capital equipment double also in the same period. Thus in equilibrium there is no progress or change in production per head of population : growth merely consists of replication and the economic system expands like a crystal suspended in a solution of its own salt. The composition of any given volume of the crystal is at all times the same. To describe a system with uniform expansion of this kind we have introduced the term quasi-stationary state.

The model, being concerned only with a quasi-stationary state, can throw no direct light on problems of economic development and changes in the standard of living. The model has only one advantage over the strictly stationary state and that is that the community has an outlet for its savings in providing for the uniform expansion of the community and its stock of equipment.

In order to make it possible for quasi-stationary state equilibrium to exist in the model, several drastic simplifying assumptions had to be introduced. Constant returns were assumed in the sense that any economic process could be carried out at half, double, or in general x times its given scale, without any increase in costs per unit of output. Conditions of perfect competition in the long period were also assumed, and it was supposed that the natural factors of production (including labour) were available in unlimited quantities. One other very important assumption was implied, although it was nowhere very clearly stated : this was that no saving was carried out

by the workers whereas the propertied class saved the whole of their income. These various simplifying assumptions, although necessary if a rigorous proof of the existence of equilibrium was to be possible, evidently render the model unsuitable for examining problems connected with monopoly, economies of mass production, technical progress, or with land.[1] Since monetary problems are also assumed away, the reader may begin to wonder in what way the model has interesting relevance to conditions in the real world.

Prof. v. Neumann's model does however exhibit certain features of a competitive capitalist economy which tend to be obscured in the more traditional approach and can deal with the consequences of the circular nature of the production process (*i.e.* that commodities are largely produced out of each other) in a way that is not possible under it. By reducing the role of the worker-consumer to that of a farm animal, he can focus attention on those parts of the mechanism determining prices and the rate of interest, which depend on supply conditions alone and not on the tastes of consumers. This emphasis is important because the orthodox analysis has distributed attention evenly between marginal utility and conditions of supply ; since supply is often more elastic than demand, prices in the long run do over a wide field reflect contrasts in cost rather than conditions of consumers' demands: a price-theory focussing attention on costs can give a very clear and yet an approximately true account. We may first consider v. Neumann's approach to the problem of prices.

Consider a good which may be manufactured out of a lot of other goods : in the simplified conditions of the model the cost price of the good will consist of the values of the goods of which it is composed plus an interest charge on the fixed and working capital involved in the process. If a good is a joint product, then the value of the other products must be subtracted from the cost in arriving at the cost price of the good. Competition will ensure that where a good may be produced by many different processes, its cost price will correspond to the costs in the cheapest process.

Wage costs are not considered as such, for labourers are not separately considered any more than are farm animals. It is supposed that they will do their work in return for rations of shelter, fuel, food and clothing, just as a horse works when it is fed and cared for. The costs of labour thus consist of the goods which maintain the workers, just as the costs of a horse's work consist of his fodder, stabling, etc. The essential point about v. Neumann's theory of prices is that goods are made out of goods alone and that the cost price of any good or collection of goods consists of the value of the goods from which they are made plus an interest charge.

Prof. v. Neumann's approach to the theory of the rate of interest is interesting. He makes no reference to marginal products or to the marginal efficiency of capital : nor does he regard the rate of interest as depending on the relative efficiency of production processes involving different " periods of production " : the rate of interest is not determined as the supply price of waiting, abstinence or saving, for it is assumed that the propertied class save all their income and that the working class consume all theirs. Nor is the rate of interest determined as the measure of liquidity preference, for money as such plays no part in v. Neumann's article. The rate of interest appears as the natural and optimum rate of organic expansion[2] of the system, and depends on the technical processes of production which are available. If these processes enable the system to expand at 5 per cent per annum at most, then 5 per cent per annum will be the rate of interest.

In its concern with a quasi-stationary state, in its theory of prices as determined

[1] By assuming both constant returns and perfect competition, v. Neumann also implies that the division of the total output (by means of a given process) between firms (using that process) is indeterminate.

[2] See pp. 14–15 for a fuller explanation of this concept.

by the minimum cost of goods made from other goods alone, and in its theory of the rate of interest determined by the greatest possible rate of expansion of the economic system, this paper approaches the problems of economics in an extremely original and stimulating fashion : it can claim, quite apart from the beautiful mathematical proof of the existence of an equilibrium position, to make a substantial contribution to the economic theory of interest, prices and production.

THE PROOF OF THE PROPERTIES OF THE SYSTEM IN EQUILIBRIUM

It would not be profitable to comment on the large part of the paper which deals with the proof of the existence of an equilibrium, since the argument is essentially one of advanced mathematics which cannot be economically expressed in words. But those readers who prefer to think in words rather than symbols may be interested in the following comments on v. Neumann's proof of the properties of the economic system in equilibrium.

Prof. v. Neumann defines a process of production as an operation lasting for one unit of time which converts one bundle of goods into another bundle of goods. He includes the fixed capital equipment used both in the bundle of goods which is converted and again in the bundle of goods which emerges from the operation. He supposes that there are available always a very large number of possible production processes, and that constant returns prevail in the sense that any operation can be carried out at any scale without affecting the relation of the output to the input. In any given process of production, the relative quantities of goods put in are absolutely fixed, and so are the relative quantities of the goods emerging from it : thus the only way in which an entrepreneur can alter the proportions between the goods which he uses is to change from one process of production to another : similarly he must do this in order to change the proportion of goods in his output. What would normally be regarded as two different forms of the same process of production, the one involving slightly different proportions of the factors of production than the other, is thus treated by v. Neumann as being two different processes of production.

It is fairly obvious that with given prices, some production processes are likely to be more profitable than others, in the sense that they could afford to pay a higher rate of interest without making losses. One of v. Neumann's conditions for equilibrium is in fact that every process in use should make zero profits : for under perfect competition, positive profits would attract competitors to use the same process and negative profits would deter people from using the process at all. He thus obtains the following rule for equilibrium :

> *Profitability Rule.*—Only those processes will be used which, with the actual prices and rate of interest, yield zero profits after payment of interest. These processes will be the most profitable ones available.

The second part of the rule follows from the fact that if there were any processes which could earn positive profits they would not have continued out of use under competition.

The usual point of view in economic theory is that free goods play no part in the economic system : but part of v. Neumann's problem is to determine which goods will be free goods in equilibrium. It is an essential property of his equilibrium that the physical outputs of all goods, whether free or not, remain in the same proportions to each other throughout time : so do the physical inputs. Suppose that the system in equilibrium is expanding by k per cent per unit of time : then the input of each good at any moment must be exactly k per cent greater than the input for the previous unit of time. Clearly this can only continue indefinitely if the output of every good is at least k per cent greater than the input of every good, since the source of the input at

any moment is simply the output of the previous unit of time. But there may be goods for which the output exceeds the input by more than k per cent : there will then be more than enough of these goods to supply the input of the next moment of time, and if the equilibrium continues it is clear that in the case of these goods, larger and larger surplus stocks will be built up. v. Neumann concludes that the prices of these goods can only remain in equilibrium if they have become free goods in the sense that these prices are zero. Hence he obtains his rule about free goods :

> *Free goods rule.*—In an equilibrium production system, those goods whose output exceeds their input by more than the expansion rate of k per cent will be free goods. Only those goods whose output exceeds their input by the minimum, namely k per cent, will have prices (other than zero).

Prof. v. Neumann is also able to tell us quite a lot about the relative intensities with which the various profitable processes will be used. We may refer to the organisation of processes of production in any given proportions as being a system of production. We may define the expansion rate of any given system of production in terms of the relation of the output to the input of the various goods. We may, in short, define the rate of expansion of the system to be equal to the least rate of expansion of any good involved in the system. For instance, if there is a good whose output under the system exceeds its input by 2 per cent, and if there is no good whose output under the system exceeds its input by less than 2 per cent, then we say that the rate of expansion under this system is 2 per cent. Thus defined, the term rate of expansion may be applied to any production system whether it is in equilibrium or not.

Prof. v. Neumann obtains the following remarkable rule :

> *System of production rule.*—In equilibrium, the system of production actually used will have the greatest rate of expansion of all possible productive systems.

It should be noticed that this comprehensive rule does not involve prices at all : it shows that the system of production actually used has a maximum property depending only on what processes of production are in fact available.

A little reflection will confirm the validity of the rule. The reason for it is roughly this : if any system of production with a higher expansion rate were available, then it would pay all entrepreneurs to adopt this other system in place of the processes they are supposed to be using in equilibrium, and in this case the equilibrium could not continue. This point requires further explanation. In the conditions of the model, the input of any process is the same as the capital involved in the process : interest payments have to be made out of the excess of the value of output over the value of input. A little reflection will confirm that the rate of interest which any process can afford to pay per unit of time must therefore be the percentage by which the value of its output exceeds the value of its input. In equilibrium, we know that each process makes zero profits and hence that each process used can just afford to pay the actual rate of interest. It follows that in every process actually used the value of output exceeds the value of input by exactly the rate of interest. From this it follows in turn that the value of output for the system as a whole in equilibrium exceeds the value of input by a proportion equal to the rate of interest : in other words, the rate of expansion of the system is equal to the rate of interest. This equilibrium rate of interest is, as we have seen, the maximum rate of interest which the equilibrium system can afford to pay. Suppose that there were some other system with a larger rate of expansion, then it is clear that it could afford to pay a higher rate of interest : since it is open to any entrepreneur to adopt this other production system, he would be able to make a profit by so doing because he would then be able to afford more than the actual rate of interest : equilibrium would therefore be impossible if there were any other

production system with a rate of expansion greater than that of the actual production system. That is why in equilibrium the actual production system must have the greatest possible rate of expansion, as stated in the system-of-production rule.

In the course of this argument we have incidentally demonstrated another of v. Neumann's results which may be summed up in the following rule :

> *Rate of interest rule.*—In equilibrium, the rate of interest equals the rate of expansion.

We still need to obtain a rule for determining the system of prices under equilibrium. The only result about prices which we have so far considered is that which gives the price of one good in terms of the prices of other goods : this in itself is not immediately helpful if we suppose the prices of all goods to be unknown. In order to understand the rule which prices must obey in equilibrium, it is useful to consider a new concept. This concept is the rate of interest possible under a given system of prices. With given prices, any particular production process will be able to pay a rate of interest equal to the percentage excess of the value of its output with those prices over the value of its input. In particular, with these prices, there will be one or more production processes which can afford to pay a rate of interest higher than can any other production processes. This particular rate of interest will be called the rate of interest possible under the given system of prices. The rule for determining prices may be set out in the following terms* :

> *Price system rule.*—The price system in equilibrium will have a possible rate of interest smaller than or as small as that of any other price system.

The reader will notice that this rule is closely analogous to the production system rule. Its validity may be confirmed by the following argument. In equilibrium, we have seen that the production system actually used must have an expansion rate equal to the actual rate of interest : it follows that the actual production system could afford to pay at least the actual rate of interest, whatever system of prices were ruling.[1] *A fortiori*, whatever the price system might be, at least one of the production processes actually used would be able to afford at least the actual rate of interest. This implies that no price system can have a possible rate of interest less than the actual rate of interest ruling in an equilibrium position. This result is embodied in the price system rule given above.

ECONOMIC IMPLICATIONS OF THE RESULTS

Since v. Neumann's results only relate to a quasi-stationary state, the utmost caution is needed in drawing from them any conclusions about the determination of prices, production or the rate of interest in the real world. Since, in the real world, land is limited in supply, the only possible quasi-stationary state is a strictly stationary state (or conceivably a contracting state[2]) : for an expanding quasi-stationary state would eventually be confronted with a shortage of land and its equilibrium would be destroyed. Hence v. Neumann's " quasi-stationary " state does not in fact bring his model any nearer to reality than would be the case with a strictly stationary state.

In spite of this v. Neumann's results are highly suggestive ; and it is interesting to explore in what respects the operation of his model may be relevant to the real world.

* Prof. v. Neumann does not use this particular rule in his article: He does, however, use the property that given the equilibrium intensities of the processes of production, the ratio of the value of the system's output to that of its input will be a minimum with respect to prices.

[1] This follows almost immediately from the definition of the expansion rate of a production system.

[2] Allowing for the existence of exhaustible resources, *e.g.* minerals, or for a system unable to provide the subsistence wages of the workers except by using up its stocks.

(1) As a first example, we may take the property that competition will ensure that equilibrium can only be reached if the maximum technically possible rate of expansion is achieved. This may immediately suggest an argument in favour of free enterprise in the real world. But quite apart from the point already mentioned that in a world with non-augmentable resources like land the maximum rate of expansion that is ultimately possible is zero (and hence competition would merely lead to an equilibrium position with no growth or contraction and with a zero rate of interest) the claim is strictly valid only if, as in v. Neumann's model, there is a slave-system and the object of production is mere enlargement without any advance in the standard of living. v. Neumann's model certainly does not suggest that competition secures the highest possible standard of living or the greatest possible rate of advance for living standards : for, on the assumptions of his model, the living standard is simply the minimum needed to persuade people to work.

(2) This point brings us to a second interesting implication of v. Neumann's results. He has successfully constructed an economic model in which the equilibrium level for real wages is simply whatever is needed to persuade people to work : it does not apparently depend on what industry can " afford to pay ". Suppose that the working-class effectively insists on a higher real wage, then this has the effect of increasing the input needed in any process (to secure a given output) by the amount of the extra fodder which the workers demand. Hence, there will be a change in the equilibrium conditions, and the position of quasi-stationary equilibrium will change to one with a lower rate of interest and a lower rate of expansion. This might suggest an argument for vigorous trade union activity : for in the model the result of standing out for higher real wages is to secure higher living standards at the expense of the owners of property : it is true that it is also at the expense of the rate of expansion of the system, but that is because in the model it is assumed that the propertied class save the whole of their income ; in the real world, where the propertied class also consume, it may be obtained at the expense of the consumption of the propertied class. Such an argument is suggested, but it is not certain whether it could be developed by means of any simple extension of the model.

(3) The question of consumption by the propertied class is also relevant to the theory of the rate of interest. The rate of interest will be determined as the greatest rate of expansion possible if all income from property is saved. A *rigorous* proof of this proposition is only possible if we assume that all income from property is in fact saved : this could happen, for example, if all property was owned by the State. On the other hand, even if part of the income from property were spent on consumption, and not saved, the rate of interest would not necessarily be much affected : it might still be *approximately* equal to the greatest expansion rate that *would* have been possible *if* all income from property had been saved. At the same time, the spending of part of the income from property would, of course, reduce the actual rate of expansion of the system ; this would now be well below the rate of interest and the maximum possible expansion rate.[1]

(4) An interesting feature of the model is that both prices and the outputs of the individual commodities are determined solely by the technical conditions of production. As was explained above, v. Neumann has proved (*a*) that competition will allow the system to be in equilibrium only if the five rules given above on pp. 13–15 are satisfied: these five rules of competitive equilibrium determine both the intensities of production of the individual commodities and their relative prices where all production processes

[1] The equality of the rate of interest and the rate of expansion in the model is, in fact, (once the existence of an equilibrium is proved) fairly obvious on the assumption that workers spend all their income and capitalists save all theirs.

are given. The model, it is true, ignores the possibility of increasing returns in the production of individual commodities, and does not allow for consumers' choice as an independent factor in the direction of productive activity. There is no room in the theory for an increase in population to make books cheaper and for a shift in demand from cotton to wool and from mutton to beef to send wool prices up and mutton prices down. But the important point is that these may conveniently be considered as the " special cases " of price-theory, to be introduced in the *second approximation ;* and not, as is common in traditional economics, at the centre of the theory. For the basic influences determining equilibrium prices v. Neumann's model provides a novel approach ; here, perhaps for the first time, is a self-contained theory of the determination of prices, ignoring the second approximation.

The role played by consumers' tastes in the determination of prices is suggested by considering how consumers' choice may be introduced into v. Neumann's model. The method is to allow several alternative production processes for obtaining " labour ", each process requiring a different bundle of goods as " real wages ", between which the labourer may be supposed indifferent. A change in the labourers' tastes will then be reflected in a change in the input required in the various processes producing " labour " : this in turn will react on the equilibrium position of the system and hence on relative prices. But the latter effect may be trivial, even if the change in tastes is significant ; and one is left with the impression that consumers' tastes play, in fact, a comparatively minor role in the determination of equilibrium prices.

It may be objected that the assumption that the propertied class save the whole of their income further restricts the scope which " marginal utility " can play in the determination of prices. This may be granted ; but this restriction is not so serious as it may appear to be : indeed the novelty of the distribution of emphasis which it implies is, from some points of view, an advantage. For even in the actual world the great bulk of productive activity (as measured, for example, by the distribution of labour between industries) is devoted to the production of intermediary products of one sort or another, which are mainly used as inputs in a series of other products. The prices (and relative outputs) of these intermediary goods can best be explained in terms of the considerations covered in v. Neumann's model.[1]

(5) Land is assumed by v. Neumann to be available in unlimited quantities. It is, however, possible to introduce land into his model by including the land used both in the input and the output of each process using it. In this case, since the quantity of " land " cannot be increased (or decreased), equilibrium is only possible in a stationary state. In such a state, the rate of interest will be zero and the workers will get the whole income. This suggests that if the assumption that all property income is saved is abandoned, the equilibrium in a system containing land may be a stationary state with a positive rate of interest and all income consumed. During the approach to this equilibrium the rate of interest will presumably fall as the increasing scarcity of land lowers the *potential* rate of expansion, and the *actual* rate of expansion may fall even faster owing to monetary complications. These considerations take us however, outside the assumptions made by v. Neumann, and away from the possibility of rigorous proof.

(6) In a world where the scarcity of non-augmentable resources exerts a major influence on the productive system, v. Neumann's model ceases to be so interesting. But even ignoring the complications due to " land ", there is still danger of another kind of complication. The rate of expansion of the system is determined, as we have

[1] And even in the case of final-consumers' goods, the prices (though not of course the relative intensities of production) are *largely* to be explained by the technical conditions of production, rather than " marginal utility ". (The exceptions being joint products, or commodities with largely increasing or decreasing cost.)

seen, by the goods whose supply can be expanded least rapidly. These may well be those goods which are created largely out of themselves, (*i.e.* in whose production processes input and output mainly consist of the same commodities), as, for example, whales or mathematical wranglers. The point of these examples is that the commodities with the lowest rate of expansion may be trivial goods. Yet, if it is impossible for the expansion of these goods to keep pace with the rest of commodities, it is they who, on v. Neumann's model, will rule the roost and determine the rate of expansion of the whole system !

The reason for this unnatural result is that there is no room in the model for processes which do *not* involve whales and wranglers ! It is expressly assumed that every good is involved (either as input or as output) in every process. Hence it is not possible in the model to reduce below a certain proportion the part played in the economy by such goods as whales and wranglers, and eventually the expansion of the system must be slowed down to their own pace. v. Neumann states that his assumption that every good enters every process does not really matter because they may be supposed to do so in very small quantities ; nevertheless the implications of this assumption need bearing in mind.

(7) It should be noted that although in the model the equilibrium rate of interest is uniquely determined, the system of prices and outputs are not *uniquely* determined : there may be any number of possible equilibrium positions. But each must satisfy the rules set out in section 2 above.

The ease with which these rules could be established once the existence of an equilibrium position was known, was due to the choice of assumptions which enabled constant prices and stable relative outputs to exist together under competition. The whole process of mathematics would become greatly complicated if increasing returns or monopoly were introduced.

It will be noted, of course, that the " equilibrium " of v. Neumann's model is a very long run equilibrium ; it may take many decades or even centuries for the system to settle down to the rate of expansion of the least expandable goods ; and over this period, the basic assumption of known technical possibilities remaining unchanged loses all reality. An important question, therefore, is how far v. Neumann's results are applicable to systems which are only in an approach to equilibrium ; and any rigorous examination of the properties of such a system would be bound to be most complicated.

Yet it is in the problems of the approach to equilibrium that economists are most interested. How can a country acquire the equipment needed to achieve the best system of production ? What prices should be used in its accounting system by a planning authority seeking to make the best use of its resources ? Here is a fruitful field for extending the powerful methods developed in Prof. v. Neumann's paper.

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