Proportions, Prices and Planning

A Mathematical Restatement of the Labor Theory of Value

by

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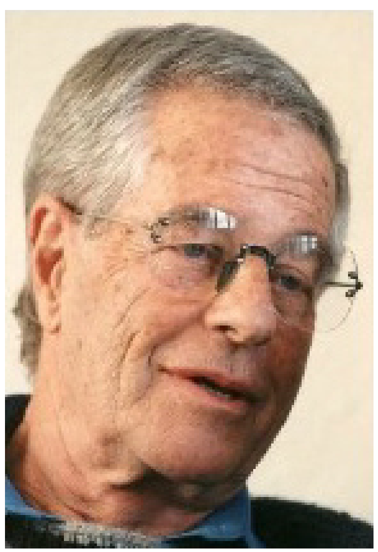

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Preface

Although it rightly claims to be the most rigorous of social sciences, economics does not progress — as a typical natural science does — in a straight line. Like a broad river slowly winding its way across a flat plain, economic thought advances in curves and loops. It turns left and right and divides from time to time into separate branches, some of which end up in stagnant pools, while others unite again into a single stream.

One of the divisions of this kind occurred between the East-European and Western theoretical thought. Mathematical economics in the U.S. and western Europe began to resemble in its playful elegance the artificial fountains of Versailles, while Marxist thought in the East became under its smooth surface rather shallow.

However, in recent years the power of the mathematical method has been rapidly gaining recognition in socialist countries; and at the same time the builders of theoretical growth models in the West become conscious of the fact that their approach has more in common with Ricardo, Marx and other classical economists than with Marshall or with Keynes.

While the driving and the steering mechanisms of centrally planned socialist and quasi-competitive free-enterprise economics are, in principle at least, entirely different, the basic structures of both systems can be described in terms of the same kind of parameters. Karl Marx, employing esoteric Hegelian terminology, distinguished universal "logical" from the transitory "historical" aspect of economic phenomena. Oscar Lange was the first among the eastern Marxist scholars to recognize that it is the first type of relationships that determines the possible growth paths of socialist and capitalist economies alike. He also was the first to introduce input-output analysis in the East.

András Bródy's book carries on from where Lange left off. He advances in this book the solution of theoretical questions discussed in current issues of western economic journals, but in doing so he shows how both the questions and the answers go back to Karl Marx and other classical economists. He makes effective use of powerful tools of formal mathematical reasoning, but also of intuitive conjecture that, after all, is the ultimate source of all analytical insight. Engaged in theoretical inquiry, he is aware — and makes the reader aware — of the peculiar problems that arise whenever we have to pass from the observed facts to mathematical formulae and from mathematical formulae back again to observable facts.

A theorist will find in this volume an original and interesting discussion of the fundamental problems of economic growth. To a general economist not familiar with input-output analysis or the modern mathematical theory of economic growth, it offers a systematic introduction to both subjects.

Cambridge, Massachusetts
August 1969

Wassily Leontief

Introduction

Since the publication of Wassily W. Leontief's first papers on Input–Output Analysis, communication among economists around the world has become easier and more fruitful. It turned out that his and a number of related methods of modern mathematical economics are not only important and useful in application but also serve to generalize a very wide set of problems. Mathematics has acted as a welcome and friendly translator of diverse verbal theorems and theories into a common language that is internationally understood.

With increasing technical penetration of the subject matter of economics we begin to realize that its deepest questions have much in common everywhere. This unity was obscured for a long time because the different economic schools used different approaches and different terminology to answer them. Until very recently these differences seemed to be irreconcilable. Yet slowly and laboriously we are becoming aware that widely differing views may be crystallized into similar mathematical models; that mathematical transformations can carry over one method of reasoning into another that at first seemed alien.

My task here is to probe a little further into these interconnections and to try to bridge the gap from one side: labor theory of value, or more precisely, Marxian economic thought. The purpose of this book is to translate Marx's original approach into mathematical terms and to indicate the path leading from it to modern quantitative economic reasoning. Once this is done it is possible to prove strict mathematical equivalence of a whole family of theories and models: the labor theory of value, game theory, open and closed static and dynamic Leontief systems, linear programming, the mathematical theory of optimal processes and other general equilibrium models. Their common basis becomes all the clearer when they are applied to everyday economic tasks: analysis, forecasting, planning and control of economic systems.

The scope of the material considered here is restricted. Theories of money and rent are not discussed, although a parallel mathematical approach to them is much needed and indeed within reach. Neither do we enter deeply into problems of technological change. Limited to questions of freely reproducible goods, the text may serve as an introduction to a mathematical labor theory of economics.

The methodology will draw heavily on the eigenvalue–eigenvector resolution of matrices. This particular mathematical representation is all the more appealing that it helps to unify various theoretical approaches. The eigenequation can represent deterministic or causal relations of the sort that the classical economists, Smith, Ricardo and Marx, set up. It can also be used in a teleological and optimizing approach such as that of the marginalist schools.

The book is divided into three parts. The first sets up the model. Full quotations of Marx's writings are required to provide correct documentation. The second one elaborates the theoretical implications of the model set up in the first part. The third part takes up questions of implementation, application and planning. The more complicated mathematical theorems and proofs are relegated to the Appendices.

I am indebted to many members of my Institute where I was free and indeed stimulated to do my research. I am also grateful to the Ford Foundation for a research fellowship at the Harvard Economic Research Project in 1964–65. I am particularly thankful to Anne P. Carter who encouraged me to translate and partly rewrite the Hungarian text, who understood what I had on my mind and helped to express it in English.

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Symbols

a_{ik}	flow coefficient
$A = \{a_{ik}\}$	flow matrix
$\mathbf{A} = \begin{bmatrix} A, & c \\ v, & 0 \end{bmatrix}$	complete flow matrix
$\lvert A \rvert = \alpha$	maximal eigenvalue of matrix A
$(1-A)^{-1} = Q$	Leontief-inverse
β	mark-up factor
b_{ik}	stock coefficient
$\mathbf{b}$	stock vector
$\mathbf{B} = \{b_{ik}\}$	stock matrix
c	consumption vector
g	resources tied up in reproducing manpower
λ	average rate of profit, rate of growth
π	rate of interest
p	value or price vector
$\mathbf{p}$	complete value or price vector
s	surplus labor
$\lvert BQ \rvert = \varrho$	maximal eigenvalue of the matrix BQ

t_{ik}	turnover time
v	labor input vector
w	wages
x	output vector
$\dot{x}$	derivative of the output vector
$\mathbf{x}$	complete output vector
y	final demand
z	variable capital

Part 1

Setting up of the Model

This first part of the book discusses a mathematical model of value and production theory. Three chapters are devoted in turn to Simple Reproduction, Extended Reproduction and Related Models.

Value theory and production theory or, to stress the continuous renewal of the processes, reproduction theory are dual reflections of society's great metabolic process by which mankind expropriates and assimilates nature's resources. They can be stated mathematically in two systems of equations, two models. But these two models will be tied together by a close interdependence and symmetry, usually called duality. This duality stems from the fact that both models or systems of equations have the same coefficients. These coefficients represent the structural interdependence of the whole economic process. Value theory and reproduction theory will be thus developed in parallel as dual interpretations of a single central structure.

The models of value and reproduction that we study are similar to a family of models now well known in theoretical and applied economic analysis throughout the world. Its intellectual roots are traced back to Leontief, Neumann, Walras, even Quesnay. It is not generally recognized that many of the central concepts originate in Karl Marx. A prime goal of this book is to point out their logical roots in Marx and show that his analysis is not only compatible with these newer forms but also provides a firm and consistent theoretical basis for their development.

In the Marxian tradition we emphasize the historical frame of reference for abstractions. Our exposition begins with the definition of Simple Reproduction appropriate to prehistoric and ancient forms of production, yielding no surplus, or almost none. This idealized model of production plays a crucial role in Marx's system of thoughts, as the following quotation shows:

"It is evident that when the laborer needed his whole day to produce his own means of subsistence . . . no surplus value was possible, and therefore no capitalist production and no wage labor. In order for the latter to exist, the productivity of society's labor must be sufficiently developed to create . . . surplus labor of some amount . . . [But] the existence of that necessary minimum productivity of labor does not in itself make it [surplus work] actual. The laborer must first be compelled to work in excess . . .

At a lower stage in the development of the social productive power of labor, when therefore the surplus labor is relatively small, the class of those who live on the labor of others is in general small in relation to the number of laborers" [T. 308].*

Thus he considers production without surplus, Simple Reproduction, a logical prototype of production before the advent of capitalism. It is a state of economic stagnation. Engels ties the "law of value" to this phase of history:

"In a word: the Marxian law of value holds generally, as far as economic laws are valid at all, for the whole period of simple commodity production, that is, up to the time when the latter suffers a modification through the appearance of the capitalist form of production. Up to that time prices gravitate towards the values fixed according to the Marxian law . . . " [III. 900].

Or quoting Marx himself:

"The exchange of commodities at their values, or approximately at their values, thus requires a much lower stage than their exchange at their prices of production, which requires a definite level of capitalist development . . . it is quite appropriate to regard the values of commodities as not only theoretically but also historically *prius* to the prices of production" [III. 177].

He also stressed the inappropriateness of Simple Reproduction under capitalism:

"Simple reproduction, reproduction on the same scale, appears as an abstraction, inasmuch as the absence of all accumulation or reproduction on an extended scale is a strange assumption in capitalist conditions . . . However, as far as accumulation does take place, simple reproduction is always a part of it, and can therefore be studied by itself . . . " [II. 399].

Later history brings capitalism and growth, more accurately characterized by Extended Reproduction and prices of production. Let us now define all these concepts in turn, stressing their historically and logically parallel evolution – a characteristic feature of Marx's explanation – from the very outset.

* The brackets refer to Karl Marx's writings as indicated in the References, p. 187

1.1. Simple Reproduction

The central task of every economy – whatever its specific institutional form – is to allocate society's labor, manpower to particular activities or areas of employment. In the course of history this task has been and will be accomplished under many different varieties of social organization. Robinson Crusoe's economy illustrates a very clear and simple form of allocation.

This Boy-Scout economy is one of the oldest thought-experiments of our science. It abstracts from the perplexing welter of institutional forms and concentrates on the theoretical problems of human production and consumption in a one-man closed economy. Robinson is technologically sophisticated: his work can create diverse products. Nevertheless this thought-experiment studies division of labor in a simple, highly idealized social environment. Robinson's "economy" is division of one person's labor, the organization of his diverse functions and capacities. But all the many diverse activities are centered around himself. Robinson is the manager, the aggregate producer and aggregate consumer of his economy.

Analyzing Robinson's deceptively simple economy Marx writes:

"Necessity itself compels him to apportion his time accurately between his different kinds of work. Whether one kind occupies a greater space in his general activity than another, depends on the difficulties, greater or less as the case may be, to be overcome in attaining the useful effect aimed at. This our friend Robinson soon learns by experience, and having rescued a watch, ledger and pen and ink from the wreck, commences, like a trueborn Briton, to keep a set of books. His stock-book contains a list of the objects of utility that belong to him, of the operations necessary for their production; and lastly, of the labor-time that definite quantities of those objects have, on an average, cost him. All the relations between Robinson and the objects that form his wealth of his own creation, are here . . . simple and clear . . . and yet those relations contain all that is essential to the determination of value" [I. 76–7].

This quotation singles out important concepts in Marx. First it states that the chief "measurable" in economic science is time. The second is the concept of value. In a theoretical sense "those relations contain all that is essential to the determination of value" because, as Marx puts it, ". . . that which determines the magnitude of the value of any article is the amount of labor socially necessary, or the labor-time socially necessary for its production" [I. 39].

In Marx, the notion of value becomes meaningful the moment there is a choice among diverse activities and diverse products. This notion, according to him, may remain latent and hidden in history for long periods. It comes to the surface only with the advent of commodity-production, that is, when products are produced explicitly as commodities for exchange or sale, satisfying other people's wants and allocated to them in exchange for and in proportion to their respective products. He believes that in absence of commodity-production there will be no value-in-exchange. Nevertheless, the underlying, deeper notion, value itself will remain with us as long as there is division of labor, as long as there are different activities to compare. As long as we have to economize society's labor, the notion of value is helpful whether there is a market (where values are expressed in prices) or not. As Marx puts it:

"Secondly, after the abolition of the capitalist mode of production, but still retaining social production, the determination of value continues to prevail in the sense that the regulation of labor-time and the distribution of social labor among the various production groups, ultimately the book-keeping encompassing all this, become more essential than ever" [III. 851].

The whole process of production and allocation can be described, analyzed and even solved in principle without open recourse to the notion of value. Let us see how Marx pictured this to himself. Continuing his analysis he speaks of economic problems of the future "community of free individuals":

"All the characteristics of Robinson's labor are here repeated, but with the difference, that they are social, instead of individual ... The total product of our community is a social product. One portion serves as fresh means of production and remains social. But another portion is consumed by the members as means of subsistence ... Labor-time ... apportionment, in accordance with a definite social plan maintains the proper proportion between the different kinds of work to be done and the various wants of the community" [I. 78–9].

This second quotation makes it clear that for Marx it is not enough to measure direct labor expended on particular products. One has to take into account the quantity of products expended on production of the final, consumable products, too. One has therefore to account for those parts which remain "social" as means of production. In principle Robinson ought to do this too, because it matters whether he uses up few or many tools and other means of production in the course of his "goat taming, fishing and hunting".

Marx makes this even more clear and explicit when setting up his tables of reproduction. He writes about the exchange going on inside the so-called "department I", responsible for producing means of production:

"Products which do not serve directly as means of production in their own sphere are transferred from their place of production to another and thus mutually replace one another ... If production were socialized instead of capitalistic, these products of department I would evidently just as regularly be redistributed as means of production to the various branches of this department, for purposes of reproduction, one portion remaining directly in that sphere of production from which it emerged as a product, another passing over to other places of production, thereby giving rise to a constant to-and-fro movement between the various places of production in this department" [II. 428–9].

And here we arrive at a crucial and tricky question. It concerns not only Marx's thoughts and the mathematical model to be built on them but also the general problem of planning and conscious management of human activity. Can those expenditures, those "to-and-fro movements", be measured? Are they a stable enough basis for anticipations? Are they reliable at all – and how can one rely on them?

1.1.1. *Input coefficients*

I believe these basic questions can only be answered in a historical perspective. The proportionate expenditures of labor and of means of production "that definite quantities of objects cost on an average" take shape very slowly in the course of history. Gradually they do evolve to more or less stable proportions. This does not mean they become rigid or unchangeable. To suppose their constancy under the conditions of rapid technical change so characteristic of our age would be flagrant nonsense.

However, one can observe average proportions for any given historical moment. This average has a certain stability and is considered "normal". The actual spread around this "norm" may well be shrinking all the time. The lower limit of expenditure is fairly strictly given by technical possibilities existing at every given date – and the upper limit is determined by considerations of efficiency. The upper limit will be the closer to the lower one the more efficiently and economically the society is organized. Thus average proportions of expenditure, average "input coefficients", will be fixed technically and institutionally at a given time and place not only as to their order of magnitude but also as to their possible "elbow room" around their "normal" magnitude.

Marx himself had a lot to say about the historical process shaping these norms and gradually making them stricter. First he stresses the historical role of division of labor in the "Manufacture" – the historical forerunner of the modern factory.

"The labor-time necessary in each partial process for attaining the desired effect, is learnt by experience, and the mechanisms of Manufacture, as a whole, is based on the assumption that a given result will be obtained in a given time ... Thus a continuity, uniformity, regularity, order, and even intensity of labor, of quite a different kind, is begotten than is to be found in an independent handicraft or even in simple cooperation" [I. 345]. And: "In Manufacture ... the turning out of a given quantum of product in a given time is a technical law of the process of production itself ... The division of labor, as carried out in Manufacture, not only simplifies and multiplies the qualitatively different parts of the social collective laborer, but also creates a fixed mathematical relation or ratio which regulates the quantitative extent of those parts – i.e. the relative number of laborers, or the relative size of the group of laborers, for each detail operation. It develops, along with the qualitative subdivision of the social labor-process, a quantitative rule and proportionality for that process" [I. 345–6].

He must add in footnote: "Nevertheless, the manufacturing system, in many branches of industry, attains this result but very imperfectly because it knows not how to control with certainty the general chemical and physical conditions of the process of production."

The situation develops further with the advent of machinery and modern industry:

"Just as in Manufacture the direct co-operation of the detail laborers establishes a numerical proportion between the special groups, so in an organized system of machinery, where one detail machine is constantly kept employed by another, a fixed relation is established between their numbers, their size, and their speed" [I. 380].

And nowadays, a century after Marx, we see this process in bolder relief. These "fixed mathematical relations or ratios of production" prevail on a far broader scale. Newer developments: large-scale production, interchange of parts, industry-wide standardization, "scientific management", assembly-line-balancing, continuous fabrication and finally automation, operations research and systems engineering have imposed greater and greater limits on the flexibility of proportions for any given process. The fabrication process of a modern enterprise once determined, it will enforce rigorous proportions among the expenditures for different sorts of manpower, raw and auxiliary materials, machine speeds and temperatures. It even requires the exact measurement and control of chance deviations from mean values.

Nowadays the sociologist mourns already over the uniformity and standardization of the most individual social product: human life. And sometimes even this mourning and complaint seem to be prefabricated.

How much expenditure is necessary "on an average" under given circumstances to produce "definite quantities of those objects" needed by Robinson or the community is only one question. How much expenditure would be necessary to produce more or less of those objects, and how much expenditure will be necessary to produce the same amount (or more or less) tomorrow or ten years hence, are separate questions. We do not have to answer these additional questions here, since we are concerned with the model of Simple Reproduction. Simple Reproduction in its strict and rigorous sense precludes the possibility of technical change. It excludes *per definitionem* change of proportions, alterations in the scale of production. Thus it requires no simplification beyond what is already implicit in the notion of simple, that is, not expanding reproduction. This abstraction of Simple Reproduction is analogous to concepts in other sciences as, say, "frictionless free fall" or "ideal gas". In reality these do not exist but they help to understand the interdependencies and regularities of real gases, or real gravitational phenomena.

Even in the case of Extended Reproduction we circumvent the question of changing input coefficients and we set up and solve the model neglecting technical change. Thus in both the first and the second part of this book fixed proportions are assumed. They are real under given circumstances and at the given moment, and may be measured with more or less accuracy. Thus they can be treated with known tools of economic statistics and expressed with the necessary precision for present purposes.

By assuming approximate measurability we do not assume that input coefficients are stable. By assuming the existence and measurability of the speed of a car we do not deny acceleration. We are concerned only in the interdependence

and regularities among the coefficients themselves, emerging in the context of a given moment. The problem of change will not be considered until the third part where problems of historical description and experiments in explaining and planning this very change are our subject.

In Parts 1 and 2 we are not yet considering whole processes of economic growth, but only given states of a system. Thus the usual objections against linear models of production – that they assume constant returns to scale etc. – are not really relevant.

1.1.2. *Output proportions*

The notion of value, the emergence of exchange and value in exchange, is a recent phenomenon in the history of mankind. Let us begin by analyzing proportions of production, volumes or scales of outputs.

First, we consider the "quantitative rules and proportionalities" of a very simple fictitious economy. In our example we simplify even Robinson's economy and imagine that he produces only two products. To name them somehow we call them "Tools" and "Materials".

In choosing these names we do not mean to distinguish between means of production and consumption, or between consumers' and producers' goods, or between the Marxian "department I" and "department II". Our distinction is only superficial. Later we shall extend it to the general case of n products, that is, to deal with an optional but finite number of different products. Under modern production conditions we are not generally able to distinguish *ex ante* between producers' and consumers' goods. Distinction is made *ex post*: an article of consumption is that which is already consumed. A great variety of important new developments and products (electricity, electronic devices, oil and its byproducts, chemicals and synthetics, cars and motors, etc.) can either enter personal consumption or lend themselves to productive (that is, *re*productive) use as intermediate goods. Their quality, form and appearance do not determine their economic role.

Let us now assume we measure "Tools" by number and "Materials" in kilograms. The free disposition over units of measurement is more apparent than real, because "To discover the various uses of things is the work of history. So also is the establishment of socially recognized standards of measure for the quantities of these useful objects" [I. 35–6].

Robinson now, as thoughtful accountant and diligent economic statistician, observes that the expenditures necessary on the average to produce

1 Tool		*1 kilogram of Material*
	are the following:	
0.2 Tools		0.7 Tools
0.2 kilograms of Material		0.2 kilograms of Material
1 hour work		1 hour work.

If he knows his consumption needs from experience he now is ready to allocate yearly labor power (the manpower of his society) among competing activities. This allocation problem might be solved without recourse to any notion of value, yet value is already implicit in the measurement of the necessary labor-time.

Let us assume further that for keeping body and soul together Robinson needs 100 Tools and 600 kilograms of Material for himself each year. The question now is: How much ought he to produce to satisfy his needs and to reproduce all the means of production used up in the yearly production of his needs? We state his problem in the language of matrix-calculus.

Let the n by n matrix A designate the input coefficients. Each element a_{ik} stands for the amount of product i used to produce one unit of product k.

In our example $A = \begin{bmatrix} 0.2 & 0.7 \\ 0.2 & 0.2 \end{bmatrix}$.

Let $y = (y_1, \ldots, y_n)$ be the vector of Robinson's needs, i.e., the personal consumption necessary to reproduce the manpower of society. In our example $y = (100, 600)$. Finally let $x = (x_1, \ldots, x_n)$ stand for gross outputs, volumes of production in the different branches of economic activity. We will speak about this vector as the output or gross output vector whenever its absolute magnitude concerns us and, interchangeably, as output proportions when we are interested only in the proportions of economic activities.

Now Robinson's problem will be solved if he determines the output that, after covering the inputs necessary to this output (in Marxist terminology "replacement fund", in Keynesian, "user cost"), yields the necessary final bill of goods, the necessities of life:

$$x - Ax = y. \tag{1}$$

Given y, this equation may be solved for x if the matrix $(I - A)$ is regular. Here I stands for the n by n unit matrix. We provisionally assume (and later prove) this regularity and thus the existence of the inverse $Q = (I - A)^{-1}$. In this way we are ready to solve equation (1).

$$x = Qy. \tag{2}$$

In our numerical example $Q = \begin{bmatrix} 1.6 & 1.4 \\ 0.4 & 1.6 \end{bmatrix}$, and

therefore $x = Qy = \begin{bmatrix} 1.6 & 1.4 \\ 0.4 & 1.6 \end{bmatrix} \begin{bmatrix} 100 \\ 600 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1000 \end{bmatrix}$.

Robinson has to produce one thousand Tools and one thousand kilograms of Material and expend on this production $1000 + 1000 = 2000$ manhours each year. Thus he must allocate his labor-time in equal proportions between its two functions, tool- and material-making.

By this ingenious method – simple equation solving – Robinson, and every closed community, can allocate the different kinds of labor available, provided that they can make their wants or needs explicit. Although the notion of value remains implicit, the knowledge of the input coefficients reflecting the structure of production is sufficient to determine "in accordance with a definite social plan" "the proper proportions between the different kinds of work". Depending on its data processing and computing facilities – clay tablet, rune, quipu, abacus or electronic equipment – society may solve allocation problems of increasing complexity.

Accounting in kind – with "use values" as classical economists called them – is characteristic of division of labor in primitive communities, inside tribes or families, in ancient Greek, Mexican or Asian societies, even in feudal economies before money entered to blur the original setup. All these examples have social organizations and technologies best characterized by Simple Reproduction. Production yields no significant surplus. When anything does remain after providing for the everyday needs of society it is not accumulated and invested in production for economic growth. Simple Reproduction will usually entail a certain traditional rigidity of wants and needs. This makes "planning" and "anticipating" relatively easy, as it was for the biblical Joseph in the years of the seven fat and seven lean cows.

Allocation of labor requires some organizational skill, which seems to develop simultaneously with mathematical knowledge and the art of writing and accounting. I suspect the prehistoric forms of mathematics – counting, measuring, cardinal numbers, the four operations of arithmetic – came into being and developed as natural notions of "mathematical economics" for primitive and rough economic formations. See, e.g., Chadwick [1958], particularly chapter 7, where economic data reminding of fixed proportions emerge for Mycenaean Greece.

Our rough and ready allocation model can do more than simply allocate. It can also establish rigorous conditions for the feasibility of Simple Reproduction in terms of the input coefficients. With our model's help we can define the criterion of Simple Reproduction in exact mathematical terms. We can state the quantitative relations among the input coefficients necessary for a qualitative condition, Simple Reproduction.

Robinson's economy will be in a state of Simple Reproduction if and only if his net product, those 100 Tools and 600 kilograms of Material, suffice to restore his labor power for a year, keeping him healthy and sane enough to carry on with his usual work. Thus performing the same functions each year he is rendered able to continue the same tedious and boring process the next year.

If, at the given consumption level, he could only work less than the necessary 2000 hours, his economy would deteriorate. In that case only Diminishing, Restricted Reproduction could be carried on and he would eventually starve. But if this consumption gives him strength enough to toil more than 2000 hours, then he can accumulate some surplus and may even enlarge his economy. Extended, Expanded Reproduction is now possible and economic growth may take place.

We may then define Simple Reproduction as the condition where the final bill of goods, the net product, is just sufficient to reproduce the primary factor, labor power, on a constant scale.

Simple Reproduction is not just unaltered reproduction of productive activity on a constant scale, nor simply conservation of the same rates of output. It also implies the unaltered maintenance of the prime mover of production, manpower, in the same, never-changing routine.

"Labor created the human being itself", said Engels. Human labor and production remain the means of creating and maintaining humanity. We should thus specify a third product with the two products of Robinson's economy already enumerated. The most important product, purpose and reason of production, its prime mover and ultimate beneficiary, is Robinson himself.

This peculiar and perishable product requires certain inputs for its production. We assumed maintenance of his 2000 hours per year of labor power required 100 Tools and 600 kilograms of Materials. Now he may pass the remaining 6000 hours of the year relaxing, digesting and performing other cultural activities. On the average the expenditure needed to maintain him will be 0.05 Tools and 0.3 kilograms of Material per manhour. These inputs are necessary costs of this particular product, as are the respective inputs for the other products.

Now Robinson, when in danger of his life, might temporarily subsist on less. He may work even when hungry and cold – for some time. Thus his usual input structure might be temporarily distorted. But even a small change in accustomed proportions has been known to cause great political waves in modern societies where the consumer is more delicate and susceptible. This causes a certain stability in input coefficients that may well exceed the stability of industrial input coefficients – making change in the structure of consumption slower and smoother. We will return to this question later. Meanwhile we suppose that Robinson takes his own input data from his stockbook as he does for other products.

We also assume that he can and will exert his labor power in full and without obstacles. Unemployment seems to be a gift of Extended Reproduction and there is no need to raise the question here.

Let us denote these input coefficients by the vector $c = (c_1, \ldots, c_n)$. This vector expresses personal consumption per manhour expended, and is in Robinson's case $1/2000\ y = (0.05,\ 0.3)$. Let us also specify direct manhour coefficients into production as the vector $v = (v_1, \ldots, v_n)$. In our example $v = (1, 1)$.

With these symbols we are ready to spell out conditions of Simple Reproduction as a mathematical equation:

$$vQc = 1. \qquad (3)$$

Under Simple Reproduction the consumption expenditure necessary to maintain 1 hour of labor power (c), needs a gross output (Qc), which can be produced in exactly one hour (vQc).

If $vQc < 1$, Expanded Reproduction is possible because reproduction of one hour of labor power costs less than one hour. Part of the product can be removed from the great carousel of reproduction in each round without jeopardizing Simple Reproduction. It can be invested anew in production to make it grow. But the potential for Extended Reproduction might alternatively be dissipated in consumption by others or massed into monuments, pyramids or cathedrals.

If $vQc > 1$, then the scale of production can by no means be maintained. The net product is inadequate to reproduce labor power unimpaired. The system needs outside help. Without it, it will deteriorate to Restricted Reproduction.

Mathematically we may define Simple Reproduction even more concisely. This definition will introduce the central mathematical tools used in this book: eigenvectors and eigenvalues of matrices. We begin with a brief characterization of these concepts.

The (right hand) eigenvectors of the matrix A are those vectors, x, which satisfy equations of the form $Ax = \alpha x$, where α is a scalar. The respective scalar quantities are called eigenvalues. We can transcribe the definitional equation to $(\alpha 1 - A)\,x = 0$. This shows that the eigenvalues are those values, α, that make the determinant of the matrix $(\alpha 1 - A)$ singular. But expanding the determinant we get an equation of degree n in α. This equation then will have n, not necessarily distinct, roots. (Here n is the order of the matrix A.) We can compute the respective eigenvectors with the aid of the different singular matrices.

Here we are interested only in the maximal eigenvalue which, in our case, is positive and has a totally positive eigenvector associated with it. The most important theorems about all this are relegated to Appendix I. For present purposes we need only to know that such a maximal eigenvalue always exists for non-negative and irreducible matrices and that it can be determined unequivocally.

We single out the special case where the maximal eigenvalue equals one and thus the eigenequation is $Ax = x$. The vector, x, remains unaltered after multiplication with A. The vector, x, is then called the fixed-point of the transformation A. It is a right-hand eigenvector and we will later define the lefthand eigenvector $pA = p$ analogously.

Our matrix A contained only the input coefficients for intermediate products. To describe the total, closed system we needed the information supplied by the vectors v and c, representing inputs of and consumption needs of manpower. It is straightforward to complement the matrix A by these vectors, adding the last sector, manpower, to the picture. Our new "complete" or "full" matrix will contain all the input coefficients, thus subsuming all the information characteristic of our production system. Let us designate this complete matrix

$$\mathbf{A} = \begin{bmatrix} A, & c \\ v, & o \end{bmatrix}. \qquad (4)$$

The inner proportions given by the coefficients, a_{ik}, determine whether there is Simple Reproduction in this closed system, Simple Reproduction is thus an intrinsic feature of the matrix $\mathbf{A}$. In effect the following theorem may be stated:

The condition of Simple Reproduction is that the maximal eigenvalue of the complete matrix, $\mathbf{A}$, be equal to one, $|\,\mathbf{A}\,| = 1$. If $|\,\mathbf{A}\,| < 1$ it is possible to extend the production process. If $|\mathbf{A}|>1$ reproduction ceases to be complete, only Restricted Reproduction is possible.

This theorem is fundamental to the mathematical treatment that follows. It can be proved in the following way:

From theorems of Perron and Frobenius (see Appendix I) we know that a non-negative irreducible matrix has only one positive eigenvector and this must belong to the maximal and positive eigenvalue. Thus if we find a positive eigenvector to our matrix, the eigenvalue belonging to it must be the maximal one. We can prove now that the vector, given by the prescription $\mathbf{x} = (Qc, 1)$, is such a right-hand eigenvector. It is a positive vector and from equations (4) and (3) and from the identity $AQ \equiv Q - 1$ it follows that*

$$\mathbf{Ax} = \begin{bmatrix} A, & c \\ v, & 0 \end{bmatrix} \begin{bmatrix} Qc \\ 1 \end{bmatrix} = \begin{bmatrix} AQc + c \\ vQc \end{bmatrix} = \begin{bmatrix} (Q-1)c + c \\ 1 \end{bmatrix} = \begin{bmatrix} Qc \\ 1 \end{bmatrix} = \mathbf{x}\,.$$

The vector $\mathbf{x}$, given by the prescription above, must be a right-hand eigenvector of the matrix $\mathbf{A}$, its elements being unchanged by multiplication with the matrix. At the same time it was shown that the maximal eigenvalue is equal to one.

Numerical example

In our case $\mathbf{A} = \begin{bmatrix} 0.2 & 0.7 & 0.05 \\ 0.2 & 0.2 & 0.3 \\ 1 & 1 & 1 \end{bmatrix}$ $\qquad Qc = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$

thus

$$\begin{bmatrix} 0.2 & 0.7 & 0.05 \\ 0.2 & 0.2 & 0.3 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \\ 1 \end{bmatrix}$$

The fundamental theorem distinguishing quality of reproduction can also be formulated in the following way:

Given the non-negative and irreducible matrix, $\mathbf{A}$, comprising the input coefficients of a closed and complete system of production

(a) if there is a positive output vector, $\mathbf{x}$, for which $\mathbf{Ax} = \mathbf{x}$, then Simple Reproduction is possible in this system;

(b) if there is a positive output vector, $\mathbf{x}$, for which $\mathbf{Ax} < \mathbf{x}$, then Extended Reproduction is possible. In this case, the surplus product is non-negative, $\mathbf{x}-\mathbf{Ax} > \mathbf{0}$, and may be used to increase production, or it may be withdrawn from the system without jeopardizing Simple Reproduction;

* $(1-A)^{-1} \equiv Q$, thus $1 \equiv (1-A)\,Q \equiv Q - AQ$.
This identity is a special case of the resolvent equation discussed in Appendix II.

(c) if finally neither (a) nor (b) is fulfilled, then only Restricted Reproduction is possible. Simple Reproduction would be possible only if the "negative surplus", $\mathbf{x}-\mathbf{Ax} < \mathbf{0}$, were supplemented from outside sources.

This already proves that equation (1) has a solution in our two cases of Simple or Extended Reproduction. In these two cases the maximal eigenvalue of $\mathbf{A}$ is equal to or less than one. The matrix A being a minor of the matrix $\mathbf{A}$ (with one row and one column, manpower, deleted) will have a maximal eigenvalue strictly less than one. Now the eigenvalues of the matrix $(1-A)$ will be the identical rational functions of the matrix A (see Bodewig, 1962). If therefore the matrix A has the eigenvalues $\alpha_1, \ldots, \alpha_n$, the matrix $(1-A)$ will have the eigenvalues $1-\alpha_1, \ldots, 1-\alpha_n$. As $|\alpha_i| < 1$, $1-\alpha_i \neq 0$ for all i. Thus no eigenvalue of $(1-A)$ can be equal to zero; the determinant of the matrix $(1-A)$, being the product of these eigenvalues, cannot be equal to zero; hence the matrix $(1-A)$ must be regular and have an inverse.

But the important fact that the matrix $\mathbf{A}$ is irreducible, besides being *per definitionem* non-negative, still awaits proof. If, now, the matrix were reducible then there would be a part – certain branches – of the economy that forms a closed and complete system in itself and does not need inputs from other branches of the economy. But the existence of such a closed and complete subsystem is impossible. This closed subsystem would necessarily contain the manpower sector since every branch requires labor input. On the other hand the manpower sector cannot be separated from the other sectors because, directly or indirectly, it needs the outputs of all the productive branches of the economy.* Since the economy cannot be separated into two independent parts, one of them not needing inputs from the other, it forms an irreducible system.

The irreducibility of matrix $\mathbf{A}$ is a consequence of the fact observed by the classical economists: the purpose of production is to satisfy human wants directly or indirectly: "directly as means of subsistence, or indirectly as means of production" [I. 35].

On the other side, the fact that every product needs direct or indirect input connects these branches into an interdependent whole. This fundamental observation leading to the labor theories of value is expressed very clearly by Marx:

"This common, 'something' cannot be either a geometrical, a chemical, or any other natural property of commodities ... the exchange of commodities is evidently an act characterized by a total abstraction from use-value. Then one use-value is just as good as another, provided only it be present in sufficient quantity ... If then we leave out of consideration the use-value of commodities, they have only one common property left, that of being products of labor" [I. 37–8].

Finally, note that in determining the gross output, $\mathbf{x}$, as an eigenvector, we determined only the proportions of outputs. Every scalar multiple of $\mathbf{x}$, say $\varrho\mathbf{x}$, will be an eigenvector. Thus, for example, the vector (1, 1, 2) or (24, 24, 48) will

* Here we disregard certain blessings of Extended Reproduction, such as military and government expenditures that might be separable from the rest of the system.

solve the same eigenequation equally well as the former vector (0.5, 0.5, 1). This can be verified by multiplication by the matrix **A**.

It is only output proportions and not the absolute scale of production that are determined by the matrix **A** of the closed system. The system has one degree of freedom – one product's gross output can be chosen arbitrarily. Once this is done, for instance if the amount of labor hours at our disposal is given, all the other gross outputs become definite.

Thus Simple Reproduction is possible if total (personal and productive) consumption, **Ax**, is equal to total production, **x**. Total consumption now means all demand whether personal or not. Thus total production is severely double-counted. It counts not only the total turnover, (all the supply coming to the market), but the supply of reproduced manpower, too. This notion is to be distinguished from both the Western "total product" and the Eastern "social without its product". It is doubly double-counted.

1.1.3. *Values*

Why do we need the notion of value, (or its more specific variants, value-in-exchange and price) if the production process can be balanced in principle without its help?

As division of labor progresses the diverse functions of labor become separated from each other and attain an apparent independence. The original clarity of the production process becomes blurred. Some means of making partial processes comparable without knowledge of the whole process, without requiring the whole jig-saw puzzle be put together every time, becomes necessary. This means is money, and the common something it represents is value. Money, once introduced, will enhance and accelerate the progressive division of labor, the diversification and apparent self-sufficiency of the different forms of labor.

As division of labor and commodity production (that is, production for an impersonal market) makes direct and conscious regulation of production more difficult, the labor of society will be dominated increasingly by prices, that is by exchange-value.

The labor process that creates use-values is at the same time the process that creates exchange-values. Therefore the notion of value should be developed from the interdependencies already discussed. Let us examine the production process with this view in mind. We resume the line of thought taken by Marx:

"The various factors of the labor process play different parts in forming the value of the product.

The laborer adds fresh value to the subject of his labor by expending upon it a given amount of additional labor, no matter what the specific character and utility of that labor may be. On the other hand, the values of the means of production used up in the process are preserved, and present themselves afresh as constituent parts of the value of the product ... The value of the means of production is therefore preserved, by being transferred to the product" [I. 199].

And to determine more exactly the value transferred by the means of production:

"If the time socially necessary for the production of any commodity alters ... all previously existing commodities of the same class are affected, because they are, as it were, only individuals of the species, and their value at any given time is measured by the labor socially necessary, i.e., by the labor necessary for their production under the then existing social conditions" [I. 210].

Thus the value transferred by the means of production should be reckoned in terms of the present value of the expenditure without taking into consideration the fact that at the times of their actual production they might have cost more or less. It is not the expenses of actual production but costs of potential reproduction, replacement costs, that settle the accounts.

If we supplement our symbols with the vector $p = (p_1, \ldots, p_n)$ standing for the values of the respective products, then we may determine their magnitudes in the following way:

$$p = v + pA\,. \tag{5}$$

Value = new value added by the laborer + value of the means of production used up in the process.

Our input coefficients play a new role in this equation. Formerly the coefficient a_{ik} measured the amount of product i, necessary to produce one unit of product k. The total amount of product i used up in the production process k was given by $a_{ik}x_k$. Now we deduce from the same coefficient how much of the value of product k can be ascribed to the product i, what is the original value, $p_i a_{ik}$, preserved in the process.

If equation (1) shows the flow of use-values in the metabolic process of production, then the new equation (5) presents the flow of exchange-values in the process of value creation. Equation (5) depicts the flow of "money" paid for the products used up in the processes. These money-flows go in the opposite direction to the product flows, and represent a dual view of the process. While the magnitudes x_k, measuring outputs of use-values, were not comparable among each other (because their units of measurement usually differ), the magnitudes p_i, measuring values of the products, all have the same unit for measurement. Hence, they are directly comparable.

Equation (5) can be solved in the same way and under the same conditions as equation (1). Its solution, using the inverse $(1-A)^{-1} = Q$, can be written

$$p = vQ. \tag{6}$$

Numerical example

$$p = vQ = (1, 1)\begin{bmatrix} 1.6 & 1.4 \\ 0.4 & 1.6 \end{bmatrix} = (2, 3).$$

In Robinson's economy 2 kilograms of Material are worth 3 Tools. The value of the products exists and is computable in Robinson's economy even though there is no value in exchange, because there is no exchange. This fact is independent of whether or not Robinson luxuriates in computations.

From the definition of value, given above, and from the mathematical equation (5), it follows that value is nothing but the total amount for labor, present and past, direct and indirect, used up in the production of the product.

It is well known that the inverse of the matrix $(1-A)$ can be written as an infinite series. If we use this form for equation (6):

$$p = v + vA + vA^2 + \ldots + vA^n + \ldots$$

it becomes clear that value is direct labor expended on the products, v, plus labor expended on means of production used up in this process, vA, plus labor expended on means of production used up in the process of producing these latter means of production, vA^2, and so forth to infinity – collecting all the labor expended at all the past stages of production leading to the present output. The series is infinite but convergent and its sum is finite because from $|A| = \alpha < 1$ it follows that

$$\left| \sum_{n=0}^{\infty} A^n \right| = \frac{1}{1-\alpha}.$$

This was, then, the original definition of values as given by Marx. Let us deduce an equivalent definition based again on eigenvectors and eigenvalues of the respective matrices. We proceed analogously to the dual exposition given for outputs.

The essence of the former procedure was to augment the matrix A to arrive at the complete matrix **A**. Is this legitimate? Can manpower be handled as any other product even from this dual viewpoint, with value creation in mind? Certainly this conception is not alien to the classical economists and Marx:

"The value of labor power is determined, as in the case of every other commodity, by the labor time necessary for the production, and consequently also the reproduction, of this special article. So far it has value, it represents no more than a definite quantity of the average labor of society incorporated in it . . . The labor time requisite for the production of labor power reduces itself to that necessary for the production of . . . means of subsistence; in other words, the value of labor power is the value of the means of subsistence necessary for the maintenance of the laborer" [I. 170–1].

In the general case the value of manpower is certainly less than the new value it is apt to create. At advanced stages of technological development labor can produce more than it needs to consume. But our special assumptions concerning Simple Reproduction entail equality of the value of labor and the value created by it. Therefore we may compute the value of labor in the same way as the value of any other product. Symbolically the yearly value of manpower (the value of manpower exerted during a year) is py, the value of the means of subsistence consumed per year. The hourly value will be pc, the consumption necessary to maintain labor power to be exerted in 1 hour.

Numerical example

Considering the input coefficients of Simple Reproduction in Robinson's economy

$py = (2,\ 3)\begin{bmatrix} 100 \\ 600 \end{bmatrix} = 2000$. This is the yearly value of Robinson's manpower.

Its hourly value is $pc = (2,\ 3)\begin{bmatrix} 0.05 \\ 0.3 \end{bmatrix} = 1.$

The example now illustrates the conditions of Simple Reproduction already pointed out, but in a new form. Formerly equation (3) $vQc = 1$, proved to be a necessary condition for Simple Reproduction. Now, taking into account equation (6), the hourly value of manpower can be computed, by the same formula, as $pc = vQc$. Thus the criterion of Simple Reproduction from the viewpoint of value creation is equally

$$vQc = 1. \tag{7}$$

The interpretation of this latter equation, constructed from the dual, value, side, is slightly different from that of the former. Formerly we interpreted the form as $v(Qc)$; now it is interpreted as $(vQ)c$. Surely the difference is only in the order of performing the mathematical operations, and the value of such a so-called bilinear form is insensitive to the order of the operations. In a strict economic sense, however, there is a difference – because we are computing the magnitudes of different economic variables in the two cases.

Formerly we computed total production (Qc) necessary for supplying consumption with adequate net product. This was a vector of use-values and its elements were measured in different units. Now we compute (vQ), value of products, and all elements are measured in value units.

Formerly we defined Simple Reproduction as a state where the total labor input into production necessary to maintain the labor power is equal to the labor power maintained. Now we define it as the case where the value of the means of subsistence is equal to the value the labor power built up on those means. If labor creates more value than its means of subsistence are worth, then Extended Reproduction is possible. If it creates less, the process deteriorates and we have Diminishing Reproduction.

We continue the parallel, dual development of the output analysis, and show that under conditions of Simple Reproduction the vector $\mathbf{p} = (p,\ 1)$ is the left-hand eigenvector of the complete matrix, **A**. The last element, 1, is now the value of the "last product", manpower. In accordance with our earlier proof, considering equations (5), (6) and (7):

$$\mathbf{pA} = (p,\ 1)\begin{bmatrix} A, & c \\ v, & 0 \end{bmatrix} = (pA + v,\ pc) = (p,\ 1) = \mathbf{p}.$$

Numerical example

Again with data of Robinson's economy:

$$(2,\ 3,\ 1)\begin{bmatrix} 0.2 & 0.7 & 0.05 \\ 0.2 & 0.2 & 0.3 \\ 1 & 1 & 0 \end{bmatrix} = (2,\ 3,\ 1).$$

A Tool is worth 2, a kilogram of Material, 3, hours of labor. The value vector, under conditions of Simple Reproduction, is the left-hand eigenvector belonging to the maximal eigenvalue of the complete input coefficient matrix, **A**. The maximal eigenvalue equals 1. This eigenvector always exists and is unique, the matrix **A** having no other positive left-hand eigenvector. The existence and uniqueness of the solution are guaranteed in both cases by the non-negativity and irreducibility of matrix **A**.

It is easy to transform our fundamental theorem to its dual form:

Given the non-negative and irreducible matrix, **A**, comprising input coefficients of a closed and complete system of production

(a) if there is a positive value vector, **p**, for which $\mathbf{pA} = \mathbf{p}$, then Simple Reproduction is possible in this system;

(b) if there is a positive value vector, **p**, for which $\mathbf{pA} < \mathbf{p}$, then Extended Reproduction is possible. In this case, the surplus value is non-negative, $\mathbf{p} - \mathbf{pA} > 0$, and may be used to increase production; or it may be withdrawn from the system without jeopardizing Simple Reproduction;

(c) if neither (a) nor (b) is fulfilled, then only Restricted Reproduction is possible. Simple Reproduction would be possible only if the "negative surplus", $\mathbf{p} - \mathbf{pA} < 0$, were supplemented from outside sources.

Finally, note that in determining the value vector, **p**, as an eigenvector, we determined only the proportions of values. Every scalar multiple, of **p**, say $\varrho\mathbf{p}$, will be an eigenvector. Thus, for example, the vector (2/3, 1, 1/3) or (10, 15, 5) will solve the same eigenequation equally well, as the former vector (2, 3, 1). This can be verified by multiplication by the matrix **A**.

It is only value proportions and not their absolute magnitude that are determined by the matrix **A** of the closed system. The system has one degree of freedom – one product's value can be chosen arbitrarily. Once this is done, for instance the gold or silver content of money is given, all other values will become definite.

Economists traditionally circumvent this difficulty by fixing the value of labor power, choosing it as the "numeraire". If labor power is the n-th product, we reckon with the values $p_1/p_n, p_2/p_n, \ldots, p_n/p_n = 1$. This is really the reason we set the last element of our value vector equal to 1.

Thus Simple Reproduction is possible if the value of the product, **p**, equals the value of its constituent parts, **pA**.

1.1.4. *Surplus*

If the increased productivity of labor allows the creation of surplus, then the new value added by the laborer – according to Ricardo and Marx – will be greater than the value of his labor power. Although this is a necessary condition for Extended Reproduction, it is by no means sufficient. The surplus may be wasted or consumed unproductively, without any feed-back to production. In practice it is difficult to draw the line between productive and unproductive consumption. In theory we will take the distinction as given.

Let us see whether the possibility of surplus alters anything said thus far. First assume that there is unproductive consumption, for instance the consumption of the feudal landlords, constraining a production system, otherwise ready for growth, to carry on Simple Reproduction. In this case we view our complete system as having two different consumption vectors. One is the consumption of the laborer, c_e, the other the unproductive consumption of the landlords, c_i. These two consumptions taken together, $c_e + c_i$, absorb the total net product.

In this case $pc_e < 1$. The value of labor power must be less than the new value added by it; otherwise there could be no surplus. Hence we may separate the hours of work into two parts. The first part is necessary to reproduce the value of manpower. The second part is devoted to producing surplus. As Marx writes:

"That portion of the working day, then, during which this reproduction takes place I call '*necessary*' labor time, and the labor expended during that time I call '*necessary*' labor. Necessary, as regards the laborer, because independent of the particular social form of his labor... During the second period of the labor process, that in which his labor is no longer necessary labor, the workman, it is true, labors, expends labor power; but his labor, being no longer necessary labor, he creates no value for himself. He creates surplus value ... This portion of the working day I name surplus labor time, and to the labor expended during that time, I give the name of surplus labor... The essential difference between the various economic forms of society, between, for instance, a society based on slave labor, and one based on wage labor, lies only in the mode in which this surplus labor is in each case extracted from the actual producer, the laborer" [I. 216–7].

Let us separate labor time into necessary labor, w, and surplus labor, s. Again $w + s = v$, the two parts amounting to the total hours of work, the new value added to the product. Under conditions of capitalism w takes the form of wages and s is the well-known surplus value of Marx.

Since we partitioned only the vectors v and c, the form of the matrix **A** is not changed. What changes is only the interpretation to be given to our theorems.

First, it is clear that $|\mathbf{A}| = 1$ is the necessary criterion for Simple Reproduction whether there is unproductive consumption or not. In the present example Simple Reproduction reproduces unproductive consumption, too.

The right-hand eigenvector, **x**, in this case gives outputs, that may include great luxury. The left-hand eigenvector, **p**, still gives the value proportions. These value proportions will not be affected at all by the allocation of net product between productive and unproductive consumption. The value of the products is insensitive to whether the laborer is paid in full for the value created by him or not.

The last element of the value vector, 1, which formerly designated the value of manpower as equivalent to the value created by it, will now represent only the value created, the value of manpower being less than 1. If we want to determine the exact value of manpower, we have to disaggregate the manpower sector explicitly:

$$A' = \begin{bmatrix} A, & c_e, & c_i \\ w, & 0, & 0 \\ s, & 0, & 0 \end{bmatrix}.$$

Later, in connection with error and sensitivity analysis, we shall prove that this disaggregation does not change any characteristics of the matrix, at least not those essential for us. Its maximal eigenvalue remains the same and the eigenvectors belonging to it will be the similarly disaggregated eigenvectors of the matrix **A**.

Numerical example

Let us change Robinson's economy to make surplus possible. We suppose a twofold increase of labor productivity and unproductive consumption eating up half of Robinson's net produce. Our matrix **A'** becomes:

	Tools	Materials	Robinson's Consumption	Unproductive Consumption
Tools	0.2	0.7	0.05	0.05
Materials	0.2	0.2	0.3	0.3
Necessary labor	0.5	0.5	0	0
Surplus labor	0.5	0.5	0	0

We may easily verify that the left-hand eigenvector (2, 3, 0.5, 0.5) is unchanged by multiplication by this matrix and still yields the eigenvalue 1.

Our numeraire here is still the new value added by one hour of work. The last two elements of the eigenvector stand for the value of manhour and the amount of surplus produced per hour. Their sum is the unit of new value added to the product. Thus these two magnitudes express the rates of "paid" and of "unpaid" labor.

If we return to wages as the numeraire, our vector will now be (4, 6, 1, 1). In this case the last element is Marx's "rate of exploitation", the quotient of necessary labor/surplus labor [I. 218].

Thus far, all of our concepts, starting points, benchmarks and interdependencies are borrowed from Marx's writings. One might still ask: Is this model, constructed from his elements and concepts, still to be viewed as his? How much of this model did he himself perceive?

Much of the mathematical symbolism used here is new; the application of matrix calculus to macroeconomic problems stems mainly from Neumann and Leontief. But the underlying quantitative interdependencies, the dual viewpoint, the array-like conception of the process was well within Marx's grasp.

The following table from one of his lesser-known manuscripts is reproduced here in its original form and language to substantiate this claim:

	für Arbeit	Rohstoff	Maschinerie	Surplus-produce	
A) Rohstoff-Fabrikanten	20	40	20	20	= 100
B) Ditto	20	40	20	20	= 100
C) Maschinist	20	40	20	20	= 100
D) Arbeiter-necessaries	20	40	20	20	= 100
E) Surplusproduzent	20	40	20	20	= 100

This may well be the first (fictive) input-output tabulation in economic science. It is interesting to note that he starts from input coefficients in constructing his table:

"According to the supposed proportions – 2/5 raw material, 1/5 machines, 1/5 necessities of labor, and 1/5 surplus product from which Mr. Capitalist lives and realizes the surplus – we need (letting total product in each case of A, B, C, D E = 100) one producer, E, for necessities for laborers, two capitalists, A and B, to produce raw materials for everybody else, one C, producing machines, one D, producing the surplus product. The accounts are the following (the machine-producer, etc. has to produce one part of his commodities for himself):..." [G. 345] (then comes the table quoted above).

The table (giving coefficients as percentages) surely adds up to 1 in every row, giving a maximal eigenvalue equal to one, and is very similar to our former matrix **A**.

Marx now uses this table to analyze product flows on one hand and value flows from the dual viewpoint (for instance he inquires into the situation when one of the capitalists sells his products for less than their value) and even takes steps to carry the inquiry further into problems of Extended Reproduction (he discusses the modifications of output proportions if the surplus is spent on additional means of production instead of luxuries).

From the quantitative point of view his analysis certainly falls short of that made possible by modern matrix methods. This is probably one of the main reasons he fell back on an aggregated, two-sector, model in his main treatise, *Capital*. Nevertheless we may view the model as a system of interdependencies, seen, and even spelled out by Marx. My task simply consisted in modernizing its mathematical form, using the shorthand of matrix algebra developed and applied to economics after Marx's time.

Does the model, as stated, adequately represent the general case of Simple Reproduction? Perhaps not only landlords and capitalists but also land and capital, that is, stocks, should be included somehow? From the following it will be clear that this is not necessary. It is enough to assume that the input coefficients cover wear and tear of stocks. Those parts which must be reproduced in kind are therefore already taken into consideration in the matrix **A** itself. As long as there is no real growth we do not need to deal with stocks. No matter how much we economize on stocks, Simple Reproduction will continue to be Simple. It can be made Extended Reproduction only if we economize on flows. Only in the latter case, with the decrease of one or more flow coefficients, a_{ik}, can the maximal eigenvalue of the matrix **A** be made less than one -- which is the necessary criterion of Extended Reproduction.

1.2. Extended Reproduction

We begin to discuss growth by analyzing turnover time, connecting flows and stocks. Matrix **A** has already been defined as a flow matrix. Now our problem is to find a satisfactory definition of the stock matrix **B**, as well. The problem of stocks is analyzed in detail in the second volume of *Capital*. We will follow the lines established there although a certain departure from the classical viewpoint, apparently at variance with, but in fact broadening or complementing it, will be stressed.

On the basis of these concepts the dual aspects of Extended Reproduction are developed: prices of production, yielding an average rate of profit; and output proportions, yielding an average rate of growth.

The price side of this dual model was developed by economists much earlier and with more care than the side dealing with output proportions and use-values. The monetary and market relations of Extended Reproduction concealed the hard inner core of society's production process for a long time. The market phenomena – the tendency toward the equalization of profit, the balancing of supply and demand, competition, etc. – attracted the attention of economists relatively early. After Adam Smith described and analyzed this mechanism of competition, the main line of economic thinking continued to be preoccupied with market relations. Apart from the exceptional works of Quesnay, Marx and Walras, only the deep depression of the thirties and the first successful results of Soviet planning drew economists' attention toward macroeconomic production processes. Along with this came implicit and explicit revival of both Walras and Marx.

Thus it happened that only after two centuries of economists' concern with the average rate of profit did Neumann first recognize its theoretical duality with the average rate of growth. The subsequent development will follow the historical evolution of these ideas.

1.2.1. *Turnover time*

Formally we can define the stock coefficient matrix **B** to correspond with the flow coefficient matrix **A**. While a_{ik} stands for the amount of product i used up to produce one unit of product k, let b_{ik} stand for the amount of product i tied up in the same process. Essentially this is how Marx defines the technical requirements of the amount of means of production tied up in production ("constant capital" in his terminology).

"So far as its material elements are concerned . . . the constant capital consists of the material requisites – the means of labor and materials of labor – needed to materialize labor. It is necessary to have a certain quantity of means and materials of labor for a specific quantity of labor to materialize in commodities and thereby to produce value. A definite technical relation depending on the special nature of the labor applied is established between the quantity of labor and the quantity of means of production to which this labor is to be applied . . . value is here altogether immaterial; it is only a matter of the technically required quantity. It does not matter whether the raw materials or means of labor are cheap or dear, as long as they have the required use-value and are available in technically prescribed proportion to the labor to be applied" [III. 45–6].

In spite of the apparent independence of their respective elements the flow coefficient matrix, **A**, and the stock coefficient matrix, **B**, are implicitly connected. Product flows, represented by matrix **A**, and product stocks, represented by matrix **B**, do not come into being independently. Flows and stocks are only two aspects of the same economic transaction. These two notions express on the one hand motion, on the other hand the state of the same phenomenon.

The phenomenon observed is that some "buyer", say, sector k, buys a certain amount of product from the "seller", say, sector i. This exchange is motion because the product moves from one sector to the other. It is described by the flow coefficient, a_{ik}. But the same transaction also changes the state of the product. It will stay in the new sector until its use-value is used up entirely in the production process, until its value is transferred to the product of the process. As long as it stays it is fixed, tied up in the process. The ratio of the product i required as stock per unit of output per year of product k is the stock coefficient, b_{ik}.

This stock-flow distinction in economics first appeared in the "avances annuelles" and "avances primitives" of the physiocrats. The former expressed the flow, the latter the stock aspect. The notion became more polished in the writings of Smith and Ricardo, and Marx devoted most of the second volume of *Capital* to clearing up related questions: fixed and circulating capital, different types of capital, capital and income, total product, replacement fund and accumulation, turnover time, etc. The central link connecting the flow and stock concepts is turnover time. He defines it as follows:

"From the point of view of the capitalist, the time of turnover of his capital is the time for which he must advance his capital in order to create surplus value with it and receive it back in its original shape" [II. 159]. Further: "The aggregate turnover [time] of an advanced capital is the average turnover [time] of its various constituent parts . . . " [II. 186].

In these passages, turnover time is specified as characteristic of capitalist production or, at least, of commodity production only. But the more general character of the concept will be apparent from the following quotation:

"On the basis of socialized production the scale must be ascertained on which those operations – which withdraw labor power and means of production for a long time without supplying any product as a useful effect in the interim – can be carried on without injuring branches of production which not only withdraw labor power and means of production continually, or several times a year, but also supply means of subsistence and of production. Under socialized as well as capitalist production, the laborers in branches of business with shorter working periods will as before withdraw products only for a short time without giving any products in return, while branches of business with long working periods continually withdraw products for longer time before they return anything. This circumstance, then, arises from the material character of the particular labor process, not from its social form" [II. 362].

As soon as society enters the stage of Extended Reproduction it must economize not only labor time but time as such. This economy of time might be achieved through the market mechanism or by conscious regulation of production. In either case time becomes valuable in itself as an essential ingredient of growth. This is just another way of expressing the fact that growth takes time.

Since the notion of turnover time connects the notions of flow and stock, motion and state, it establishes a mathematical relation between the matrices **A** and **B**. If the amount a_{ik} is tied up in sector k for a given turnover time t_{ik}, then we can express the stock coefficient b_{ik} by

$$\{ b_{ik} \} = \{ a_{ik} t_{ik} \} . \tag{8}$$

This interdependence has a very important consequence. Turnover time is always positive. It may be very short as in the case of electric energy or services. Still, for a shorter or longer period every purchase will tie up resources. Therefore, $a_{ik} > 0$ entails $b_{ik} > 0$. This implies that the structural patterns of the two matrices must be analogous. Both are non-negative and the irreducibility of **A** carries over to the irreducibility of **B**.

Equation (8) (which in essence – but not in mathematical symbols – can be found in the second volume of *Capital*) was first explicitly written out by Lange [1952].* His definition being not quite precise, we have to quote him verbatim and comment. Lange's definition for t_{ij} is:

"Let the durability of the part of the output of the i-th sector allocated to the j-th sector as additional means of production be T_{ij} units of time. T_{ij} is taken as a parameter given by the technological conditions of production and may be called the 'turnover period' of the particular type of productive equipment."

Lange does not distinguish between durability, a physical characteristic of capital goods, and turnover period, the length of time it takes to recover money capital advanced. He also neglects inventory investment.

It is tempting to substitute life span for turnover time. The former is more easily measurable and independent of the current price system. But unfortunately the two notions are not directly equivalent and their exact relation needs further elaboration. This relation assumes different forms for particular parts of the capital stock. Let us therefore first subdivide the total capital employed, following

* He referred to D. Hawkins [1948] who analyzed a closely related model and noted that the quotient of the respective elements of the two matrices, b_{ik}/a_{ik}, "is of the dimensionality of time, and is simply the time required for capital from i to turn over in the productive processes k".

Table 1

Marx's Terminology for Stocks

Primal Objects	**Dual Definitions**	
Items as displayed on the Balance Sheet	For Capital in Production *Productive Capital*	For Capital in Circulation *Commodity Capital*
Plant Equipment	Constant Capital	Fixed Capital
Raw materials Auxiliary materials Semifinished goods Finished goods Accounts receivable Cash	also "means of production"	Circulating Capital
Labor	Variable Capital	

Marx's original terminology. Since his categories overlap, it helps to display them in tabular form. (See Table 1.)

Let us now look into the matter more closely. Capital consists of fixed and circulating portions. Circulating capital in turn consists of variable capital (the capital invested in buying labor) and circulating capital proper (the capital invested in materials, semifinished goods, etc.).

Variable capital and other funds tied up in reproducing manpower (they may be funds of the capitalists or of the family and of society) will be analyzed later, in Section 2.1.3. It will be shown that life span does play a certain role there, too.

Circulating capital proper consists of raw materials, semifinished and finished goods. Together they are called production inventory:

"... the magnitude of this productive supply depends on the greater or lesser difficulties of its renewal, the relative nearness of markets of supply, the development of transportation and communication facilities, etc. All these circumstances affect the minimum of capital which must be available in the form of a productive supply, hence affect the length of time for which the capital must be advanced and the amount of capital to be advanced at one time. This amount, which affects also the turnover [time], is determined by the longer or shorter time during which a circulating capital is tied up in the form of a productive supply ..." [II. 249–50].

For this inventory it is reasonable to speak about "life span" instead of turnover time. As long as a product "lives" somewhere (as an inventory item), as long as it exists at all, it will tie down some funds: its embodied value. This value will most often be recouped in one lump sum at the end of its life span. The inventory's life span ends when its value is transferred to a new product for which it is an input.

Yet this point of view blurs the financial details of the process. We do not know how much capital a given sector will fix into a particular product. For a given sector turnover time ranges from buying to selling and not from process to process. From society's point of view, funds will be tied down somewhere, but the life span might be spent in two or more sectors in the course of the product's circulation. When we speak of life span we are looking at the process from society's point of view. When we speak of turnover time we look at the process from a particular producer's point of view. Thus $a_{ik}t_{ik} = b_{ik}$ will measure capital intensity not of a given sector but of society. This solution may be reasonable on an economy-wide scale but sectoral capital intensities might be biased.

To include that part of circulating capital which usually exists in the form of cash, banking and other receivable accounts, etc., would further complicate the question. We can neglect this problem for a socialist economy. "In the case of socialized production the money-capital is eliminated" [II. 362] – paper-money and bank-accounts do not absorb any real resources, and we do not analyze monetary problems here. Although the more precise measure of capital intensity in a sectoral or enterprise detail will still be turnover time, from an economy-wide standpoint, say for purposes of planning, we can use the concept of life span for inventories.

Life span has a characteristic relation to turnover time in the case of plant and equipment, that is, fixed capital. The classical standpoint, reflected in old-fashioned bookkeeping practice, assumes that fixed assets transfer their value gradually in proportion of wear and tear. Depreciation, then, caused by physical attrition, is added drop by drop to the cost or value of the goods produced. This entails a different relation between life span and turnover time, depending on accounting practice.

Marx, analyzing the replacement of fixed capital, writes:

"Money plays a specific role in it which finds expression particularly in the manner in which the value of the fixed capital is reproduced. (How different the matter would present itself if production were collective and no longer possessed the form of commodity production is left to later analysis)" [II. 455].

Marx did not finish the analysis to which he refers parenthetically. Yet some clues can be found in later dated passages of the text.

How much capital is embodied in plant and equipment of a given life span? Under circumstances of classical capitalism the value transferred by the fixed assets was accumulated in money-form. Funds were literally tied up.

"... the money proceeds realized from the sale of commodities, so far as they turn into money that part of the commodity value which is equal to the wear and tear of fixed capital, are not reconverted into that component part of the productive capital whose diminution in value they cover. They settle down beside the productive capital and persist in the form of money. This precipitation of money is repeated, until the period of reproduction consisting of great or small numbers of years has elapsed, during which the fixed element of constant capital continues to function in the process of production in its old bodily form. As soon as the fixed element, such as buildings, machinery, etc., has been worn out, and can no longer function in the process of production, its value exists alongside it fully replaced by money, by the sum of money precipitations, the values which had been gradually transferred from the fixed capital to the commodities in whose production it participated and which had assumed the form of money as a result of the sale of these commodities. This money then serves to replace the fixed capital (or its elements, since its various elements have different durabilities) in kind and thus really to renew this component part of the productive capital. This money is therefore the money-form of a part of the constant capital-value, namely of its fixed part" [II. 454–5].

Under the above circumstances one part of the value of the fixed capital is the residual value of the working machine and the other part is accumulated to replace it. The sum of the two parts must be equal to the original value of equipment. In this special case, turnover time equals life span. In this sense:

". . . the different constituents of the fixed capital of a business have different periods of turnover, depending on their different durabilities and therefore on their different times of reproduction" [II. 186].

Nowadays we do not generally accumulate sinking funds in this rigid way. Thus value recouped can be invested anew.

Marx foresaw the possibility of more flexible financial management:

"This part of the value of the fixed capital transformed into money may serve to extend the business or to make improvements in the machinery which will increase the efficiency of the latter. Thus reproduction takes place . . . reproduction on an extended scale . . . This reproduction on an extended scale does not result from accumulation – transformation of surplus value into capital – but from reconversion of the value which has branched off, detached itself in the form of money from the body of the fixed capital into new additional or at least more effective fixed capital of the same kind" [II. 175].

Thus, part of the capital necessary to increase production can be borrowed from the sinking fund. Domar [1957] presented the first rigorous model of this more flexible business behaviour. He shows that depreciation can be a source of growth, the more so the longer the life span and the higher the growth rate. This changes the relation of life span to turnover time.

With straight line depreciation and immediate reinvestment of depreciation turnover time is one-half of life span. Under circumstances of growth the age distribution of assets will not be uniform. With rapid growth, then, turnover time will approach life span as its maximal upper limit. (See Appendix III.)

Appendix III replaces Domar's fixed life span with a probabilistic one using an exponential density function for its representation. With this latter assumption turnover time is always equal to expected life span. Therefore, for the economy as a whole, we can consider fixed capital as transferring its value in one lump sum at the end of its life. Hence circulating capital and fixed capital can be combined and represented by the same capital matrix.

Since the probabilistic treatment is at variance with the classical standpoint, we will speak only about turnover time – and not consider its relation to life span. Following classical theory, we shall say that the elements of matrix **B** are given by the elements of matrix **A** multiplied by the proper turnover times. As in the

case of **A** we do not assume constancy of the elements of matrix **B**. We only postulate that such proportions exist at a given time and place, and that they are statistically measurable with the precision required for practical purposes.

1.2.2. *Production prices*

"The price of a commodity, which is equal to its cost-price plus the share of the annual average profit on the total capital invested (not merely consumed) in its production . . . is called its price of production" [III. 158].

This concise and exact definition suggests, by its very wording, a mathematical equation. This equation, however, cannot be found in the published text of *Capital*. The equation appearing in the text following the definition contradicts the formulation given above in a peculiar fashion:

"The formula that the price of production of a commodity $= k + p$, i.e. equals its cost-price plus profit, is now more precisely defined with $p = kp'$ (p' being the general rate of profit)" [III. 165]. Or: ". . . cost-price plus the average rate of profit multiplied by the cost-price . . . " [III. 173].

Thus, the mathematical formula given multiplies rate of profit with cost-price, that is, with capital consumed, and not with capital invested – in clear contradiction to the definition given at the outset. Naturally, this contradiction has led to confusion in the development of a theory of production prices.*

It became all the more severe because in Volumes II and III of *Capital*, Marx sometimes assumed a turnover time of one year. This assumption was warranted only for theoretical speculation. It makes consumed and invested capital equal and thus indistinguishable. We begin afresh with the mathematical formulation of Marx's correct definition of production prices, as quoted at the outset.

We may speak about production prices in the case of Extended Reproduction only.

Production price can exist only if there is surplus value to distribute among prices. This distribution must be in proportion to the capital tied down in production. Therefore the flow coefficient matrix must have a maximum eigenvalue less than one, $|\mathbf{A}| < 1$.

The surplus realized in the branches of production, given a price system, $\mathbf{p}$, will be $\mathbf{p} - \mathbf{pA} = \mathbf{p}(\mathbf{I} - \mathbf{A})$. If $|A| < 1$, there will be a positive price system, $\mathbf{p}$, yielding a positive surplus, $\mathbf{p}(\mathbf{I} - \mathbf{A}) > 0$, in every branch.

We might for instance find a price system distributing surplus in proportion to cost of production

$$\mathbf{p} - \mathbf{pA} = \varrho\mathbf{pA}\,.$$

* Marx's original manuscript contained ". . . a series of uncompleted mathematical calculations . . . as well as a whole, almost complete note-book . . . which presents the relation of the rate of surplus value to the rate of profit in the form of equations". These unedited parts may contain the correct formula. This seems likely because there are quite a few correct numerical examples in the text where Marx does distinguish invested and consumed capital.

This is the equation giving the price system of the second, incorrect, definition. If $|A| = \alpha < 1$, then from $\mathbf{p} = (1 + \varrho)\,\mathbf{pA}$, $\alpha = \dfrac{1}{1+\varrho}$ and $\varrho = \dfrac{1-\alpha}{\alpha}$ follows.

This points to the futility of a price system (as established in the fifties in most socialist countries) that attempts to prescribe a rate of "profit" after cost-price externally. This rate cannot be prescribed from outside – it must stay in the relation with the maximal eigenvalue indicated above. Otherwise no consistent price system may be found.* And it is equally important to stress that this price system (advocated as being close to the value proportions) has nothing to do with value proportions. In a value price system, surplus (or "mark-up") is in proportion to labor (the vector v or wages, w) and in the former price system, the so-called cost-price system, mark-up is on cost (that is, in proportion to $\mathbf{pA}$).

We are now looking for a price system where surplus is proportional to capital invested. For the time being let us designate the capital invested per unit of production in sector i by b_i. Thus the vector $\mathbf{b} = (b_1, \ldots, b_n, O)$ will be total capital invested per unit, better known as the capital output ratio. The last element of this vector is zero because in the last sector, manpower or "households", we do not reckon with "capital". Of course there are resources invested in reproducing manpower. But the classical notion of production prices did not consider them. The production of manpower did not follow the usual rules of the capitalist game. There were no business firms investing in the production of this particular product and the laborer was not a capitalist, expecting a profit on funds tied up in his winter coat and other consumer durables. The resources tied up in reproducing manpower do not shape the average rate of profit according to the classical view.

Let us designate average rate of profit by λ. Then the equation for Marx's correct definition will be

$$\mathbf{p} = \mathbf{pA} + \lambda\mathbf{b}. \tag{9}$$

Production price = cost price + average rate of profit on capital invested.

If $(1 - \mathbf{A})$ is regular and has an inverse $(1 - \mathbf{A})^{-1} = \mathbf{Q}$, the solution of equation (9) will be

$$\mathbf{p} = \lambda\mathbf{bQ}. \tag{10}$$

But in the state of Extended Reproduction $|A| < 1$, and thus $(1 - \mathbf{A})$ must be regular. The inverse, $\mathbf{Q}$, is of course not equal to the former inverse Q, because now the matrix A is bordered by the row and column of the manpower sector. It will have much larger elements – but its economic significance, its meaning and interpretation, remains the same. It sums expenditures incurred in different phases of production. Thus matrix $\mathbf{Q}$, multiplied by any direct input vector, yields total (direct plus indirect) expenditure levels in every sector, including households.

* The Hungarian price reform of 1959, aimed at setting a uniform 5 per cent "net gain", was frustrated for this reason. Computation had to be stopped without satisfactory convergence – and the actual spread of "profits" thereafter proved to be enormous: from net loss to 20–25 per cent gain.

The product $\mathbf{bQ}$ therefore can be interpreted as total capital invested in the respective production processes. For the time being we neglect the scalar factor λ and formulate the system of production prices. It is a valuation system that weighs each product in proportion to total capital directly and indirectly tied up in its production process.

This characterization of production prices is not easy to grasp without mathematics. However Marx did have a fairly clear picture of it:

"The whole difficulty arises from the fact that commodities are not exchanged simply as *commodities* but as *products of capitals*" [III. 175].

Thus it may be reasonable to summarize Marx's point of view as follows: As long as there is simple commodity production (Simple Reproduction) the "law of value" states that there is a tendency of commodities to be exchanged on the market according to their "labor content". Exchange is then regulated by the proportions of total labor necessary to produce the diverse commodities.

This law changes under capitalism (Extended Reproduction). Here the "law" states that there is a tendency for commodities to be exchanged on the market according to their "capital content" as products not of labor but of capital. Under Extended Reproduction exchange is regulated by the proportions of total capital tied up in the production of the diverse commodities.

But how do we measure "capital"? This question was a headache for both Ricardo and Marx. They tried to reduce capital to labor in several ways, some of which later proved ambiguous and incorrect. This was the famous problem of the "unchanging standard of value" or the "transformation problem of values into prices" to which we shall return in Part 2.

The existence and uniqueness of production prices has to be proven first. Then we can ask whether there is a "transformation" of values, an algorithm for the correct computation of those prices.

Equation (10), $\mathbf{p} = \lambda\mathbf{bQ}$, does not help much in solving the problem because we assumed $\mathbf{b}$ to be given. It is specified not only as a bundle of goods (which it is legitimate to assume) but as funds already measured by some price system. Now production prices and the magnitude of λ can be determined rigorously only if we determine the capital output ratio as a value ratio at the same time. It would be illegitimate to assume any *ex ante* valuation of the resources tied up. If commodities are all measured in production prices then the fixed capital, consisting, as it does, of commodities, must be measured by the very price system to be determined.

Here the matrix $B = \{b_{ik}\} = \{a_{ik}t_{ik}\}$ plays an important role. Its elements, at least in theory, can be measured as physical proportions without the intervention of any price system at all, then the capital output ratio in this very price system can be expressed as $\mathbf{b} = \mathbf{pB}$.

Substituting this expression in equation (9) we get

$$\mathbf{p} = \mathbf{pA} + \lambda\mathbf{pB} = \mathbf{p}(\mathbf{A} + \lambda\mathbf{B}). \tag{11}$$

This equation is analogous to the equation $\mathbf{p} = \mathbf{pA}$ that defines value proportions under Simple Reproduction. It is the same sort of eigenequation. But under Simple Reproduction $|A|$ had to be equal to 1; now under Extended Reproduction $|\mathbf{A} + \lambda \mathbf{B}|$ has to be equal to 1.

Because we are considering Extended Reproduction, $\mathbf{A}$ must be non-negative and irreducible, with a maximum eigenvalue less than 1. $\mathbf{B}$ will also be non-negative and irreducible. Thus equation (11) will have one and only one positive solution for λ and $\mathbf{p}$. Equation (11) can be transformed to $\mathbf{p}\,[1 - \lambda\mathbf{B}\,(1 - \mathbf{A})^{-1}] = 0$. The matrix $\mathbf{B}\,(1 - \mathbf{A})^{-1} = \mathbf{BQ}$ is a Frobenius matrix (it is positive). Thus it has a positive eigenvector and a positive maximal eigenvalue equal to the reciprocal of λ. Thus λ must be positive.*

Therefore, given the two non-negative and irreducible matrices $\mathbf{A}$ and $\mathbf{B}$, with $|\mathbf{A}| < 1$, there is one and only one positive price system, $\mathbf{p}$, and average rate of profit, λ, determined by equation (11).

This shows that production prices can be determined unambiguously in terms of an eigenequation.

Numerical example

In our old Robinsonian economy let us suppose that unproductive consumption is discontinued. Then Extended Reproduction is possible according to the matrix:

$$\mathbf{A} = \begin{bmatrix} 0.2 & 0.7 & 0.05 \\ 0.2 & 0.2 & 0.3 \\ 0.5 & 0.5 & 0 \end{bmatrix}.$$

Let us assume a stock coefficient matrix

$$\mathbf{B} = \begin{bmatrix} 0.2 & 0.7 & 0 \\ 1 & 1 & 0 \\ 1.6 & 0.6 & 0 \end{bmatrix}.$$

$\mathbf{B}$ depends on matrix $\mathbf{A}$ and turnover times as follows: Turnover time for Materials (first row) is 1 year. Thus the first rows of the two matrices are equal. Turnover time for Tools is 5 years, that is, fivefold yearly production is always held in stock. Thus, in row 2, coefficients of $\mathbf{B}$ are five times those of $\mathbf{A}$. Finally the labor tied up in semi-finished products (this is the "variable capital" of Marx) is assumed to be 3.2 years' labor in the first and 1.2 years' labor in the second sector. All these values are of course entirely fictitious and chosen to make the example easy to solve.

* Proofs of the theorems are to be found in Appendix I.

The solution tells us that a 10 per cent per year profit is secured in both sectors

$$\mathbf{p} = \mathbf{p}(\mathbf{A} + 0.1\,\mathbf{B}) = (2,\ 3,\ 1)\begin{bmatrix} 0.22 & 0.77 & 0.05 \\ 0.3 & 0.3 & 0.3 \\ 0.66 & 0.56 & 0 \end{bmatrix} = (2, 3, 1).$$

It happens that in this example production prices equal value prices. This is by no means necessary, but it may happen under special circumstances.

Of course it does not make any difference whether there is any unproductive consumption left. It does not alter production prices if the surplus is spent entirely or mostly on luxuries and not on growth.

1.2.3. *Output proportions*

Now we write the equation for the dual of the production price system

$$(1 - \mathbf{A})x = \lambda\mathbf{Bx} \quad \text{or} \quad \mathbf{x} = \mathbf{Ax} + \lambda\mathbf{Bx} \tag{12}$$

and set ourselves a double task. First, we must give an economic interpretation to this equation, just arrived at by purely formal reasoning. Second, we set out to show its close resemblance to the famous table of reproduction in the second volume of *Capital*.

To interpret the equation we start from already defined relationships. $(1 - \mathbf{A})\mathbf{x}$ is surplus product created in the respective sectors, expressed as a bundle of goods and measured in diverse physical units of measurement. This is the net product, the final bill of goods of the system. If Extended Reproduction is possible, that is, if $|\mathbf{A}| < 1$, we always can have $(1 - \mathbf{A})\mathbf{x} > 0$, a positive surplus in every sector.

Yet we are interested in distributing the surplus in special proportions. They should be proportional to $\mathbf{Bx}$, that is, total resources tied up in production. Again $\mathbf{Bx}$ is a bundle of goods measured in physical units. Net product should be so structured as to make possible a balanced growth in all sectors' stocks. λ is the rate of increase in productive capacity. The solution of equation (12) for $\mathbf{x}$ therefore gives output proportions that, after covering the necessary flows, $\mathbf{Ax}$, for Simple Reproduction, allow for growth in every sector at the same rate, λ. The growth rate, λ, is a dual expression for the average rate of profit. Like the rate of profit, the growth rate is determined in terms of a certain unit of time — the same unit used to measure turnover time, and hence the unit implicit in the coefficients of the stock matrix $\mathbf{B}$.

Numerical example

Returning to the example of Robinson, we may compute the right-hand eigenvector. λ will be 0.1 as formerly, making yearly 10 per cent growth attainable.

Here we can no longer confine ourselves to round numbers. Computation must be truncated somewhere and rounding errors emerge.

$$\mathbf{x} = (\mathbf{A} + 0.1\ \mathbf{B})\mathbf{x} = \begin{bmatrix} 0.22 & 0.77 & 0.05 \\ 0.3 & 0.3 & 0.3 \\ 0.66 & 0.56 & 0 \end{bmatrix} \begin{bmatrix} 1000 \\ 936 \\ 1184 \end{bmatrix} = \begin{bmatrix} 999.92 \\ 936 \\ 1184.16 \end{bmatrix}.$$

These output proportions secure a surplus

$$(\mathbf{I} - \mathbf{A})\mathbf{x} = \begin{bmatrix} 1000 \\ 936 \\ 1184 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.7 & 0.05 \\ 0.2 & 0.2 & 0.3 \\ 0.5 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 1000 \\ 936 \\ 1184 \end{bmatrix} =$$

$$= \begin{bmatrix} 1000 \\ 936 \\ 1184 \end{bmatrix} - \begin{bmatrix} 914.4 \\ 742.4 \\ 968 \end{bmatrix} = \begin{bmatrix} 85.6 \\ 193.6 \\ 216 \end{bmatrix}$$

which is approximately equal to the investment needed for a 10 per cent growth:

$$0.1\ \mathbf{Bx} = \begin{bmatrix} 0.2 & 0.7 & 0 \\ 1 & 1 & 0 \\ 1.6 & 0.6 & 0 \end{bmatrix} \begin{bmatrix} 100 \\ 93.6 \\ 118.4 \end{bmatrix} = \begin{bmatrix} 85.52 \\ 193.6 \\ 216.16 \end{bmatrix}.$$

(Differences in the last digits are because of rounding.)

The numerical example brings out some essential assumptions in equation (12):

1. Output can be increased only by investing — that is, by building up new capacities. The economy pictured in the model always works at full capacity or, at least, there is no way to change the proportion of reserve capacity.

2. New investment is made according to the same coefficients as the old technology. There are no technological improvements. Thus growth is purely extensive, to use Marx's term ("extensive if the field of production is extended; intensive if the means of production is made more effective" [II. 175]). Only scale of production is increased, its inner proportions remaining unchanged.

3. Every branch of production, every sector, every product is augmented by the same factor, the universal growth rate.

These assumptions contradict actual growth experience in particular countries. In practice, stand-by capacity is exploited to a greater or lesser degree, in accordance with the everyday market situation and the business cycle. In real life new investment usually brings new technology, new inner relations of production. Investment is often a means of improving production processes. Finally the various sectors usually develop at different rates — there are characteristically slow-, fast-growing and even declining sectors, depending on historical circumstances.

Here, then, we have modeled a very special and not a really general case of economic growth. It is almost as special as the model of Simple Reproduction. Before we turn to possible generalizations we examine Marx's own version of this abstract model of Extended Reproduction.

Numerical examples of Extended Reproduction set out in the second volume of *Capital* are based on the same assumptions as those implicit in equation (12). His schemata can easily be described in the form developed as the dual of production prices.

Marx uses the following symbols in his *tableau économique*

c constant capital
v variable capital
s surplus value, divided among
  Δc increase of constant capital
  Δv increase of variable capital
  and consumption of capitalists which we denote by e.

Subscripts 1 and 2 denote departments I (means of production) and II (articles of consumption). Marx assumes turnover time equal to one year, thus capital advanced (invested) and capital consumed (cost-price) are equal. Hence constant capital, c, equals the means of production used up in the process, and variable capital, v, equals annual wages.

Marx's schemata are not given in coefficients — but for our purposes it does not matter whether we deal with coefficients or annual flows. The flows can now be written in a 4-sector input–output table.

The flows for the matrix **A** will be

	I. dept.	II. dept.	Laborers	Capitalists
I. dept.	c_1	c_2	—	—
II. dept.	—	—	v_1+v_2	e_1+e_2
Laborers	v_1	v_2	—	—
Capitalists	—	—	—	—

This table contains 10 zeros — which made a computation very easy for Marx's purposes. The row for capitalists is empty: they do not increase the value of the product or add anything to the process, according to the labor theory of value. Thus we could reduce our table to 3 sectors, handling capitalists' consumption as the final bill of goods. The remaining 3 sectors form an irreducible, self-maintaining system.

The flows for annual increase of stocks will be

	I. dept.	II. dept.	Laborers	Capitalists
I. dept.	Δc_1	Δc_2	—	—
II. dept.	—	—	—	—
Laborers	Δv_1	Δv_2	—	—
Capitalists	—	—	—	—

If we implement this scheme with numerical values given for Extended Reproduction [II. 514–7] we arrive at Table 2.

Table 2

Tableau Economique of Marx

	A				B			
1st year	4000	1500	—	—	400	100	—	—
	—	—	1750	1100	—	—	—	—
	1000	750	—	—	100	50	—	—
	—	—	—	—	—	—	—	—
2nd year	4400	1600	—	—	440	160	—	—
	—	—	1900	1110	—	—	—	—
	1100	800	—	—	110	80	—	—
	—	—	—	—	—	—	—	—
3rd year	4840	1760	—	—	484	176	—	—
	—	—	2090	1221	—	—	—	—
	1210	880	—	—	121	88	—	—
	—	—	—	—	—	—	—	—
4th year	5324	1936	—	—	532	193	—	—
	—	—	2299	1344	—	—	—	—
	1331	968	—	—	133	97	—	—
	—	—	—	—	—	—	—	—
5th year	5856	2129	—	—	586	213	—	—
	—	—	2529	1477	—	—	—	—
	1464	1065	—	—	146	107	—	—
	—	—	—	—	—	—	—	—

From the second year on, every magnitude increases uniformly 10 per cent each period. The same increase takes place for output of all the sectors, surplus, investment, wages, etc. — hence the numerical example shows the same implications we analyzed in connection with equation (12).

Because of the uniform one-year turnover time the flows of matrix **B** are exactly one-tenth of the corresponding flow elements of matrix A. (Only the first year shows a little computational lapse of Marx.)

To avoid misunderstanding: this parallel of equation (12) and the schemata and tables of Marx for Extended Reproduction are not meant to substantiate any claim for the model's being valid or realistic or useful. It only serves to show that the conception underlying equation (12) and that expressed in Marx's schemata and numerical example are the same. In short our model is really an n-sectoral generalization of Extended Reproduction with Organic Composition of Capital Remaining the Same.*

Equation (12) does not figure as explicitly in Marx's writings as the model of production prices or the model of values or Simple Reproduction. Nevertheless, what Marx has to say about this particular form of Extended Reproduction can be brought into agreement with the model.

* See [I.] Chapter XXV, Section 1, where Marx elaborates accumulation without technical change.

This model of Extended Reproduction surely is not sufficiently general for applied work. Some practical suggestions for generalization are discussed in Part 3 in the context of applications of the model. The main problems to be solved for this generalization are the following.

In the course of real growth the matrices **A** and **B** do change. For the time being we even lack adequate description of the changes, experienced historically. Only after gathering data enough will it be possible to set up and test any theory concerning regularity of changes. Marx himself stressed the rising composition of capital caused by technical change. This might have been entirely appropriate to his age — transition from manufactures to large-scale enterprise and mass production. But it is no longer true in the age of capital-saving inventions. After all the rate of profit could not decrease indefinitely, and the remedy was found in new technology where more could be produced with less investment.

For the time being let us think about the constant matrices **A** and **B** as giving only momentary values of matrices $\mathbf{A}_t$ and $\mathbf{B}_t$ which are actually changing through time. We draw a momentary tangent to a more or less curved and twisted evolutionary path. If change in the coefficients is not too fast — and generally it is not — our approximation will be good enough.

The second problem is that we do not fully understand: what happens when real output proportions and real prices do not correspond to their theoretical magnitudes, when they deviate from the eigenvectors defined above. We can anticipate some inventory and profit changes. But we still have no model that explains the size and direction of price changes under specific conditions. Here again we clearly lack factual information with which to test theories.

To sum up: the model as expressed in equations (11) and (12) is only a step in the direction of developing a more general theory of Extended Reproduction. It represents not the whole process but a momentary state of growth. This is the reason we did not need to assume constancy of coefficients. In a given state given intrinsic proportions exist — and this is all that is needed to set up the model.

On this level of abstraction, average rate of profit and growth rate are equal. But what happens if there is unproductive consumption out of those profits? Naturally this has to be subtracted from the funds to be fed back into production. The accumulated surplus will be less than the surplus produced and expressed in the profit rate and thus the growth rate will be less, too. Growth rate can equal profit rate only in the absence of unproductive consumption. (Hence the classical crusades against unproductive classes.)

But there is a much more important factor to be taken into account. In the classical theory of production prices, resources invested in reproducing manpower will have no effect on the rate of profit. But if we want growth we have to increase those inputs also. This can severely limit the growth rate. These differences will be emphasized in Part 2, where we discuss further implications of the labor theory of value.

1.3. Related Models

The model, based on Marx and transcribed into matrix algebra in the previous sections, has various close relatives. Some of them are explicitly built on the same theoretical foundation — for instance the growth models of Feldmann. Yet, these are heavily aggregated models, based directly on Volume II of *Capital* and not displaying any duality. We return to aggregated models in Part 3 in connection with the analysis of historical trends of growth.

Here we review models that are superficially alien but actually very close, in their essential logic, to the one we have been discussing. These are detailed linear models of production, disaggregated, multi-sectoral linear models of the economy. The individual models represent widely differing schools of economic thought that ignored or opposed each other for some time. However, insofar as they reflect reality, they cannot avoid its unifying force and their common basis is becoming clearer in the recent theoretical and empirical literature. In the following I should like to show how despite their very different backgrounds and interpretations they can still be brought to a common mathematical form.

That apparently contradictory views may lead to a common mathematical model is not without precedent in the history of science. To quote Neumann, whose models we discuss later: "Indeed, in classical mechanics there are two absolutely equivalent ways to state the same theory, and one of them is causal and the other one is teleological. Both describe the same thing ... Newton's description is causal and d'Alembert's description is teleological ... All the difference between the two is a purely mathematical transformation ... This is very important, since it proves that if one has really technically penetrated a subject, things that previously seemed in complete contrast, might be purely mathematical transformations of each other. Things which appear to represent deep differences of principle and of interpretation, in this way may turn out not to affect any significant statements and any predictions. They mean nothing to the content of the theory" [1963, 496].

Several types of models will be presented briefly in their historical order. Then they will be expressed in a common, fundamental form and their formal similarities and differences analyzed. Finally we review the economic purport and usability of the individual models and their common feature: duality.

1.3.1. *Description of the models*

a) Theory of games

The fundamental model of the theory of games constructed by Neumann in 1926, the so-called two-person, zero-sum game, seems to lie far from our subject and

can hardly be called a production model. It is presented here for two reasons. First, because its usefulness for the solution of production problems has been proved.* Second, as will be shown below, its mathematical equivalence with two important production models was recognized very early. The inspiration of this model was not economic at all, but abstract mathematical speculation.

The mathematical problem can be stated briefly as follows. There are two "players", J_1 and J_2, who are free to choose among various "strategies"; J_1 may choose strategies $i = 1, 2, \ldots, n$ and J_2 the strategies $j = 1, 2, \ldots, m$. Now, if J_1 has chosen strategy i and J_2 strategy j, then in this game, J_1 has to "pay" the sum c_{ij} to J_2. If c_{ij} is positive, it will be a loss to J_1; if it is negative, it will be a gain.

The question is whether, given matrix $C = \{c_{ij}\}$, consisting of n rows and m columns, the "value" of the game can be unequivocally determined, i.e. whether there is a "mix" of strategies for J_1 and J_2 from which they have no logical reason to deviate and whether, if they adhere to the strategy mix, the gains and losses paid will converge to a constant sum: the "value" of the game.

Let J_1 choose the strategies $1, 2, \ldots, n$ with frequencies $u_1, u_2, \ldots, u_n$ ($u_i \geq 0$, $\sum_{i=1}^{n} u_i = 1$), and J_2 the strategies $1, 2, \ldots, m$ with frequencies $v_1, v_2, \ldots, v_m$ ($v_j \geq 0$, $\sum_{j=1}^{m} v_j = 1$). The sum of gains and losses to be paid in the course of the game can be given by a bilinear form:

$$uCv = \gamma(u, v). \tag{13}$$

One of the players will endeavour to minimize the value of γ by properly choosing u, while the other one will try to maximize it by the proper choice of v. Now, Neumann has proven that $\min/u \max/v \gamma(u, v) = \min/v \max/u \gamma(u, v)$ and thus an "equilibrium" situation exists. Since then, several proofs and convergent computation methods have been found for the solution of this basic model.

It was known to Neumann (and elaborated in fuller detail later in collaboration with Morgenstern) that this model represents the rational choice or decision-making process of the *homo oeconomicus*. Still, the general model for production decisions was developed some years later.

b) The Neumann model

This model undoubtedly has roots in marginal analysis, and particularly in the theory of general economic equilibrium of Walras. In the 1930's an econometric seminar led by Karl Menger suggested that the proof of Walras was naive and

* Suffice it to mention here that the first method of solution for the Hungarian model of two-level planning was arrived at by T. Lipták on the basis of ideas from the theory of games. (See Kornai [1967].)

nsufficient.* The existence of a unique solution to a dynamic system of inequalities, based on the static system of Walras, but substantially modified, was rigorously proven by Neumann. This work of Neumann, though for a long time barely noticed, meant a turning point in the history of mathematical economics. There were two important reasons.

On the one hand, this opened the way for the application of more up-to-date mathematical methods in economics. Neumann found a felicitous mathematical form for economic problems. His language, the formulation in terms of a system of linear inequalities, proved to be decisive in promoting further development, particularly with the advent of computers. On the other hand, the model has a decisive feature for economic theory. For Neumann, as opposed to Walras, the production relations become the hub of the model and he deduces market relations from them. It is not clear whether this "concession" to Marxian political economy was conscious on the part of Neumann. Reviewing his assumptions, he laconically remarks: "It is obvious to what kind of theoretical models the above assumptions correspond." Had the Marxian influence been conscious, however, he very likely would not have wondered at the remarkable "dual symmetry" of his model with regard to money variables and technical variables. Marx had already belaboured the point that value relations are only dual reflections of the social division of labor. In any case, Neumann's model has become one of the meeting points of economists trained in marginal analysis and in the labor theory of value, allowing and indeed demanding interpretation in both schools.

In his model we have n products ($i = 1, 2, \ldots, n$) to be made by means of m production processes ($j = 1, 2, \ldots, m$). The processes all take place during a unit of time; if they take longer in reality, they may be subdivided into several parts in the model. Unit level performance of the jth process turns out t_{ij} quantities of the products $i = 1, 2, \ldots, n$ and uses up the quantities f_{ij}, $i = 1, 2, \ldots, n$.

The model is flexible enough to embrace joint production. This assumption was really indispensable for treating stocks: in different phases of their life span they are considered different products. Thus the process of spinning consumes machines and cotton at the beginning of the process and turns out a joint product at the end of the process: yarn and machines one period older.

The question is whether, given matrices $F = \{f_{ij}\}$ and $T = \{t_{ij}\}$, the following unknowns can be determined:

$$\begin{array}{lll} x = (x_1, x_2, \ldots, x_m) & \text{production levels} & x_i \geq 0 \\ p = (p_1, p_2, \ldots, p_n) & \text{prices} & p_i \geq 0 \\ & \alpha \text{ expansion coefficient} & \\ & \beta \text{ interest factor} & \end{array}$$

with the following constraints:

I. $\alpha Fx \leq Tx$
and if for some i, the relation $<$ holds, then $p_i = 0$

* Walras thought that he had proved the existence of a unique solution by simply counting the equations: he found as many equations as unknowns.

II. $\beta pF \geq pT$
and if for some j, the relation $>$ holds, then $x_j = 0$.

These two constraints can be interpreted as follows:

I. In a given period we cannot consume or use more of a product than was produced in the previous period. Should a surplus of product i arise, even with a maximum rate of growth, its price will fall to zero, i.e., it becomes a "free good", $p_i = 0$.

II. In an equilibrium situation there can be no profit above the average rate of interest β; if that were possible, either prices or the rate of interest itself would grow. If, however, some process j is characterized by losses, that process will be abandoned and thus $x_j = 0$.

With some special restrictions,* Neumann proved the existence of a solution for the system of inequalities and also showed that $\alpha = \beta$.

After the solution of the model we eliminate the abandoned processes and free goods from the system of inequalities. Making use of the equality $\alpha = \beta$, we obtain the following dual equality (the elimination can take place without upsetting the interrelations since for the processes and goods to be eliminated $x_j = 0$ and $p_i = 0$):

$$(T - \alpha F)x = 0 \quad \text{and} \quad p(T - \alpha F) = 0. \tag{14}$$

Note that, apart from the above-mentioned restriction, which can be dropped, Neumann's proof makes use of the positiveness of the matrices F and T only in a single place where he does not permit the fraction on the right-hand side of the expression $\alpha(p, x) \leq \dfrac{pTx}{pFx}$ to take the undefined form $\dfrac{0}{0}$ (for non-negative p and x). If we exclude this last problem, his theorems will be valid for matrices in general, i.e. for those with positive, negative or zero elements. Since it is hardly conceivable that in reality (with any kind of non-negative price system and non-negative production levels) the price sums of either production or consumption will become zero, this possibility can be excluded *ab initio* on economic grounds.

At the end of his study Neumann called attention to the fact that the model, constructed to reflect an equilibrium situation is interpretable on the basis of the minimax strategy of the theory of games. Furthermore, it can be solved either for production levels (primal solution) or for prices (dual solution) by simple maximization of the expansion factor or minimization of the rate of interest.

c) The dynamic model of Leontief

The roots of Leontief's model in the history of theory are the most varied. Leontief [1949] refers to the equilibrium model of Walras, but Walras' model is funda-

* He assumed, namely, that $t_{ik} + f_{ik} > 0$, i.e. that all processes use or produce all products. Such a strong restriction proved to be unnecessary; it is sufficient to assume that the matrices are irreducible. With the exception of some "degenerated" cases, irreducibility will secure a unique solution in α, β, p and x. (See Gale [1960].)

mentally static. To construct his dynamic model, Leontief adds new elements also found in the Harrod–Domar growth model. In the latter, investment is the sole source of changes in production. The Leontief model might be conceived of as a multi-sector Harrod–Domar model. He might have been also influenced by the first chessboard tables constructed in the Soviet Union and by the growth models of Feldmann as well. Thus Leontief's model has a close relation to the Neumann model and to the original Marxian concept of the reproduction process.

In the Leontief dynamic model total production of individual sectors must cover both intermediate consumption and the investment needed to increase production. The model can be written as either a system of difference equations or of differential equations; we start with the former:

$$x = Ax + B\Delta x$$

where, using the familiar symbols

x = the vector of total outputs
A = matrix of flow coefficients
B = matrix of stock coefficients
Δx = incremental production.

Seeking the "equilibrium" solution of the model, we assume as usual that production develops proportionately in all sectors $\Delta x = \lambda x$, where λ is the growth factor and thus

$$x = (A + \lambda B)x. \tag{15}$$

It strikes us immediately that the model is mathematically equivalent to that of equation (12).

While Neumann developed both the primal and dual aspects of his model, Marxian production prices and the dynamic Leontief model were long treated as things apart, although at this level of abstraction they are only two aspects – the primal and dual – of one and the same model.

Recently a number of scholars have elaborated the dual form of the Leontief model, and Seton [1951], Morishima [1964] and Johanssen [1965] pointed out its equivalence to Marx's production prices.

The solution of the Leontief dynamic system is well known and can be given in a closed mathematical form. Now, as Neumann suggested, maximization or minimization problems can also be interpreted in terms of minimax strategy. It therefore becomes apparent that this closed model, built along deterministic lines, can be reinterpreted as maximization. The solution of equation (15) may also be conceived of as a maximum problem, to find those proportions that will maximize the growth rate in the long run. This question will be taken up in more detail in Part 3.

d) Linear programming

The origins of this model too are technical and mathematical rather than economic. At its cradle stand two mathematicians, Kantorovich [1939] and Dantzig [1947]. Both of them developed the mathematical apparatus in order to solve technical supply problems. Since then, attempts have been made to interpret the model from the points of view of both marginalist and labor theory. It is not surprising that both approaches have been essentially successful.

The aim of the model is to allocate limited resources among competing activities in an optimal way, i.e. to achieve maximal results through choosing the best allocation of resources.

The well-known mathematical formulation of the model is: maximize $c'x$ subject to the constraints $Ax \leq b$, $x \geq 0$ where

b = vector of resources
A = matrix of technological coefficients; the element a_{ik} specifies the amount of the resource i used by a unit of activity k;
c' = vector of coefficients of the objective function; its element c_k specifies the weight of a unit of activity k in the objective function.

It is usual to transform the inequalities into equations by introducing so-called slack variables, fictitious activities using resources without affecting the objective function. In general we can state the linear programming problem as:

$$\text{maximize } \delta = c'x \tag{16}$$

$$\text{subject to the constraints } Ax = b,\ x \geq 0.$$

The dual form serves to determine the so-called shadow prices. It can be stated as

$$\text{minimize } \varrho = p'b \tag{17}$$

$$\text{subject to the constraints } p'A = c',\ p \geq 0$$

where p' is the vector of shadow prices.

There are several algorithms for solving the model; the best known are variations of the simplex method by Dantzig. The simplex method yields solutions for both the primal and the dual models simultaneously. It has also been proven that if the model has any solution at all, then

$$\max \delta = \min \varrho.$$

This means that the values of the objective functions for the maximization and the minimization problem are identical.

1.3.2. *Equivalence*

It has long been recognized in mathematics that the same problem can often be formulated in both a "deterministic" and a "teleological" form. Thus, the solution of the deterministic equation $Ax = b$ is $x = A^{-1}b$. But the same numerical result is reached if, instead of that, we seek an x that minimizes the residual value of $(b - Ax) = r$. If the first formulation has a solution it will be a solution of the second, too. The second formulation is somewhat more general in that it yields an answer in cases when the first has no solution at all – for instance when the matrix A is singular or the equation system is inconsistent, etc.

If models have identical solutions notwithstanding differences in their formulation and in the interpretation of the results, we call them equivalent. Thus, equivalence can be proven if – assuming they have solutions – the models can be transformed into a common form.

The models described thus far are based on diverse approaches. The production price model has a deterministic-causal character. It defines production prices as those that yield equal rates of profit. The same model can be understood as an equilibrium model, too. For production prices, supply and demand coincide and thus the market will not trigger any deviations from existing proportions. No branches will attract capital from other branches because of their higher profit rates or repel capital because of lower rates. This interpretation, of course, necessitates certain assumptions concerning the character of the economic mechanism – say, capitalist-market relations.

Analogously the Neumann model is an equilibrium model, too. Its market mechanism is simplified to the utmost. If there is any surplus at all, the product concerned will have a zero price. When supply exceeds demand, prices simply tumble to zero. And the same for scale of production: losses entail abrupt discontinuation of the process.

Another starting point for model building – and we have already touched upon it by raising the question of market rules – is teleological. There is a goal, to maximize some number considered favorable, or to minimize some loss. The logic of linear programming runs along these lines and every extremum-seeking model will have this feature. And it is interesting to note that both (primal and dual) aspects of the dynamic Leontief model and of the Neumann model, taken separately, can be reinterpreted as teleological models.

The Neumann model and the theory of games started from a very special and somewhat more complex viewpoint: that of the minimax principle. This is a mixture of an optimizing and an equilibrium approach. It poses the interesting question: will a function, maximized with respect to some of its independent variables and minimized with respect to others, attain any equilibrium?

A well-known common feature of all these models is their linearity. This seems to be a common shortcoming. They cannot reflect economic reality that is "not linear", and where interrelations take more complicated forms. It is no justification that the assumption of linearity is "comfortable", that it requires fewer observations for a quantitative measurement of relationships. (The argument has some practical bearing, since, if a relation is not linear, one should specify more precisely what form it does take. Complex forms necessitate broader data and it is questionable whether the increase in exactness to be expected is commensurate with the costs of collection.)

There is, however, a much more powerful argument in favour of using linear relationships. In the immediate vicinity of the point to be examined (e.g. the equilibrium situation, or the point determined by real price and volume relations) most complex relations can be approximated by linear ones with the required accuracy. Assume, for example, that the elements of the technological matrix A are complicated functions of production levels and prices as well as of time. The real question is not what functions they are in general but whether they can be approximated by the matrix A with the accuracy required in economic decisions, which is not too great in most cases. We should ask whether with changed prices and volumes and at another date the given A matrix is still sufficiently exact. Since economic decision and policy based on such models do not change the relations too rapidly, the linear models can often be safely applied subject to additional checking; their linearity is not generally a serious problem.

But linearity is not the central issue here. The models have much deeper features in common. For two, this was rigorously proved by the model-builders themselves. When constructing his equilibrium model Neumann noted that the model of the theory of games may be conceived of as a special case of the growth model, where $f_{ij} \equiv 1$. Then the "utilization" matrix, F, takes the form:

$$\begin{bmatrix} 1, & 1, & \ldots, & 1 \\ 1, & 1, & \ldots, & 1 \\ \ldots & \ldots & \ldots & \ldots \\ 1, & 1, & \ldots, & 1 \end{bmatrix}$$

He did not, however, mention that the matrix C of the theory of games can have any real elements while the corresponding matrix T of the growth model is strictly non-negative. It still may be accepted that the theory of games is a special case of the mentioned "general" Neumann model with unrestricted matrices. Other models will be reduced to the "general" form of the Neumann model below.

Dantzig also proved equivalence. He showed that all problems in the theory of games can be written as linear programming problems with matrices of the same order but that not all programming problems can be written in the form of a model of the theory of games having the same dimensions.*

Dantzig's findings of equivalence can be shown by a very simple transformation of the linear programming problem.

As we have seen, the model of linear programming is: $\max \delta = c'x$, with the constraints $Ax = b$, $x \geq 0$.

This can be transformed into a Neumann model by the following means. We assume that the programming problem has a solution. Hence $\max c'x$ exists and is finite. Let us form the diadic matrix $bc' = \{b_i c_j\}$. It will have as many

* He did prove later also that linear programming models can be written in the form of skew-symmetric game-matrices of double dimensions. (See Dantzig [1963].)

rows as we have resources and as many columns as activities. Now we can write:

$$\max \delta, \qquad x \geq 0 \tag{18}$$

$$(bc' - \delta A)x = 0.$$

This is mathematically equivalent to our equation (14). Instead of matrix T, which was not restricted in any respect, we have a diadic-matrix of rank 1. The matrix $\{f_{ij} \equiv 1\}$ of the theory of games is also such a diadic matrix and a very special one at that.

Equation (18) is equivalent to the programming model as well. This can be shown by multiplying it by x arriving at $bc'x - \delta Ax = \delta(b - Ax) = 0$. If δ is not zero, this necessarily leads to the equation $b = Ax$.

Similarly the dual form may be reduced to the equivalent form of $p'(bc' - \delta A) = 0$. Thus, any programming model can be written in the form of a Neumann model but only special Neumann models, namely those whose matrix T is of rank not greater than 1, can be written in the form of a programming model. Therefore, the Neumann model is more general than the programming model and the latter is more general than the fundamental model of the theory of games. Obviously, the Neumann model is more general than the Leontief model: Leontief models may be written as Neumann models but not all Neumann models can be written in the more restricted form. This becomes clear if the Leontief model is rewritten by introducing the unit-matrix 1:

$$p[(1 - A) - \pi B] = 0 \quad \text{and} \quad [(1 - A) - \lambda B]x = 0.$$

Here we have a Neumann model whose components are matrices $(1-A)$ and B.* On the other hand, this model is more special than the Neumann model since its matrices are square while those in the Neumann model may be rectangular.

However, note that the solution of the Neumann model has as many optimal processes as there are products with positive prices and is therefore equivalent to an equation written with square matrices. Precisely this feature of the model will serve later as the basis for a solution algorithm.

We cannot say whether the Leontief model or the programming model is more "general". In the former the two matrices must be square while programming models operate with any kind of matrices. At the same time the bc' matrix used in the programming procedure is a strictly diadic matrix whose rank is at most 1. This is less general than the corresponding matrix $(1 - A)$ whose rank is always n.

Both of these models can be transformed into a Neumann model. For the Leontief model this entails augmentation with relationships describing possible processes (not, or not yet, employed in reality). The programming model must be

* The inverse of the matrix $(1-A)$ is strictly positive but the matrix itself, being a so-called Metzler matrix, may also have negative elements. Only the generalized Neumann model comprises this model, too.

rendered homogeneous. This is somewhat more complicated. It means that all processes (activities) by means of which the resources can be expanded must be fitted into the model — the resources must be made into variables. Obviously, this makes sense only for long-run analysis. In the short run, resource conditions are realistically fixed.

To make the programming model homogeneous is to make it into a closed model. "Exogenous" resources disappear and therefore the "exogenous" objective function may also be dropped. When the scope of reality represented by the model both over space and time is expanded, the role of exogenous "endowments" and "objectives" will diminish.

Kornai [1965] points out that in a programming model there is a close interdependence between the "necessities" expressed by the constraints and the "wishes" contained in the objective function: "There is no self-evident and natural criterion for separating in each case the relationships which are to be enforced within the system of constraints . . . from those coming under the objective function." Thus, even in an open model, the "constraints" and "objectives" may — to a certain extent — substitute for each other. They serve to express the same deeper requirement which may be often expressed in an alternative manner (either as a "constraint" or as an "objective").*

And this is only natural, since the "constraint" and the "objective" appear when we "open" a closed self-reproducing system. Where we cut is apt to be somewhat arbitrary, and so, thus, is the distinction between beginning (resources) and end (objectives). This is a fundamental feature of the teleological approach which leads from a valuation of the final objective to the valuation of resources through optimization.

What, then, are the optimization features of the closed model? That the closed model may be constructed both in a strictly deterministic manner and in terms of extreme values has already been noted. Thus we could "maximize" the rate of growth (viewing the model from the production levels) or the rate of profits or interest (considering the problem from the dual aspect of prices). In both cases we economize on the only remaining "scarce resource" — time. It was in this sense that Marx made the remark: "Alle Ökonomie ist Ökonomie der Zeit" [G. 89].

In summary, all the models examined here can be written in the form of a general Neumann model. With the qualifications cited, their "generality"* is ranked by the following diagram:

Neumann
/ \
Marx–Leontief Programming
\
Theory of games

* Along the same lines, note that the diadic matrix bc' obtained in the course of transformation combines the constraints b and the coefficients c' of the objective function.

Even the open static Leontief model may be interpreted as a "parallel" to the theory of games for the case of a $B = \{b_{ik} \equiv 1\}$ matrix. In this case: $p(1 - A)x = \gamma(p, x)$ and the value of the "Leontief-games" will be net surplus. A possible economic interpretation of the game is: there are two players, the Price Office and the Planning Office. The Price Office regulates prices – chooses p – so that the enterprises earn the minimum possible profits. The Planning Office regulates production so as to maximize profits. The two "strategies" will lead to stable output proportions and prices while a surplus emerges as the "value" of the game.

Various different approaches – equilibrium analysis, minimax strategy, computation of extreme values (optimization), as well as the construction of a deterministic model – could be used interchangeably as a basis for all the models treated. These theoretical orientations do not influence the essential contents of the models nor the numerical values of their solutions.

The causal, deterministic concept of the labor theory of value and the optimization concept of marginalism have led to the same conclusions here. The fundamental question is whether real relationships are represented correctly in the model. If the model reflects the essence of reality, both approaches will yield identical solutions and identical practical instructions. Should the two views conflict in some practical questions, the difference can lie only in the relations specified. When these are reconciled the practical solutions (as Neumann puts it, the "foresight") must coincide.

While they can be rendered equivalent there are significant structural differences among the models in their traditional forms. This certainly affects their usefulness – they are not all equally well suited to the same practical purposes. Compare, for example, the uses of linear programming and the Leontief model in planning application.

Strictly given constraints on resources are entirely real in many technical problems. If we program the cutting of mill plates, their specifications will be strictly given and unyielding. As we increase the scope of programming to a whole shop or enterprise, the constraints begin to lose their rigidity. In the short run the number of machines or men in a shop is given – and this constrains the possible activities. Similarly the shop faces a given price system over which it has little control. But as the scope of programming is broadened in territory and in time, constraints and outside prices are less rigidly fixed. The number of machines can be increased by investment, the price system can be changed. While the number of workers, at a given moment, is limited in any actual economic system, working hours, or intensity of labor, or productivity of labor or all of them can be changed albeit at some expense and at limited rates. The area of arable land is limited – but the Dutch have succeeded in altering these limits by reclaiming land with skill, money and time. The same goes for all "capital goods": they may be scarce but the limits are not set forever.

By setting finite limits to resources, the programming model assumes them given once for all, in spite of all the possibilities of their increase. This fault in reflecting reality is also one of the great merits of linear programming. By artificially dividing the world into changing and rigid parts it can neglect all that is not easy to alter in the given context and time span. This is the reason that it shows a promise for the analysis of the problems of monopoly, scarce resources, land and mining rent, and rent theory in general.

* The yardstick here is the quantity of information subsumed in a model of given order.

There would be much to be gained by extending the reach of closed and homogeneous growth models by some cunning marriage with a programming model. Because there is a formal equivalence between the two models one need not choose but can use them simultaneously, in conjunction. And really this is the path chosen in practical applications not so long ago. Joining forces of the two models might remedy their individual deficiencies.

The Leontief model does not limit expansion possibilities. Certainly there are some real limits that it does not specify. Thus it might well claim some goal as achievable that actually cannot be attained or can be reached only if special measures are taken. On the other hand, allowing no leeway for favorable choices of technology it may underestimate the possible speed of development.

Linear programming can optimize. But it cannot take into account restrictions that are not introduced explicitly. Thus if there are territories of the economy left out of the model, the solution will be a suboptimum that disregards them. If it spans a given time horizon it will neglect posterity. It may favor one group or sector while neglecting other interests, neglect the future in favor of the present, or even throw the economy into some sort of cycle by oversteering it.

The development of planning methods will certainly bring models which connect the two possible approaches more firmly. Here we have only tried to clear away some illusory and doctrinaire theoretical obstacles. Linear programming, Neumann and Leontief models have more in common than appears on the surface. Joining forces will not solve all problems – but I believe it is the way to go.

1.3.3. *Duality*

Not so long ago the view was generally held that Marxian political economy and the mathematical approach to economics were at best alien to each other and that Marx's *Capital* provided but little guidance to the unsolved problems of practical management and planning under socialism.

But we have just shown that one of the fundamental instruments of modern mathematical economics and, for that matter, of operation research and control theory – namely the principle of duality – was formulated and elaborated a hundred years ago by Marx; moreover, this principle was a keystone in the theoretical foundation of his approach.

Primarily for lack of adequate mathematical tools, 60 years had to elapse before the first precise mathematical formulation of the principle of economic duality; as for its application in economic practice almost a hundred years had to go by, because of lags of computation techniques and statistics. Now, a hundred years later, it is becoming clear that duality is a general clue to controlling complicated systems, and economic systems are especially so. By carefully elaborating the concept of duality Marx has contributed to the evolution of mathematical economics and to the solution of the management and control problems. It is not altogether clear whether these concepts were adopted with open or tacit acknowledgement of the priority and merits of Marx or whether his contribution was recognized consciously at all. In any case there is firm evidence of the priority of Marx in the history of this important area of theory. There are major links that connect him to modern mathematical economics and to the solution of the fundamental theoretical problems of the socialist economy.

It should hardly need proving, at this stage, that Marx and mathematical economics are compatible with each other – suffice it to mention the names of Feldmann, Nyemchinov, Kantorovich, Strumilin, Kalecki, Lange, and their followers. It has also become clear that certain aspects of socialist commodity production do not differ so fundamentally from the economic categories of *Capital* as we might have liked to believe. Nor is this surprising: Marx stressed and clarified precisely those categories which point in the direction of socialism and the conscious control of a whole economy.

Let us return to the primary subject of our investigation: the mathematical formulation and development of the concept of duality, its logical and economic content, and the form it took in *Capital*.

Duality in mathematical economics

Very simply the economic principle of duality means that all intricate productive processes can be examined from two aspects: as physical processes creating use values and as processes simultaneously assigning values to them. The first conscious and precise mathematical formulation of the principle of duality seems to be found in Neumann's writings, already described and discussed. From the fifties on it became more and more obvious that many successful multi-sector models of production were very similar to the Neumann models and were indebted to the mathematical notions worked out and applied by him.

If all these models are special cases of an "arch-type", the principle of duality must play a central part in all of them. And this is actually so: all these models deduce in a similar manner from the same structural relations, on the one hand, the desired (optimal or equilibrium) proportions of social production and, on the other hand, the valuations (values, prices or shadow prices) belonging to these proportions.

Although the principle of duality first became widely known as the connection between the primal and dual solutions of linear programming, duality is not an exclusive feature of optimization models. Duality may be interpreted similarly in other models as well.

In all these models duality means a strict symmetry of the two aspects of the economic system presented, of the two sides of the production process, its physical and its value pattern, its "use value" and "value" aspects. There is not only symmetry but close interdependence as well. The mathematical equation systems describing the process of reproduction, the huge metabolism of society, determine, "as viewed from one aspect", the proportions of products and labor processes, the relations between useful things and the special sorts of labor creating them, while the same system of equations, "as viewed from the other aspect", explains valuation, the flow of values or monetary transactions, of homogeneous exchange values.

This leading principle of duality may govern non-linear as well as linear models, since non-linearity is, in this context, only a characteristic of the equation system describing the internal functioning of a model. Linear or non-linear, a system can always be examined from two points of view, from that of heterogeneous use values and from that of homogeneous exchange values. For example, the mathematical theory of optimal processes formulated by Pontryagin and associates [1962] again utilizes the laws of the dual system for the mathematical solution of rather complicated problems in the field of control techniques. The economic application of Pontryagin's mathematical model may be regarded as a non-linear generalization of the Neumann model. It provides a powerful tool for the treatment of some problems of economic control and regulation, to which we will return in Part 3.

From the purely mathematical point of view, the principle of duality is very simple in all these models. It establishes a connection between the solutions of a given system of equations and those of an adjoint (or transposed) system.* Despite its mathematical simplicity, duality characterizes many varied real phenomena. The principle already is or will be useful in numerous problems of classical and quantum mechanics, in physics and biology and increasingly in economics. It is most helpful in mathematical description of the movements and laws of movement of highly complicated systems.

In the analysis of such complicated systems certain parts of the system (its physical parameters or – in economic systems – certain activities, types of labor and of product) may not be directly commensurable because of their naturally heterogeneous character. However, for a clearer description and understanding of the system's operation, and, later, for the control of these processes, a common denominator, a homogeneous measure becomes necessary. This problem of order, measurement and control can be solved by taking into account those very interrelations that connect the parts of the system. Thus the system provides its own measuring instrument based on its own intrinsic laws and interrelations.

So long as the mutual relations of the parts of a system lend themselves to mathematical description, the dual will yield an instrument suitable for ordering and measurement. With the dual solutions as "weights", "valuations" or "prices",

* The principle of mathematical dualism may be formulated in still more general terms. E.g.: "A central idea of analysis . . . can be expressed in the following simple form: An element of a linear space S can often be characterized most readily and revealingly in terms of its interaction with a suitably chosen set of elements in a dual space S". This principle rather naturally finds its source in geometry . . . permeates algebra . . . and analysis. In all likelihood it is the single most ramified concept in mathematics" (Beckenbach–Bellmann [1961]).

it becomes possible to order and measure the originally heterogeneous, non-commensurable parts (states, activities, products, etc.). Measurements can now be used for "optimization" of the system, i.e. for the control of the processes themselves. We hope that technical description and analysis of the relations of a system will help us to "handle" it, to know what can be expected of it, what it is able and unable to do. Ordering, measurement and control are just steps in increasing knowledge about a given system, and the principle of duality permeates all these steps.

Duality in the economy

Obviously, the principle of duality in mathematical systems interests us only in that it reflects the real duality of the systems appearing in real life. In economic systems the substance of this duality can be (and was) understood without a knowledge of the mathematical equations describing the system. The reproductive process takes place under conditions of complex and developed division of labor. The individual parts of the system (partial processes, activities) are interdependent – because, for example, they use each other's products, share the same raw materials or productive factors, serve the same final objectives or are subject to the same legal and social rules.

Thus individual operations are interconnected by innumerable technically and socially determined relations. This complicated network of interrelations determines whether the reproduction process as a whole will operate within broad or narrow, flexible or rigid limits.

But, however these interrelationships may develop or change, they remain mutual relations. Each individual economic transaction, every exchange takes place between two parties. There is a "seller" and a "buyer". Delivery of a finished product at the same time creates the preconditions of production, or some type of consumption; thus all output is, at the same time, an input. One participant in the division of labor produces in order that the other (or several others) may consume and these, again, consume in order to be able to produce for others.

With highly developed division of labor and commodity production flows of money emerge as the "dual" of flows of goods. By inventing coins the Lydians presented mankind with a practical (though possibly inconsistent) mechanism for the "calculation" of the dual solution. The circulation of money operates as a huge analogue computer that continuously traces the dual "unknowns" of the reproduction process.

Money was invented long before the principle of duality was formulated and before the necessary computational techniques were developed. Social practice greatly preceded the advance of science in this field – "Am Anfang war die That . . . They acted and transocted before they thought." [I. 86].*

* "In the beginning was the Deed." The quotation is from Goethe's *Faust*.

Neumann's insight on duality seems to come with mathematical intuition rather than economic study. In articulating the principle of duality he looks in frightened amazement at his own intellectual child:

"Another feature of our theory, so far without interpretation, is the remarkable duality (symmetry) of the monetary variables (prices y_j, interest factor β), and the technical variables (intensities of production x_i, coefficient of expansion of the economy α)."

However, the first part of *Capital* not only gives a clear interpretation of this duality (so much missed by Neumann on the occasion of the rediscovery of the dual character of the process) but also traces the historical and logical evolution of the dual form from its beginnings to the socially established forms of money and currency.

Duality is even implicit in simple forms of random and isolated barter. With the "hair-splitting" analysis of these forms Marx's new ideas began their scientific life.

Duality in Marx's *Capital*

That commodities incorporate the duality of use value and exchange value, had been already noted by Smith and, in a somewhat clearer, more consistent form, by Ricardo. But they did not know, or at least did not show, that duality of the sysems rests on the "two-sidedness" of exchange and the dual character of labor in society.

This is one of the most important features of Marx's approach: "In this method we proceed from the first and simplest relation that historically and in fact confronts us; here, therefore, from the first economic relation to be found. We analyze this relation. Being a relation already implies that it has two sides, related to each other" [S. 369]. Marx's analysis shows that even in simple and random barter "the value of a commodity . . . is expressed by . . . the use value of another one", [I. 52] that value and use value, the dual and the primal aspects, stand on the two sides of the relationship.

In contrast to Smith or Ricardo, Marx does not speak of the duality of use value and exchange value but of that of use value and value in general. Only under certain historical conditions, namely, in the commodity-producing society, does this value appear in the form of exchange value.

"When, at the beginning of this chapter, we said, in common parlance, that a commodity is both a use-value and an exchange-value, we were, accurately speaking, wrong. A commodity is a use-value or object of utility, and a value. It manifests itself as this two-fold thing, that it is, as soon as its value assumes an independent form – viz., the form of exchange-value. It never assumes this form when isolated, but only when placed in a value or exchange relation with another commodity of a different kind. When once we know this such a mode of expression does no harm; it simply serves as an abbreviation" [I. 60].

Labor has a dual character whenever there is division of labor, regardless of the type of social organisation. The writings of Quesnay, Smith and Ricardo served to show that value and money relations help to solve some "primal" problem, for instance the allocation of use values among members of society. But Marx went further. He points out that the special form of the dual found in a commodity-producing society is a historical phenomenon. He also believed that this form is not ideal (as Smith believed it to be) but may, due to its contradictions, be plagued with inevitable disturbances in production and commerce.

This provides an important clarification of the connection between the two sides. The primal problem constitutes the fundamental economic problem of every social system, while the special solution of the dual problem characteristic of commodity production is a historical one and, therefore, subject to change in form. Indeed it is one of the most revolutionary discoveries in economics. Marx writes about it to Kugelman:

"It is self-evident that the necessity of allocating the labor of society in determined proportions will by far not be abolished by a definite form of social production, only its form of appearance will change . . . What can change with various historical conditions will be the form in which these rules assert themselves. As a matter of fact, the form in which this allocation of labor according to certain proportions asserts itself in a state of society where the interrelations of social labor are expressed as the private exchange of individual products of labor, is nothing else but the exchange value of these products . . . But there is also something else behind this fact. With an understanding of this interrelation, all theoretical faith in the eternal necessity of existing conditions will collapse – before they collapse in actual practice" [W. 552–4].

Many more passages, in fact, almost every sentence of the first part of *Capital* could be repeated here.

Duality is most clearly and consistently elaborated in the first volume of *Capital*. Suffice it here to point to such clearly dualistic pairs of concepts as labor process and realization process, surplus product and surplus value, and technical composition and value composition of capital.

Yet the principle of duality runs through the whole work connecting many different ideas and problems. This correspondence is worked out not only in rough outlines but often in minute detail as well.

In Marx's treatment the theory of value and the theory of reproduction are always corresponding pairs in strict duality with each other. For each definition, thesis or rule in the theory of value a strictly parallel definition, thesis or rule in the theory of reproduction can be given.

Marx himself felt that the consistent development of this dual character was the key not only to his book and his scientific research but, indeed, to the understanding of political economy in general. In addition to the letter written to Kugelmann and quoted above, two further letters addressed to Engels point to this fact. In the first of these, summing up his own opinion about the first volume, Marx declares:

". . . The best thing in my book is: 1. the emphasis on the dual character of labor, right in the first chapter, according to whether labor is expressed in use value or exchange value (this is the basis of the whole understanding of facts)' [W. Vol. 31, 326]. And in a second letter: "It has escaped the attention of all economists, without exception, that if the commodity is something dual – use value and exchange value – then the labor embodied in the commodity must also be of dual character . . . In fact, this is the whole secret of the critical conception" [W. Vol. 32, 11].

It is both remarkable and regrettable that this fundamental approach that Marx emphasized as the main result of his research work should have left so little impression on our general concept of *Capital*, his principal work.

However great its indirect effects, the fate of *Capital* as a scientific work is, on the whole, not enviable. Were it less praised and less denounced but more widely read, there would have been fewer false ideas about it – and economics would have made faster progress.

Part 2

Discussion of the Model

The core of the model set up in Part I is the matrix $\mathbf{A} + \lambda\mathbf{B}$, describing the interdependence of society's metabolic process. The proportions represented in the two matrices yield rigorous definitions of values, production prices and their dual concepts, output proportions of Simple and Extended Reproduction, as a natural extension of Marx's original definitions. They were defined respectively as positive left- and right-hand eigenvectors of the matrix $\mathbf{A} + \lambda\mathbf{B}$, belonging to its maximal eigenvalue. The latter is always equal to 1.

We now proceed to examine the character of the model in greater depth. We begin by showing that values and production prices are just special cases of a more generalized price system.

The mathematically well-known extremal properties of eigenvalues are helpful in demonstrating certain optimizing characteristics of each price system in its own model. Price systems defined by the labor theory of value do orient our decisions properly in certain typical situations. These questions will be discussed in the first chapter.

Mathematical eigenproblems are logically circular, but their formulation is powerful and indispensable in economic science. This prompts a review of some old arguments against circularity in labor theories of value. One of them, the so-called "transformation problem of values into prices", actually suggests a practical algorithmic solution of the model. All this will be the content of the second chapter.

The third chapter fills in a few missing points: the analysis of the correct dimensionality of the model and the discussion of the economic significance and behavior of the scalar λ. Finally, further theoretical generalization of the model is attempted through strict axiomatization and probabilistic interpretation.

2.1. Three Types of Price Systems

Values were defined in terms of relationships of Simple Reproduction, a historically ancient and simple form of production. Production prices pertain to Extended Reproduction — historically capitalistic production. Thus far the resources tied up in reproducing manpower did not participate in determining the average rate of profit under Extended Reproduction. At the end of this chapter a third, more general system of prices will be developed. It will take into account Extended Reproduction of manpower, and investment of resources serving this purpose as well. This so-called "two-channel" price system will subsume values and production prices as its two special limiting cases. Here we resume that dual solutions (that is: price-like solutions) of models describing the functioning of dynamic systems yield proper scales for aggregating, explaining and finally controlling the systems themselves.

The models to be discussed reflect only problems of freely reproducible goods. As already noted, scarce natural resources, monopolistic situations — that is, circumstances temporarily hampering free development of production — call for rent theory, and are better described by programming models.

This analysis only begins to explore the optimization properties of the models we discuss. It does not advocate the actual application of any particular price system under socialist circumstances. This latter question can be decided upon only after comparing the logical structure of the price system with actual (or suggested) rules of the economic game. "Which price system orients properly in a given model?" and "What would be the practical consequences of the same price system under given or proposed rules of the game?" are separate questions. The latter question will not be answered.

2.1.1. *Value prices*

We may reinterpret the causal-deterministic definition of value proportions in an optimizing context using the extremal characterization of eigenvalues. The analysis rests on the following mathematical inclusion theorem for eigenvalues:*

If the maximal eigenvalue of a non-negative irreducible matrix, $\mathbf{A}$, equals α, then if the matrix is premultiplied by an arbitrary positive vector $\mathbf{p} > 0$, then either

* The theorem stems from Frobenius and was sharpened by Birkhoff and Varga. Its proof may be found in Appendix I.

$$\frac{(\mathbf{pA})_i}{p_i} = \alpha \text{ for every } \quad i \ (i = 1, \ldots, n)$$

or

$$\min_i \frac{(\mathbf{pA})_i}{p_i} < \alpha < \max_i \frac{(\mathbf{pA})_i}{p_i}.$$

Let us reformulate this theorem in economic terms. $\mathbf{pA}$ is the cost of the products measured by the given price system $\mathbf{p}$. Its i-th element, $(\mathbf{pA})_i$ is cost of product i. Now p_i being the price of product i, the quotient $(\mathbf{pA})_i/p_i$ may be called the cost ratio of product i, a term frequently used in socialist accounting and planning practice.

Our theorem, now, stated in economic terms is: either all cost ratios are equal, in which case they equal the maximal eigenvalue, or they are different, in which case the maximal eigenvalue will be flanked by the maximal and minimal cost ratios.

This interpretation motivates our decision to consider the maximal eigenvalue as an indicator of efficiency of the economic system represented by the matrix $\mathbf{A}$. This was already suggested by the fact that $\alpha = 1$ is the criterion of Simple, $\alpha < 1$ of Extended, and $\alpha > 1$ of Diminishing Reproduction. This indicator is invariant with respect to similarity transformations of the matrix $\mathbf{A}$. Thus it is independent of the units of measurements chosen and of the actual price systems used to measure the matrix $\mathbf{A}$. This invariability is important. The criterion of optimality should be independent of the units or prices of the system.

Now the theorem can be used to prove that proper orientation is furnished by value proportions. Let us assume an economy under Simple Reproduction with value prices. Thus $\mathbf{p} = \mathbf{pA}$, and the cost ratios are equal to 1 for every product. Let a new technical possibility to produce product i emerge. Say, vector a_i stands for the old technology and vector a_i^* for the new one.

How can we decide upon whether to substitute the new technology for the old one? If after substitution Simple Reproduction is changed to Extended Reproduction, substitution is advantageous. If Simple Reproduction is still to be continued, substitution is optional. But if Diminishing Reproduction ensues, substitution is clearly disadvantageous.

One important function of a price system is to orient decisions about possible alternative techniques. Value proportions (taken as a price system) help us to ascertain which of the three outcomes would ensue after substitution. If we use them to compute the cost ratio of the new technology, a_i^*, the number computed will indicate the efficiency of substitution. Thus

if $\dfrac{\mathbf{p}a_i^*}{\mathbf{p}_i} = 1$, that is, $\mathbf{p}a_i^* = \mathbf{p}_i$ substitution is optional

if $\dfrac{\mathbf{p}a_i^*}{\mathbf{p}_i} < 1$, that is, $\mathbf{p}a_i^* < p_i$ substitution is advantageous

if $\dfrac{\mathbf{p}a_i^*}{\mathbf{p}_i} > 1$, that is, $\mathbf{p}a_i^* > \mathbf{p}_i$ substitution is disadvantageous.

If after substitution of a_i^* for a_i the cost ratio remains 1, the matrix, $\mathbf{A}^*$, with new technology substituted for old, will have still 1 as its maximal eigenvalue and the system remains in Simple Reproduction. The result of substitution is neutral.

If $\mathbf{p}a_i^*/p_i = \delta < 1$, then, after substitution the cost ratios of all technologies but the i-th are still 1, but the new technology has a cost ratio less than 1. Here the inclusion theorem secures an eigenvalue of the matrix $\mathbf{A}^*$ that is less than 1, namely $\delta < \alpha < 1$. Thus Extended Reproduction becomes possible. The result of substitution is favorable.

Finally, if $\mathbf{p}a_i^*/p_i = \gamma > 1$, then, after substitution, the maximal eigenvalue of the new matrix, $\mathbf{A}^*$, will be strictly greater than 1; $1 < \alpha < \gamma$. Thus only Diminishing Reproduction is possible. The result of substitution is unfavorable.

Value prices, therefore, do orient properly in this sense. Furthermore: the value vector, $\mathbf{p}$, being the left-hand eigenvector belonging to the maximal eigenvalue, α, this must be the only price system, the only set of relative prices that does orient properly under conditions of Simple Reproduction.

Certainly the scalar multiplication of the two vectors a_i^* and $\mathbf{p}$ is equivalent to everyday practical reckoning. Our decision process then amounts to the usual appraisal of comparing expenditures with results.

The above argument does not claim that the price system is appropriate for measuring the amount of saving induced by the new technology; nor does it give any orientation as to how much to produce with the new technology. But the scale of production cannot be determined by prices alone in a linear system of this sort.

The criterion of optimality used above should not be confused with the maximization of a fixed consumption structure or utility function. The consumption structure before the decision is, of course, given. But the structure after the decision is not predetermined. In our procedure, after a favorable decision, society may increase and/or change its consumption without jeopardizing Simple Reproduction. If $|\mathbf{A}^*| < 1$, then any or every element of consumption can be increased to a certain extent before the maximal eigenvalue again reaches 1. What to do with the surplus accruing after substitution is clearly a second question. It might be accumulated or consumed or both in varying proportions. To formalize this second decision we must know who will be in possession of the surplus and what are his goals. This certainly changes historically.

What happens now in the particular case where the surplus is consumed unproductively?

Now, in general, it is not value prices but the left-hand eigenvector that orients properly. Its last element shows the new value produced by labor. If there is unproductive consumption the value of labor power is less than the value produced by it. To this extent, then, value prices will undervalue labor power; it will be "cheaper" than its properly orienting value, thus decisions will tend to waste it.

This seems to explain Marx's train of thought in the following passage: "The use of machinery for the exclusive purpose of cheapening the product, is limited in this way, that less labor must be expended in producing the machinery than is displaced by the employment of machinery. For the capitalist, however, this use is still more limited. Instead of paying for the labor, he only pays the value of labor power employed" — and he adds in footnote: "Hence in a communistic society there would be a very different scope for the employment of machinery than there can be in a bourgeois society" [I. 392—3].

Value prices, therefore, will lead to inefficient use of labor resources when there is unproductive consumption. Value prices, therefore, orient properly only under conditions of simple commodity production, when the amount of surplus is negligible.

2.1.2. *Production prices*

In what sense do production prices, the left-hand eigenvector of the matrix $\mathbf{A} + \lambda\mathbf{B}$, orient properly under conditions of Extended Reproduction? Why is total capital tied up in the production process the proper measure — and why do value prices disorient in the case of growth?

Let the average rate of profit, λ, and production prices, $\mathbf{p}$, be given. We know that the cost ratio, now defined as $[\mathbf{p}(\mathbf{A} + \lambda\mathbf{B})]_i/p_i$, will be 1 for every product i because $\mathbf{p}(\mathbf{A} + \lambda\mathbf{B}) = \mathbf{p}$ by definition. This new definition of cost ratio contains beside costs proper, $\mathbf{pA}$, a second term for profit on capital, $\lambda\mathbf{pB}$. Classical economists do not consider profit *bona fide* cost of the production process. Still on the basis of our earlier analysis of production prices it makes sense to include the term for capital costs equal to the expression given. Although deviating from classical usage, we use it to make the parallel to Simple Reproduction. After proof and discussion we will reformulate our result in classical terms, too.

Cost ratios, then, equal one for all the given input structures. Let us now consider a new technical possibility to produce product i with flow coefficients a_i^* and stock coefficients b_i^*.

Analogously to the judgement for value prices,

if $\quad \mathbf{p}(a_i^* + \lambda b_i^*) = \mathbf{p}(a_i + \lambda b_i) = p_i \qquad$ substitution is optional

if $\quad \mathbf{p}(a_i^* + \lambda b_i^*) < \mathbf{p}(a_i + \lambda b_i) = p_i \qquad$ substitution is advantageous

if $\quad \mathbf{p}(a_i^* + \lambda b_i^*) > \mathbf{p}(a_i + \lambda b_i) = p_i \qquad$ substitution is disadvantageous.

If the new technology is optional, then after it is substituted for the old the cost ratio remains 1 for every i. Thus the maximal eigenvalue of the new matrix $\mathbf{A}^* + \lambda\mathbf{B}^*$ still remains 1 and yields the old rate of profit.

If $\mathbf{p}(a_i^* + \lambda b_i^*)/p_i = \delta < 1$, then, after substitution, the maximum cost ratio equals 1, the minimal equals δ, and thus the maximum eigenvalue of the new matrix $\mathbf{A}^* = \lambda\mathbf{B}^*$ will be strictly less than 1. Thus we can increase λ until we reach its former level, 1. In other words, we can now achieve a higher average rate of profit and thus average rate of growth. The same reasoning shows that a disadvantageous substitution entails decrease of these rates.

This does not mean that the saving of an advantageous substitution must be spent to increase the growth rate. It could be spent on a higher standard of living by increasing consumption coefficients without reducing the growth rate. What is essential is whether it is possible to answer the question of "worse or better?" without presupposing any special preference function. If we do not predetermine how the additional surplus will be spent we can circumvent the far more complex and intricate question of "how much worse or better?" and "better or worse for whom?".

The rentability of a new technological process can now be reformulated in classical terms. The criterion of a favorable decision can be transformed (let us drop the subscript i) from

$$\mathbf{p}a^* + \lambda\mathbf{p}b^* < \mathbf{p}a + \lambda\mathbf{p}b$$

to

$$\lambda < \frac{\mathbf{p}(a - a^*)}{\mathbf{p}(b^* - b)}\,. \tag{19}$$

Here $\mathbf{p}(a - a^*)$ stands for saving in flows resulting from the new technology and $\mathbf{p}(b^* - b)$ is additional capital spent for this purpose.* So long as the quotient of these two magnitudes exceeds the average rate of profit, the new investment project should be accepted.

In the latter form our decision-prescription is equivalent to the investment appraisal formula in current use in socialist countries. It requires the internal rate of return to exceed the average, that is, the usual or external rate. Socialist literature calls λ the "time-factor" and its reciprocal, $1/\lambda$, the "pay-off period". These expressions are thus linked conceptually to the average rate of profit and its dual, the average rate of growth.

In this formulation the production price system suggests an iterative solution algorithm for the Neumann model.** The algorithm is probably not practically useful but it is pedagogically interesting in that it gives an idealized description of perpetual optimization in a dynamic, long-range system.

Let us assume that, in the original Neumann model to be solved, we know more processes than products, that is: $m > n$. (If $n > m$ we simply perform the dual of the following operations.) Let us select an arbitrary subsystem consisting of, say, the first n processes and n products neglecting temporarily the remaining $m - n$ processes. We denote the selected submatrices by T_0 and F_0. First we compute the "production prices" of our subeconomy, that is, we solve the equation

$$p_0(T_0 - \lambda_0 F_0) = 0,$$

for p_0 and λ_0. How to solve it is a practical question taken up in Section 2.2.3. Theoretically it is possible that — for some submatrices — there will be no solution at all. But if the complete model has a solution (and that was proven by Neumann) there must be at least one subsystem of the possible $\binom{m}{n}$ subsystems that can be solved. Thus we may suppose that our T_0, F_0 subsystem has a solution.

* Or there might be dis-saving in flows $\mathbf{p}(a-a^*) = \gamma < 0$ counterbalanced by savings in stock $\mathbf{p}(b^*-b) < \gamma/\lambda$, or savings in both!

** The procedure is closely related to that suggested by Weil [1964].

"Production prices", p_0, together with our "profit rate", λ_0, supply the necessary measuring rods for judging the $(m - n)$ processes neglected in the subeconomy T_0, F_0. Thus we examine the neglected processes, characterized by the vectors t_i, f_i ($i = n + 1,\ n + 2, \ldots, m$), one by one. That is, we compute the scalars $p_0(t_i - \lambda_0 f_i) = s_i$. If we find a positive s_i it signifies a possibly advantageous substitution — because this i-th process produces more, its sales receipts exceed the necessary expenditures allowing for the old return on capital, and reckoned in the prevailing price system. After substituting the new i-th process we may improve the situation — that is, reach a higher profit rate.

Substituting the new process for the old in i, we get a new subsystem, say, T_1, F_1, and solve it for "production prices" again:

$$p_1(T_1 - \lambda_1 F_1) = 0\,.$$

Now we attain a higher λ_1 than before: $\lambda_1 > \lambda_0$.

The number of possible subsystems being finite we sooner or later find the optimal one, that with the highest λ. For the optimal subeconomy there will be no excluded process with a positive s_i, that is, there is no excluded process yielding a higher rate of return than those already in the optimal subeconomy.

We still have not entirely solved the problem of substitution. If a process is advantageous, which other process should be replaced by it? In the absence of joint products it is easy to answer this question. But if the product in question is already produced by several processes, it is not easy to see which of them should be eliminated by the favorable new process i. We must have some convention to prevent cyclic substitutions.*

But we are not really interested in computation here. Production prices can be interpreted as "shadow prices". The real economy may be viewed as a single iteration in computing a big and endless Neumann model. If the processes that are not used in reality are not efficient at production prices, then the real economy is at an optimum. If, however, some hitherto unused process (or a newly invented one) seems to be efficient at production prices, one should decide to use it. After the new process has been introduced, a new production price vector has to be worked out again.

We do not look into the question of whether this perpetual iterative process of selecting improved technology does or does not reflect actual happenings. This would involve asking whether there is any ascertainable tendency nowadays toward production prices and equalization of profits and whether innovation policies are in fact rational at all.

* It might perhaps be a good rule to replace that activity j for which t_{ij}/f_{ij} is maximal — or if $f_{ij} = 0$ for several j, hen to choose by inspecting the quotients $p_j t_{ij}/p_i$.

Under the idealized conditions of capitalism and Extended Reproduction it is production prices rather than value prices that should orient technological decisions. Our abstract model of capitalism does find its proper regulator in production prices.

The system described thus far does not take into account resources tied up in reproducing manpower. Thus, the computed average rate of profit will exceed the possible growth rate. Yet, to make growth possible, every factor must be increased, including all the resources tied up in creating manpower. This question is of utmost importance for proper economic orientation, and we devote the next section to its more thorough investigation.

2.1.3. *Two-channel prices*

Here we analyze two closely connected questions. The first is: How do resources tied up in reproducing manpower (the last column of matrix B, quite unrealistically considered thus far zero) modify the overall reproduction process and the corresponding price system? The second is the historical role of the three types of price systems.

Funds tied up in reproducing manpower are costs incurred long years before manpower becomes a skilled or unskilled agent of production. These are the costs of raising, educating and training manpower.

These costs were naturally not unknown to Marx, whose work abounds in observations about them. Thus on raising manpower:

"Hence the sum of the means of subsistence necessary for the production of labor-power must include the means necessary for the laborer's substitutes, i.e. his children, in order that this race of peculiar commodity-owners may perpetuate its appearance in the market" [I. 172]. And about training and educational expenses:

"In order to modify the human organism, so that it may acquire skill and handiness in a given branch of industry, and become labor-power of a special kind, a special education or training is requisite, and this, on its part, costs an equivalent in commodities of a greater or less amount. This amount varies according to the more or less complicated character of the labor-power. The expenses of this education (excessively small in the case of ordinary labor-power) enter *pro tanto* into the total value spent in its production" [I. 172].

"The expenses of developing that power which expenses vary with the mode of production" [I. 519] tend to decrease because "... the necessary training ... is more and more rapidly, easily, universally and cheaply reproduced with the progress of science and public education the more the capitalist mode of production directs teaching methods, etc. towards practical purposes" [III. 300]. But in spite of "... the general development which reduces the cost of production of specially trained labor-power" [III. 389] and thus reduces the gap in earning range between unskilled and highly qualified manpower; the funds tied up in reproducing manpower remain considerable.

The hundred years that have passed since the publication of Marx's *Capital* have even witnessed a considerable increase in those expenses — notwithstanding "economies" of mass-production in educational institutions, because of the very high training requirements of modern technology. Illiteracy and child labor have dwindled since Marx's age, not just because of more humanitarian ideals but because of economic requirements.

The increase of costs and the fact that they are incurred far in advance of the benefits to be gained become very conspicuous under socialism. The socialist state shoulders an increasing responsibility for family-care, health services, educational outlays — costs traditionally incurred by families themselves. This is done mainly for extra-economic reasons: whether the family is worse or better off should not determine its state of health, its educational opportunities or its size. But we have still to face the facts: after transferring heavy financial burdens to the state budget, families still carry very high expenses.

Nowadays in Hungary (and in other socialist states) government-financed "social costs" of reproducing manpower amount to approximately 20–30 per cent of wages. They are defrayed from the budget. And still the family spends approximately half of its income on rearing children. Thus a non-negligible part of national income — a part well in excess of the conventional investment ratio — serves the future generation and in this sense can and should itself be considered as investment. This investment is embodied not in the means of production but in the agent that controls them.

The gestation period of new fixed capital seldom exceeds 2–3 years. The gestation period of manpower, nowadays, is seldom less than 16–20 years.

The amount and composition of these costs, and the length of lags must certainly influence the overall pattern of reproduction and thus affect output proportions. This influence is recognized in the theory of planning — even if its full significance is seldom grasped. But planning practice has already taught planners some lessons. The demographic and employment structure of population — both very slow and expensive to influence and modify — delimits and constrains long-range goals. One of our experienced Hungarian planners, Timár [1966], writes:

"As a consequence of the above-mentioned long reproduction cycle of manpower, the available manpower supply in the next fifteen to twenty years, for the most part consisting of the able-bodied population, is substantially given ... In this manner we obtain an important starting point for long-term national economic planning ... In long-term planning the principal task of manpower plans is ... to yield information, on the basis of the available manpower supply, on the possible utilization of manpower in the national economy and to give initial data for drawing up the plans of individual branches".

And the same problem is recognized for underdeveloped countries where lack of qualified manpower is a perpetual bottle-neck to take-off and growth. All the necessary educative tasks simply cannot be accomplished at once in the case of an underdeveloped country without a developed system of schooling. For example, teachers must be trained to teach engineers, who in turn teach engineers' helpers, who train foremen to train skilled workers. A heavy influx of foreign experts and, possibly, equally strong help in educating countrymen abroad will be a *sine qua non.* Even with substantial foreign aid it remains a tedious and time consuming process.

In the framework of an already developed comprehensive system of education, all these tasks are performed simultaneously; once the educational pipeline is filled, there is parallel education in all layers. Yet, we should continue to take account of the indirect, less conspicuous time requirements for educating today's educators and so on.

These expenses and time lags will affect output proportions, and also modify the price system. Value of manpower equals the value of means of subsistence necessary to reproduce it with all its skill – and the magnitude of this value is not changed whether the expenses are defrayed by the individual, the family, the community or the state.

This too is not a new idea among planners and economists. Esze and Nagy [1963] argue for the so-called "two-channel" price system (which allocates a part of "surplus" in proportion to wages) as follows:

"Here we think about that part which accrues to the workers as indirect transfer, including the amount spent to train new workers. This part is, strictly speaking, not a net income of society but – from the viewpoint of society – as much the cost of labor as wages".

The costs of education, health and family care, shouldered by the state are usually not of a "flow" but of a "stock" nature. Even if they apparently take the form of flows (say, in the case of sick-relief or old age pension), they actually are disbursements of a collective insurance fund.

Let us estimate the order of magnitude of the funds invested in reproducing manpower and their turnover time. We will neglect consumer durables (a minor part of the total stock) and simplify the investigation, seeking only a rough estimate.

Turnover and reproduction of manpower differ characteristically from turnover and reproduction of machines. In the case of machines two periods are sharply discernible: their gestation or production period, and the second period, when they are performing their duties. In the case of manpower the two periods are not segregated as clearly: after finishing their formal education men may be trained on the job; they learn by doing. The formative period of labor will thus not be entirely completed when they enter production. Furthermore, manpower acts as consumer of goods even after it becomes productive.

Of course the separation of the two periods is an abstraction even in the case of machines. There are additions, improvements, maintenance costs for a functioning machine, too. Economics has something, but not very much, to say about maintenance. Maintenance is treated much more adequately in management science. In the case of manpower this neglect of "maintenance" is not permissible, and indeed personal consumption is considered one of the most important subjects of economics.

We begin our estimate by assuming a constant annual consumption throughout an individual's life. This assumption is very crude because the first years cost more. The years of primary and secondary schooling absorb above-average costs,

too, and old age also requires again above-average health and nursing care. We nevertheless assume constant consumption.

Our second assumption is even cruder: Manpower creates a constant amount of new value each year. On the basis of average earning potential in the different age groups it seems probable that the amount of new value created is not constant. It seems to increase very substantially after the first years and has a peak (varying with occupations) around the age group 40–45. After that it seems to decline.

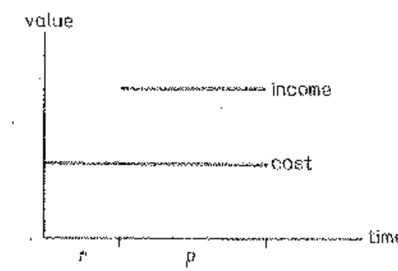

Fig. 1. Costs and incomes

Both of our assumptions are incorrect in that they will tend to understate funds tied up in reproducing manpower, as should become clear below.

Third, for present purposes we assume Simple Reproduction. Taking Extended Reproduction into account here would complicate and blur the picture without adding much to our knowledge. Thus, we claim that the amount spent to reproduce manpower and the amount of value created by manpower so reproduced are equal. This is a slightly different formulation of Simple Reproduction where value of manpower and the new value created by it are equal. There is no difference between the amounts spent and the amounts recovered. There is nevertheless a considerable difference in the timing of those two amounts. This difference, this lag in recovering the amounts spent, explains the amount of funds tied up in reproducing manpower.

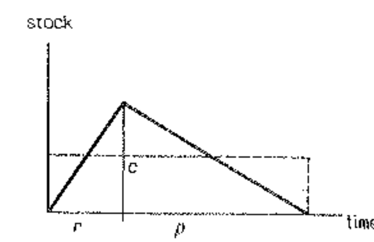

Fig. 2. Cumulated costs minus cumulated incomes

We divide the total life span of the laborer into two parts: raising time $\mathbf{r}$, and productive time $\mathbf{p}$. In the first $\mathbf{r}$ years there are only costs, in the second $\mathbf{p}$ years there are costs and "income" (new value created). The sum of costs over the $\mathbf{r} + \mathbf{p}$ life span must equal the sum of the income.

Graphically, annual costs and incomes (recovery of costs) are shown in Fig. 1. But, at the same time, cumulated costs equal cumulated income. Thus Fig. 2 shows the amount fixed.

The average amount of funds required is represented by the height of the rectangle indicated by broken lines – its area is equal to the area of our triangle. If annual costs consumption is $\mathbf{c}$, the height of the triangle must be $\mathbf{h} = \mathbf{rc}$. Therefore the height of the rectangle, funds tied up on the average, will be $\mathbf{rc}/2$.

If we compute "turnover time" of resources tied up in reproducing manpower, we arrive at a "life span" measure. The correct "life span" to reckon with is raising time, $\mathbf{r}$. This means that, for Simple Reproduction, $\mathbf{r}/2$ years of consumption are tied up as investment in human beings.

Time spent in actual production, $\mathbf{p}$, does not play any apparent role here. But this is deceptive. If we designate yearly value creation by $\mathbf{v}$, then the postulated equality of cumulated costs with cumulated income, $(\mathbf{r} + \mathbf{p})\mathbf{c} = \mathbf{pv}$, implies $\mathbf{rc} = (\mathbf{v} - \mathbf{c})\mathbf{p}$, hence $\mathbf{rc}/2 = (\mathbf{v} - \mathbf{c})\mathbf{p}/2$. Now half of production time (actual life span spent as an agent of production) is equal to the "turnover time" of another stream: the difference between annual value production and consumpion.

On the basis of this relationship it becomes clear that if raising time, $\mathbf{r}$, is 10–20 years, the order of magnitude of the funds fixed in reproducing manpower will be about 5–10 years' income. If raising time costs are above average (as they usually are nowadays), we underestimate the stock, and if income is not constant but has a peak around 40–50 years, we overestimate the speed of recovering the sums spent and thus tend to underestimate stocks again.

It is well known that the total reproducible (tangible) wealth of society (including not only means of production but "infrastructure" – highway systems, residential buildings, etc.) seldom exceeds 3–4 years' national income. Comparison with 5–10 years' income tied up in reproducing manpower clearly indicates that the man working with a machine is worth much more than the machine itself, including all the expensive auxiliary fixed equipment. This crude computation justifies the socialist commonplace: "The greatest value is man himself."* Yet this point of view remains purely theoretical until output proportions and price systems are actually adapted to it.

Everybody is shocked by waste and yet we condone everyday waste of manpower and skills. For example in Hungary, where skills and talents are in principle much esteemed, Kovács [1968] has shown that, in spite of a shortage of skilled labor, about 2000 highly qualified persons work in jobs not requiring more than skilled workers' education, 2500 graduates of universities and 25 thousand graduates of high-schools work as semi- and unskilled workers, and about 235 thousand skilled workers work at jobs not equal to their skills. (Total active population is about 6 millions.) The educational expenses wasted here are about 10 per cent of national income – and additional loss of national income caused by this dislocation exceeds one per cent annually. But this estimate covers only waste by persons working full time in jobs beneath their qualifications. Many also spend part of their working day at tasks that could be accomplished by lesser talents.

As long as we reckon labor costs in wages alone this waste will not stop. If we were to introduce human investment into the matrix $\mathbf{B}$, we would get a totally different price system, one that would give proper orientation for allocation of labor of different skills also.

"Two-channel" prices are a comprehensive system of production prices that also takes into account resources tied up in reproducing manpower. We show that this "two-channel" price system is nothing but the left-hand eigenvector of the total system (including stocks in the last column of matrix $\mathbf{B}$); this

* Abbot Galiani, a very early economist, already had an inkling of this: "La vera ricchezza . . . cel'uomo." ("Della Moneta". Custodi. Parte moderna. Vol. III, p. 229.)

serves as proof that in the new, total system it is "two-channel" prices (and not classical production prices) that orient properly.

Let us start from the definition of "two-channel" prices given by Esze and Nagy. They are prices where the mark-up on cost is partly proportional to fixed capital and partly to wages. Thus, surplus value is allocated among prices with a mark-up rate π on capital and β on wages. In this notation

$$p = pA + \pi pB + (1 + \beta)v + \pi(1 + \beta)z . \tag{20}$$

Here pA = costs of intermediate inputs
pB = capital/output ratio
$(1 + \beta)v$ = wages, increased by the mark-up
$(1 + \beta)z$ = variable capital, increased by the mark-up.

Thus wages and the part of capital tied up in advancing wages, that is, variable capital, are both increased by the mark-up and a rate of profit π is computed on all the capital tied up in production. Our stock coefficient matrix, with its last row and column identified separately, will be

$$\mathbf{B} = \begin{bmatrix} B, & g \\ z, & 0 \end{bmatrix}.$$

B stands for means of production, z for variable capital and g is resources tied up in reproducing manpower. In parallel to the augmentation of the matrix A to $\mathbf{A}$, the matrix $\mathbf{B}$ is augmented into a comprehensive system by taking into account (in place of the former zeros) the resources tied up in the production of manpower.

The definition just given in equation (20) for two-channel prices is simply the positive left-hand eigenvector of the matrix $\mathbf{A} + \pi\mathbf{B}$.

$$p(\mathbf{A} + \pi\mathbf{B}) = (p, 1 + \beta)\left\{\begin{bmatrix} A & c \\ v & 0 \end{bmatrix} + \pi \begin{bmatrix} B, & g \\ z, & 0 \end{bmatrix}\right\} =$$

$$= (pA + \pi pB + (1 + \beta)v + \pi(1 + \beta)\, z, pc + \pi pg) =$$

$$= (p, pc + \pi pg) = (p, 1 + \beta) = \mathbf{p}.$$

Therefore the "two-channel" price system is a comprehensive production price system, computed from the matrix $\mathbf{B}$ of the total system, manpower reproduction included. The magnitude of π and of the primal growth rate are again equal – average rate of profit no longer exceeds the attainable rate of growth because the very sizeable investment in human beings is no longer neglected.

There is an interesting aside: as $\beta = \pi pg$ the mark-up cannot be chosen arbitrarily. It must equal the product of the rate of profit, π, and the resources tied up in reproducing manpower, pg. That is, mark-up is the cost of using those resources. Now if g is not specified beforehand – and its measurement must be left to the future because statistical data are insufficient — we can choose the mark-up, β, "freely" but the choice implies valuation of those resources. The formula is

$$pg = \frac{\beta}{\pi}.$$

For instance a 5 per cent rate of profit and a 30 per cent mark-up on wages implies $\frac{0.3}{0.05} = 6$ years income tied up in reproduction of manpower.*

There are certainly limits on the choice of β. If $\beta = 0$, then from $pg = 0$ and $p > 0$, $g = 0$ follows. This means that we are totally neglecting human investment and the resulting price system will be the classical production price system. This is one extreme of the "two-channel" price system.

Let us now assume as a second extreme that all the resources are considered human investment. Then all the elements of the matrix **B** become zero except its last column. The corresponding "two-channel" price system is

$$(p, 1+\beta)\left\{\begin{bmatrix} A, & c \\ v, & 0 \end{bmatrix} + \pi \begin{bmatrix} 0, & g \\ 0, & 0 \end{bmatrix}\right\} = (pA + (1+\beta)v,\ pc + \pi pg).$$

Thus $p = pA + (1 + \beta)v$ from which

$$p = (1 + \beta)v\,(1 - A)^{-1} = (1 + \beta)vQ\,.$$

But this is the familiar formula for the value price system. β is now surplus labor (included into the valuation, taking paid and unpaid labor into account). The form $\beta = \pi gp$ shows Expanded Reproduction of manpower following a π rate. The second extreme of "two-channel" prices is, therefore, the classical value price system. This can be seen intuitively: if we increase the mark-up on wages, the surplus value will all be allocated in proportion to wages. If all surplus is allocated in proportion to wages and none in proportion to capital, then we reach value prices. Thus value prices can be reinterpreted as production prices reckoning only with resources tied up in reproducing manpower. Considering all investment as human investment is equivalent to ignoring investment at all when computing prices.

Resources tied up in reproducing manpower tend to grow historically — not only in absolute terms but in relation to those tied up in production. Therefore we may conjecture that the properly orienting system of prices will deviate from classical production prices more and more and become nearer and nearer to value prices. If man becomes the greatest asset — if the resources tied up in its reproduction outgrow every other sort of funds — then the "two-channel" prices will approach value prices.

* These were approximately the implications of the Hungarian price reform of 1968. I suspect human investment was still underestimated by this.

Certainly the historical phase when we can neglect all investment except that tied up in reproduction of manpower is far off.

This, then, suggests how price systems change in the course of history. The value prices of simple commodity production evolved slowly into classical production prices. Production prices develop as production requires more and more funds to invest. In the beginning of commodity-exchange "plant and equipment" is negligible as compared to variable capital, thus the dominating form remains close to value prices. As production becomes more capital-intensive, human investment requirements lag. Classical production prices prevail.

Sooner or later developing technology demands qualified manpower. As resources tied up in reproducing manpower increase, the classical production price system ceases to orient properly. As human investment begins to dominate, the price system returns toward value prices.

Summing up: the appropriate price systems defined by the labor theory of value vary with historical changes in the economy. The labor theory defines the price systems as historical phenomena.

2.2. Circularity

A circular definition, a definition that defines some concept in terms of itself, *idem per idem*, was considered a grave fault by classical logic. But self-contained (closed) systems require circular definitions and modern scientific experience has demonstrated their power.

Classical labor theory thought itself free of such "logical blunders". It defined values by reducing them to labor expenditure, explaining them as a certain amount of crystallized labor energy. The corresponding mathematical equation was $p(1 - A) = v$, which could be solved as $p = v(1 - A)^{-1} = vQ$. That is, it started from direct labor expenditure, v, the *premises*, and from these *premises*, using the operator (matrix) Q, it deduced the *conclusion*, p. The prescription was to add up all the labor directly and indirectly expended on the product. This logical operation yielded the sought values.

The eigenequation for the closed model, $\mathbf{p} = \mathbf{pA}$, leading to the same numerical results, would have been considered circular by classical logic. It starts from $\mathbf{p}$ and deduces its own premises, $\mathbf{p}$, again. It defines values by values, *idem per idem*, moving in a circle rather than reducing values to something given previously. Nevertheless it is still a logically correct and fruitful formulation and in a sense a more general one than that for the open classical model.

This peculiar feature of circularity was a very important basis for criticism of Marx's labor theory of value.

Böhm-Bawerk [1896] was the first of a long series of critics who objected to this sort of reasoning. Rebuttals were unconvincing because they attempted to deny the circularity altogether. (See for example Hilferding [1904].) A more realistic and fruitful defense would acknowledge the circularity but go on to demonstrate the merits of such definitions in scientific economic thought.

It is no accident that Marx developed this new way to approach reality. He was a well read philosopher and a former follower of Hegel. Hegel, in an ingenious clumsy and sometimes mysterious way, anticipated many problems of modern scientific thought and set up the first new tools of a more flexible, dialectical logic. In the following we start with these "circular" eigenequation definitions of value and production prices and use them to answer the very questions that troubled Böhm-Bawerk. These are the problems of skilled and unskilled labor, and the famous problem of the transformation of values into production prices. I shall try to show that Marx's standpoint was entirely logical and consistent — and that mathematical difficulties arise when we try to eliminate circular definitions.

2.2.1. *Simple and skilled labor*

Our eigenequation for values was

$$\mathbf{p} = \mathbf{pA}\,.$$

It can be interpreted in economic terms: if we add the values of all the socially necessary expenditures incurred during production we obtain the value of the finished products. Value of manpower and value created by its activity are equal under Simple Reproduction; surplus product or surplus value is nil. This eigenequation simply states the law of the conservation of value: the sum of the values of the parts is the value of the whole. Yet it subsumes some economic laws that are far from trivial.

We set out from the labor theory of value when defining the value vector, $\mathbf{p}$, above. Labor expended by manpower as a source of value and labor time as an immanent measure of the magnitude of value play a central role. Yet with the same eigenequation any other product could be cast in the same role. It guarantees the necessary expenditure to reproduce the special product, manpower, but it also guarantees all the inputs for all the other products. In the eigenequation this special thing, labor, source and measure of value, is logically indistinguishable from other products and services.

The left-hand eigenvector of the matrix $\mathbf{A}$, the value vector defined according to the labor theory of value, certainly depends on labor expenditures. But the form of the eigenequation generalizes this dependency. The eigenvector always characterizes the whole matrix — and it can be expressed equally well as a function of any row, column or element of the matrix. Hence from the point of view of the proportions of the price system attained, we might single out any other product or any bundle of products, instead of labor, and shall obtain the same answer to the valuation problem.

In economic terms this means that we may start from any "primary" input — or rather consider any input as primary: coal expenditures, electric energy consumed, etc. We might even choose ferrous or phosphate content as "source and measure" of value. Using any output as numéraire we reach the same price proportions. This really is a natural consequence of the law of conservation of value: every product (even manpower) can only transfer that amount of value into other new products that it contained originally.

Numerical example

$$\mathbf{A} = \begin{bmatrix} 0.2 & 0.7 & 0.05 \\ 0.2 & 0.2 & 0.3 \\ 1 & 1 & 0 \end{bmatrix}.$$

We have seen that value prices are $\mathbf{p} = (2, 3, 1)$.

If we consider product I ("Tools") as numéraire, we have to cancel the first row and column; the remaining submatrix is $A_I = \begin{bmatrix} 0.2 & 0.3 \\ 1 & 0 \end{bmatrix}$ and its Leontief-inverse is $(1 - A_I)^{-1} = Q_I = \begin{bmatrix} 2 & 0.6 \\ 2 & 1.6 \end{bmatrix}$.

We form now the corresponding price system by premultiplying this inverse by the deleted row:

$$(0.7, \quad 0.05) \begin{bmatrix} 2 & 0.6 \\ 2 & 1.6 \end{bmatrix} = (1.5, \quad 0.5)\,,$$

and complete the price system by inserting the numéraire in its proper place:

$$\mathbf{p}_{\mathrm{I}} = (1, \ 1.5, \ 0.5)\,.$$

This price vector is proportional to our former price system $\mathbf{p} = (2, 3, 1)$.

Taking product II ("Material") as numéraire, we proceed as follows:

$$A_{\mathrm{II}} = \begin{bmatrix} 0.2 & 0.05 \\ 1 & 0 \end{bmatrix} \qquad Q_{\mathrm{II}} = \begin{bmatrix} 1.3 & 0.06 \\ 1.3 & 1.06 \end{bmatrix}$$

$$(0.2, \quad 0.3) \begin{bmatrix} 1.3 & 0.06 \\ 1.3 & 1.06 \end{bmatrix} = (0.6, \quad 0.3)\,,$$

and, after inserting the numéraire:

$$p_{\mathrm{II}} = (0.6, \quad 1, \quad 0.3)\,.$$

The proportions being the same as those of the former price systems we may state:

In the case of Simple Reproduction the value price vector can be computed by starting from any product's inputs or any combination of several product's inputs (weighed by their proper values).

This insight now leads to a new (or rather old) possibility of solving the problem of skilled and unskilled labor. The problem – as generally stated – is that labor, the source and measure of value, is not homogeneous. Not only its use value (as the carpenter's, mason's or spinner's work) but also its "value creating energy" is heterogeneous – one hour of a sculptor's labor counts as, say, five hours of a quarry worker's labor, the former being more skilled. According to Marx:

"Skilled labor counts only as simple labor intensified, or rather, as multiplied simple labor, a given quantity of skilled being equal to a greater quantity of simple labor. Experience shows that this reduction is constantly being made. A commodity may be the product of the most skilled labor, but its value, by equating it to the product of simple, unskilled labor, represents a definite quantity of the latter labor alone. The different proportions in which different sorts of labor are reduced to unskilled labor as their standard, are established by a social process that goes on behind the backs of the producers and, consequently, appear to be fixed by custom" [I. 44].

But Böhm-Bawerk interjects:

"How does Marx explain this? He says the exchange relation is this, and no other – because one day of sculptor's work is reducible exactly to five days of unskilled work. And why is it reducible to exactly five days? Because experience shows that it is so reduced by a social process. And what is this social process? The same process that has to be explained."

Marx has to explain the exchange relations found in the market. And how does he explain them? By recourse to the market itself. The market has first to decide how skilled and unskilled work will be reckoned. Only after this process of reduction to common units are we able to explain the exchange relations found on the market.

But there is really no need to consult the market to homogenize labor inputs. It was not the market but the social division of labor that Marx meant by the "social process". It is this division of labor that dictates the structure of the system and thus regulates the process "behind the backs of the producers".

All the products, therefore, really possess a definite value before entering the market. The market cannot decide their values but only their actual prices which may diverge from their values (or production prices) under unbalanced supply and demand conditions. A certain circularity characterizes the definition of the values (or value creating powers) of skilled and unskilled labor because the theoretical prescription is based on the eigenequation approach.

All we have to do is to disaggregate (or rather not to aggregate) the labor sector in our matrix **A**. If under Simple Reproduction we have as many rows and columns for labor as the number of different skills, we will still have a non-negative and irreducible matrix yielding a unique positive left-hand eigenvector: values. The relative weights for the different skills, that is, their values, can be used thereafter to homogenize labor to a common standard.

This might well be the very solution Marx had in mind when, on a later occasion, he elaborated the problem:

"All labor of a higher or more complicated character than average labor is expenditure of labor power of a more costly kind, labor power whose production has cost more time and labor, and which therefore has a higher value than unskilled or simple labor power. This power being of higher value, its consumption is labor of a higher class, labor that creates in equal times proportionally higher values than unskilled labor does" [I. 197].

The same procedure – disaggregation of the labor sector – can be used to compute production prices. All we have to do is to secure the same detail in matrix **B**, too, and we are ready to compute the production prices of different sorts of labor power.

If there is some surplus product and hence surplus value, we might be asked how much surplus value can be derived from the different skills. But we can raise the same question for homogeneous labor. It is not easy to decide whether the "homogeneous" labor employed in different sectors is exploited at a uniform rate.

Marx assumed a uniform rate of exploitation, and we may assume the same in the case of different skills.* It is not possible to prove or disprove either assumption.

There is no urgent need to inquire into differences in exploitation: if there is a surplus fed back into production, then we face Extended Reproduction and prices will oscillate around production prices and not around values. Under these con-

* See [III. 142–3].

ditions total surplus is divided among capitalists in proportion to the amount of capital invested. The total amount of surplus can be determined irrespective of its sectoral origin. It is "incidental and irrelevant" which sphere of production or which sort of labor yields the surplus when society is reckoning with prices of production. It is not the individual laborer but the laboring class that is exploited.

Similar reasoning applies to the problem of joint products. Thus far we have considered homogeneous products and heterogeneous labor inputs. In the case of joint production the inputs and the process are homogeneous and the product is heterogeneous.

The same method outlined above also solves the joint product problem when there are just as many products as processes.

Numerical example

Let us assume that, in our original example, one unit of product I is jointly produced with product II. To remain in the state of Simple Reproduction, Robinson has to consume it whether he likes it or not. Our previous matrix becomes:

$$\begin{bmatrix} 0.2 & 0.7 & 0.05 \\ 0.2 & 0.2 & 0.3 \\ 1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0.2 & -0.3 & 0.55 \\ 0.2 & 0.2 & 0.3 \\ 1 & 1 & 0 \end{bmatrix}.$$

The "input" coefficient -0.3 shows a consumption of 0.7 units and a production of 1 unit of product I. The values are

$$(2,\ 3,\ 1) \longrightarrow (10/7,\ 5/7,\ 1).$$

Formally, our procedure works, but now the matrix **A** is not necessarily non-negative. What really happens is that in the definitional equation $\mathbf{p1} = \mathbf{pA}$, the left-hand unit matrix has extra elements wherever joint products occur. But then it becomes a Neumann model with non-negative square matrices. The transformation into a Neumann model is necessary in the theoretically more general case, when – because of joint production – the number of products and of processes is no longer equal. To be more general, we must introduce rectangular matrices. As was noted in Part 1 the Neumann model is the most general of this family of models.

2.2.2. *The transformation problem*

The transformation problem is a traditional, almost sanctified one, settled "definitely" as many times as it has been opened and reconsidered. It is not even possible to find a rigorous statement of the problem that is generally accepted. The problem concerns the relation between value and production prices and has at least two major facets. The first is essentially computational. Given a system that produces some surplus, how are the production prices of commodities related to their values, how to effect the reduction of production prices to something already known? The second concerns the relation between economic history and the history of economic thought concerning value and surplus. What were the historical developments that led the classical economists to single out labor as the fundamental source of value and to reduce the production prices, somehow, to values? We begin with the computational problem.

For lack of adequate development of the necessary mathematics, it was not possible to pose the transformation problem rigorously. Marx's text contains correct and unassailable deductions and also undeniable weak spots. These problems were brought to light by Dmitriev [1904], Bortkiewicz [1907], Winternitz [1948], Seton and Morishima [1951]. None of their solutions is entirely adequate to the original formulations of Marx.

The weakest spot, discovered first by Bortkiewicz, is that Marx's thesis "... the sum of the profits in all spheres of production must equal the sum of surplus values, and the sum of the prices of production of the total social product equal the sum of its value" [III. 171] is not entirely correct.

Transcribed into our system, letting **p** stand for production prices and **r** for values, the thesis amounts to the double assertion

$$\mathbf{p}(1 - \mathbf{A})\mathbf{x} = \mathbf{r}(1 - \mathbf{A})\mathbf{x} \qquad \text{(for surplus)}, \tag{21}$$

and

$$\mathbf{px} = \mathbf{rx} \qquad \text{(for total product)}. \tag{22}$$

One of these equations can be always satisfied by using up the one remaining degree of freedom. For instance if we choose $(\mathbf{rx}/\mathbf{px})\mathbf{p}$ as our production price system, equation (22) will be fulfilled automatically. In this case, equation (21) might be reduced to $[(\mathbf{rx}/\mathbf{px})\mathbf{p} - \mathbf{r}]\mathbf{Ax} = 0$. This will hold only if the vector $(\mathbf{rx}/\mathbf{px})\mathbf{p} - \mathbf{r}$ is perpendicular to the vector $\mathbf{Ax}$, a particular and not a general case. Thus Bortkiewicz was right in his criticism. On the other hand $\mathbf{Ax}$ might be proportional to $\mathbf{x}$, in which case equations (21) and (22) are identically fulfilled, and Marx's thesis is right. Where the structure is close to Simple Reproduction, and $\mathbf{Ax} = \mathbf{x}$ (and this is the historical situation when the transformation happens), these circumstances might prevail.

Yet, even if the thesis is entirely right (or wrong) a deeper problem remains. The price system to be derived is not the bastard price system mentioned in Section 1.2.2, with an equal mark-up after costs, but the production price system. These two price systems are equal only if we assume one year turnover, and thus equate stock and flow. It is the same assumption playing havoc here, which was already mentioned before. In general the average rate of profit is not given by dividing total surplus by total product but by total surplus divided by total funds invested. The latter quotient yields the rate of profit we are really after.

On this point we find a strong spot in the original text:

"The average profit, determining the prices of production must always be approximately equal to that quantity of surplus value which falls to the share of the individual capital in its capacity of an aliquot part of the total social capital" [III. 179–80].

Surplus value divided by total funds will be approximately equal to the average rate of profit, whether numerator and denominator are measured in value prices or production prices. This fact (often useful in planning) has a mathematical explanation to be analyzed in Part 3. There, in connection with error analysis, we prove the relative stability, insensitivity of λ to errors both in the price system and in the structure of production. Here we are only concerned with the economic interpretation and with an appropriate algorithm for effecting the transformation, that is, for computing production prices.

The relation mentioned by Marx is simply

$$\lambda \approx \frac{\mathbf{p}(\mathbf{1}-\mathbf{A})\mathbf{x}}{\mathbf{pBx}}. \tag{23}$$

If the structure of production is proportional to the right-hand eigenvector, the balanced growth path, then $[(\mathbf{1}-\mathbf{A})\mathbf{x}]_i = \lambda[\mathbf{Bx}]_i$ for every i, and the relation exactly equals λ, whatever the price system. If the price system is proportional to the left-hand eigenvector, then again the quotient exactly equals λ, whatever distortion or deviation from the balanced growth path there may be. If the actual price system and the actual output proportions only approximate the theoretical ones (the eigenvectors), then the relation, too, will only approximate the true value of λ – but it will be a very good approximation, its error being much less than the error in the prices or outputs.*

How can we improve the price computation? Simply by iterating the procedure that leads from values to approximate production prices, transforming the approximate production prices to more accurate ones, and repeating the procedure until the necessary accuracy of the price vector is obtained.

The whole iterative procedure can be prescribed as

$$\mathbf{p}_{j+1} = \mathbf{p}_j\mathbf{A} + \frac{\mathbf{p}_j(\mathbf{1}-\mathbf{A})\mathbf{x}}{\mathbf{p}_j\mathbf{Bx}}\mathbf{p}_j\mathbf{B}, \tag{24}$$

and if after enough steps and with the precision necessary $\mathbf{p}_n = \mathbf{p}_{n+1} = \mathbf{p}$, then of course,

$$\mathbf{p} = \mathbf{pA} + \lambda\mathbf{pB}$$

and we have arrived at our solution for the price system.

In this algorithm total product, $\mathbf{px}$, is held constant. By postmultiplying both sides of equation (24) by the vector $\mathbf{x}$, we get $\mathbf{p}_{j+1}\mathbf{x} = \mathbf{p}_j\mathbf{x}$. Total product, measured by the successive approximative price systems, does not change. Here $\mathbf{x}$ can be

* A 10 per cent deviation in both (price and output) vectors can cause at most a 1 per cent relative error in the computed value of λ, and an even smaller error limit is likely because of further cancellation of errors of the data. (See Section 3.1.3. and Appendix II.)

arbitrary. However, the closer $\mathbf{x}$ to the right-hand eigenvector the more accurate the magnitude of $\lambda_j = \frac{\mathbf{p}_j(\mathbf{1}-\mathbf{A})\mathbf{x}}{\mathbf{p}_j\mathbf{Bx}}$, and if $\mathbf{x}$ is equal to the eigenvector, then λ_j is exact.

Now the transformation leading to the successive price systems being continuous and closed (because of $\mathbf{px}$ = constant, $\mathbf{p} \geq 0$) the procedure will converge to the fixed point given by the left-hand eigenvector, the production price vector.

This is a very quick and simple algorithm. Its main advantage lies in the fact that it uses the original $\mathbf{A}$ and $\mathbf{B}$ matrices throughout. Hence there is no accumulation of rounding errors.

The same procedure can be used to compute the right-hand eigenvector, the balanced growth path. Here the initial $\mathbf{p}$ can be chosen arbitrarily, but it helps a lot if the initial price system is as close to production prices as possible and thus yields an accurate initial value for the growth rate λ.

Certainly the most interesting algorithm would be one alternatively improving the left- and the right-hand eigenvectors. This procedure along the lines of the original Smith–Ricardo–Marx conception of equalizing rates of profit and regulating output proportions at the same time leads us too far from our main questions here.

The main practical lesson of the transformation problem lies in the question just discussed: how to compute production prices starting from a value price system – or more generally: how to improve a distorted price system? Planners certainly face this problem in their daily routine and are continuously seeking "more reliable" benchmarks. Hungarian planning practice (not just theory) abounds with computations in search of more meaningful guidelines than the simple current cost reckoned in the prevailing price system. Many computations resemble the procedure outlined above and arrive (by more or less naive labor-theory reasoning) at the same or similar results. Many are convinced that important proportions (say, the savings ratio, or the attainable growth rate, or the capital-output ratio, or the share of industry and agriculture in national income, etc.) are distorted by the current price system – and they usually are distorted in planned and market economies as well. The algorithm just presented may answer some practical needs in this area.

The above is only one facet of the transformation problem. Now we turn to its other facet, the alleged "antinomy" between the first and third volume of *Capital*.

2.2.3. *Value versus production price*

We have seen that, under conditions of Simple Reproduction, the role of labor as source and measure of value is indiscernible. Any other "source" may serve as well – the relative price system remains the same whether we start from labor power or any other product.

Under conditions of Extended Reproduction again, the price system can be constructed entirely on a "cost plus" principle and, once it is firmly established that this "plus" is proportional to capital, labor does not seem to play a special analytical role.

It is exactly on the threshold between stagnation and growth that the question emerges: whence the surplus? The mercantilists claim that it comes from commerce; the physiocrats from land. Each emphasizes an important element in mankind's first "take-off" period: the establishment of broader exchange relations and a primitive accumulation in agriculture ready to be fed back into production. The classical economists agreed that this surplus must be the fruit of labor and of increased labor productivity.

Classical economics is classical not because its authors were so much brighter – which they may have been – but because the contemporary historical picture was clear. Just savor Smith's deep and vivid impressions of the pin manufacture. Division of labor brought great jumps in labor productivity and opened the door to still more. The classical economics of Smith and Ricardo was indeed rooted in the labor theory of value. The same impression seems to attract minds toward a labor theory in more recent cases of take-off.

Marx, from the beginning of his economic studies, joined this strong tradition – perhaps more decisively since Ferguson's and Smith's theories were transmitted to him by no less respected a teacher than Hegel himself. The first great spontaneous clashes between laborers and capitalists also impressed him at this time. Along with his deep respect for Ricardo as a scholar, he was looking for economic facts and theories to explain the new historical events.

The following paragraph should show his esteem for Ricardo:

"At last, however, Ricardo comes on the stage, and calls to science: Halt! – The foundation, the starting point for the physiology of the bourgeois system – for the understanding of its internal organic coherence and life process – is the determination of *value by labor time*. Ricardo starts with this and compels science to leave its old beaten track and render an account of how far the rest of the categories it has developed and described – the relations of production and commerce – correspond or conflict with this foundation, with the starting point; how far in general the science that merely reflects and reproduces the phenomenal forms of the process – how far therefore also these phenomena themselves – correspond to the foundation on which the inner connections, the real physiology of bourgeois society, rests, or which forms its starting point; and what in general is the position with regard to this contradiction between the apparent and actual movement of the system. This is, therefore, the great historical significance of Ricardo for the science" [T. 203].

Still Marx finds theoretical faults to be corrected. He finds the most disturbing error in Ricardo's profit theory:

"He assumes a general rate of profit . . . Instead of assuming this general rate of profit in advance, Ricardo should rather have investigated how far its existence is in any way consistent with the determination of value by labor time; and he would then have found that instead of being consistent with it, *prima facie* it contradicts it and its existence has therefore to be explained through a number of intermediary stages – an explanation which is something very different from merely including it under the law of value" [T. 212].

Clearing up this point was one of the main tasks Marx set for himself. The problem appears very early in his thinking, and his first full length treatise written in 1857/8 already contained his final solution. [G. 339, 449, 592, 632–3, etc.]

The important questions he set about to answer were: What is the source of profit and what determines its magnitude? In Marx's time the mathematical determination of λ, as treated by Neumann, was inconceivable. Some solid groundwork was required first. The labor theory of value as applied by Marx gave an answer to these two questions. It showed that there might be a surplus even if exchange is regulated by values. With equal exchange on the market, without cheating, profits still might exist. Marx provided what we nowadays would call the "existence proof" of the profit. He points out that this proof of existence is missing in Ricardo's system and:

"Without this, the average profit is an average of *nothing*, a mere figment of the imagination. And in that case it might just as well be 1000 per cent as 10 per cent" [T. 231].

Clearly, Marx was trying to prove not only the existence but also the uniqueness of the average rate of profit. He posed the theorem of uniqueness correctly and suggested outlines for a rigorous proof, by trying to reduce production prices to values.

Yet, Marx tried to go deeper and explain the historical and logical transformation from values to production prices. An early letter to Engels clearly poses the main question, explains its solution and the necessity of relegating it to the third volume:

"How is the value of the commodity transformed into the production price of the commodity, in which

1. The whole labor appears as paid labor in the form of wages,

2. The surplus labor, on the other hand, that is, the surplus value, takes the shape of a plus above the cost price (= the price of the constant capital + wages) as interest, profit, etc.

The preconditions of the answer to this question are:

I. That the transformation of the daily value of manpower into the daily wages, that is, price of labor be explained. This takes place in Ch. 5 of this volume.

II. That the transformation of surplus value into profit, of profit into average profit, etc. be explained. This needs first the explanation of the circulation process of capital, because turnover time, etc. plays a part in this question. This question, then, can only be explained in the III. Volume" [W. 32].

The "preconditions" of this "transformation" are not simply logical ones; they describe the historical process whereby wage-labor, circulation of capital, etc. – not found under simple commodity production – gradually become everyday phenomena.

Böhm-Bawerk, possibly because he did not notice the historical aspect, objected very strongly:

"I cannot help myself: I see here no explanation and reconciliation of a contradiction but the bare contradiction itself. Marx's third volume contradicts the first." And why? Because: "Either products do actually exchange in the long run in proportion to the labor attaching to them . . . or there is an equalization of the gains of capital."

Yet, this objection might have been directed against Ricardo – but never against Marx. The different price systems belonged to different historical layers for Marx.

Still, for modern times, would it not be clearer to start not from values but from production prices? Some scholars advocate this approach. Schmidt [1892] claimed that value has no counterpart in real life. Sweezy [1946] takes a similar stand: "One might be tempted to go farther and concede that from a formal point of view it is possible to dispense with value calculation."

I felt that it is better to proceed along orthodox lines, starting with values. "The history of a thing is the thing itself" – said Hegel. Our ideas and categories are reflections of real processes and there is advantage in developing them in the same order as they appeared in history. Perhaps unwittingly, Joan Robinson [1947] furnishes the strongest argument for considering values as an acceptable historical phenomenon. She describes pricing in a future socialist economy:

"In the simplest case . . . if all incomes from surplus are abolished, prices would be regulated by wages cost *plus* depreciation.

This would be appropriate if investment has come to an end because no further increase in the stock of capital . . . In such a case capital, in orthodox language, has ceased to be a 'scarce factor of production', and the orthodox theory of prices would come to the same thing as the labor theory of value."

Now – instead of a future socialist economy – is this not a picture of the remote past before the advent of capitalism and growthmanship? Even the most minute growth rate, extrapolated backwards, makes national income dwindle in a couple of hundred years. It does not take much sense of history to see that most of mankind's history must have passed away in virtually dead calm, in Simple Reproduction.

2.3. Miscellanea

We now have to add some missing points to the discussion. The first two of them are already latent though not quite explicit in Marx's *Capital*: first, the problem of correct dimensionalities in economic science in general and in our model in particular and, second, the various interpretations given to λ, the average rate of profit, growth rate or "time factor". The goal here is the analysis of the "valuation of time" – the comparison of material expenditure and time expenditure.

The third point is not outright Marxian: mathematical formulation of economic theories encourages generalization of the underlying concepts and of the model based on them. The axiomatic approach spells out in detail all the abstractions and postulates of the model. Handled axiomatically, the model is ready for certain further generalizations. This, then, ends the discussion of the theoretical model and provides a transition to more practical problems – the application of the model to economic reality.

2.3.1. *Dimensions*

Clear and unambiguous definition of the correct dimensionalities serves three purposes. First, without spelling out the basic dimensions of measurement it is impossible to quantify scientific categories. Only if it is settled once for all that, say, the dimensionality of cubic content is $[L^3]$, where L stands for length, can we begin to measure and compute it. Dimensionality gives a correct prescription of what to do when changing units of measurement. Cubic content in feet (1/3 yards) must be $3^3 = 27$ times the cubic content measured in yards. Speed, that is $[LT^{-1}]$, where T stands for time, will be 60 times as much if measured in hours than if measured in seconds – the dimensionality T^{-1} correctly reminding us what to do when conversion of the time unit becomes necessary. Only dimensional analysis – be it simple or complex – can establish the multipliers for transition from one system of units to another one.

Second, dimensional analysis provides a check on the logic of equations. Even prominent economic models actually lack dimensional consistency. Sometimes the situation can be remedied by inserting the necessary constants of dimensionality. Consider the Cobb–Douglas function: $P = K^{\alpha}L^{1-\alpha}$ where P stands for production – measured in monetary units per year, K for capital – measured in monetary units (usually not the same ones) but without the dimension $[T^{-1}]$, that is: not per year but for a given moment, and L for labor – measured in, say, man years. Without appropriate constants of conversion, dimensional analysis raises serious questions:

$$\left[\frac{\text{Money}}{\text{Time}}\right] = [\text{Money}]^{\alpha}\ [\text{Man}\cdot\text{Time}]^{1-\alpha}$$

If α happens to be 1, this means $\frac{\text{Money}}{\text{Time}} = \text{Money}$, if $\alpha = 0$, it means $\text{Money} = \text{Man}\cdot\text{Time}^2$ and if $0 < \alpha < 1$, it does not mean anything. One can derive various other interesting imbecilities by changing units of measurement, for instance by measuring production by index-numbers.

Third, the most important outcome of dimensional analysis would be to help us to express economic laws in a way unaffected by changing units of measurement. Thus, the eigenvalues of the matrix **A** retain the same numerical magnitude regardless of changes in the physical or monetary units used for setting up the matrix. All other measures for the efficiency of an economic system will be affected by the price system or output proportions, etc.

Dimensional analysis is not unheard of in economic science, and I believe the first scholar to enter this field was Jevons [1888]. Soon after Jevons a correction was brought forward by Wicksteed [1889] who after paying tremendous lip service to Jevons' original thoughts succeeded in deriving almost exactly opposite dimensional statements. There has been little revival of interest since, except occasional meticulousness in questions of flows and stocks. Marx mocked the neglect of this question in his day:

"Capital — profit (profit of enterprise plus interest), land — ground-rent, labor — wages, this is the trinity formula . . . On closer examination of this economic trinity, we find the following:

First, the alleged sources of the annually available wealth belong to widely dissimilar spheres and are not at all analogous with one another. They have about the same relation to each other as lawyers' fees, red beets and music" [III. 814].

Let us inspect creation and flow of values. Value, as crystallized labor time is spread layer by layer continuously in the course of the productive process onto the already existing value of the object of labor. But the labor itself may be interrupted — and the value of the finished product will not flow but jump over to the next stage of production. As means of production its circulation will not be smooth but irregular. Streams, flows of means of production (or of their value) are therefore images of pure abstraction — the corresponding phenomena cannot be observed in economic life. Value circulates in quanta, quite spasmodically from one sector to another. It is the result of abstraction when weekly pay rolls, payed 52 times a year, are imagined as a stream of wages running smoothly into the pockets of laborers and then leaking out in exchange for streams of necessities.

Yet, we are pushed to this conception because, though the thing exchanged has a given, finite amount of value, stocked in it, still this value can be subdivided (at least mentally) infinitely, smoothly, continuously. The illusion stems from the money-form. Surely there is a minimal unit of money (a cent, a halfpenny, a

centime) but firstly it is very small compared to the total amount of wealth and secondly, nobody can stop us in subdividing it, as done routinely in cost estimation.

Divisibility of money helps us to think in terms of streams of value:

"The distinct attribute — whether it serves as the money-form of revenue or capital — changes nothing in the character of money as a medium of circulation; it retains this character no matter which of the two functions it performs" [III. 445].

But could we not do without this artificial separation? Extended Reproduction, growth itself seems to require it. In continuous or discrete terms we must discern change itself from what is changing. If we settle for continuity we must distinguish between a given magnitude and the speed of its change — leading us in mathematical terms to the value of some function and the value of its derivative. If we assume discrete growth, then the principal sum and the jump in it are apparently more homogeneous — but the jump is still specified for a time interval while the amount itself (which was or will be increased by the jump) remains "timeless" — insensitive to changes in the unit of time.

Thus we distinguish conceptually value from value stream. We designate the former dimension (based on German Werth and English Worth) symbolically by $[W]$, and value streams, therefore, will have the dimension $[WT^{-1}]$ — intensity of flow. Provisionally this means only that if we change unit of time from, say, a year to a month, then the numerical magnitude of a given value flow will become a twelfth of its former magnitude while all the variables of dimension $[W]$ remain unchanged.

But what is then the measure of value? How is its dimension connected with other things observed in economics? Labor theory of value offers an answer. Value is created by labor:

"Labor has incorporated itself with its subject: the former is materialized, the latter transformed. That which in the laborer appeared as movement, now appears in the product as a fixed quality without motion. The blacksmith forges and the product is a forging" [I. 189].

"While the laborer is at work his labor constantly undergoes transformation: from being motion, it becomes an object without motion; from being the laborer working it becomes the thing produced. At the end of one hour . . . a definite quantity of labor . . . has become embodied" [I. 180].

We designate this, work, motion — labor in short — by the symbol $[L]$. The quantity of labor expended $[LT]$, yields value. From this it follows that $[L] = [WT^{-1}]$, labor has the dimension of a value stream.

But how to measure labor? Could it not be measured by its number? But this is like measuring radioactivity of metals by their weight or cubic content, neglecting the specific activity itself. Manpower can work more or less intensely, on a more or less skilled job — and the result will be quite different values created. Perhaps labor of average skill could be measured, as we have seen in Section 2.2.1, by the differences of the cost of reproducing the skills needed. But the differing intensity of labor, or skills, in spite of being hard facts of everyday life, defy objective measurement and can be judged only very indirectly in terms of training costs, or of productivities.

Therefore, we cannot find the exact measuring rod by going back from value to labor and from labor to manpower. Value, it seems, has no intrinsic unit. In our model value had to be determined in a circular way. We are able to determine proportions but not absolute magnitudes. There is one degree of freedom left for the measuring rod; it has no absolute unit element. But we are in the same philosophical position about length or time units. They too are a matter of convention and there is no inbuilt, intrinsic, objective unit for them either.

What can we do? We arbitrarily fix, say, one hour of labor of average skill and intensity as the unit to be used. Marx did this for didactical reasons in the first volume of *Capital,* and only this enabled him to deduce value prices (in contrast to production prices) in a seemingly non-circular manner. We do the same here when analyzing the dimensionality of our model — we consider the unit of value to be fixed somehow at the outset, and then build further on the dimension $[W]$, now considered unequivocally measurable.

We started in setting up the model by defining input coefficients. Their dimensions cover a wide variety according to the socially accepted standards of measure of the individual products.

"The diversity of these measures has its origin partly in the diverse nature of the objects to be measured, partly in convention" [I. 36].

Though usual input–output tables are expressed in money terms, it is correct to start theoretically from physical units and derive value (or money) terms. In principle every sector could have a different unit — piece, kilogram, liter, calorie, etc. We designate it by $[i]$. Of course the same dimension and same unit must be used for a given sector (or product) throughout. The dimensionality of the input coefficient is therefore $[a_{ik}] = [i/k]$.

We then derive output proportions from these input coefficients. The elements of the output vector must be measured in the same units. The dimensionality of $[x_i]$ is $[iT^{-1}]$.

The sort of time introduced here is peculiar, it has holes in it. Suppose that a given enterprise works only one eight-hour shift per day. Its output still will be measured per 24-hour day, and for that matter, per 365-day year, irrespective of the actual number of days (or hours) worked. We use different scales for labor time, turnover time and calendar time and this complicates planning and logistics.

The product Ax has the same dimensionality as x; it is a flow. In multiplication the k dimensions cancel

$$[a_{ik}x_k] = [i/k]\,[kT^{-1}] = [iT^{-1}]\,.$$

Multiplication by the matrix A does not change the dimensionality of x in spite of the fact that the matrix contains a welter of dimensions.

The choice of a manpower unit was discussed above. It was chosen as common labor of average skill and intensity per hour. Thus the row of our labor sector coefficients will have the dimension $[v_i] = [W/i]$ and hence the column coefficients $[f_k] = [k/W]$.

A given number of laborers can create an ever-increasing amount of wealth if intensity, skill and technology develop. Certainly the unit of measurement will change when average skill and intensity increases. But this is a separate problem of long-range measurement not to be discussed here.

If we look at the last element of the product Ax its dimensionality will be

$$[v_ix_i] = [W/i]\,[iT^{-1}] = [WT^{-1}]$$

a flow dimension again.

To investigate value prices we start from the form $p = vQ = v + vA + \ldots + vA^n + \ldots$ yielding

$$[p_k] = [v_ia_{ik}] = [W/i]\,[i/k] = [W/k]$$

price is thus the value of a unit of the product.

Yet, the last element in the price vector will be different:

$$[p_n] = [p_if_i] = [W/i]\,[i/W] = [1]$$

a dimensionless number! It is a pure number determining the ratio of the price (value) of labor to the value created by it.

The bilinear form pAx has, therefore, the dimension

$$[p_ia_{ik}x_k] = [W/i]\,[i/k]\,[kT^{-1}] = [WT^{-1}]\,,$$

a flow dimension as expected.

In practice, now, we reckon not with these theoretical dimensionalities but with a matrix **A** already expressed in money terms. But it is easy to show that the operations amount to almost the same as formerly.

The elements of a practical matrix possess the form p_ia_{ik}/p_k. Their dimensionality is therefore $[W/i]\,[i/k]\,[W/k]^{-1} = [1]$, that is, they are pure numbers, ratios (the proportions of cost). Multiplication therefore is dimension-preserving as before. Yet, the value price vector changes in an interesting way because the vector of labor inputs becomes dimensionless. The price computation must then yield a dimensionless vector: it is the value price index.

The bilinear form pAx still will have flow dimensions because we reckon output intensities in money terms and the new dimensinos actually reduce to the old ones:

$$[1]\,[p_ia_{ik}/p_k]\,[x_kp_k] = [p_ia_{ik}x_k] \text{ as formerly.}$$

Capital coefficients were defined as products of input coefficients and turnover times — and here time acquires a new role, to be analyzed later in more detail:

$$[b_{ik}] = [a_{ik}t_{ik}] = [i/k]\,[T]\,.$$

The capital/output ratio is the matrix **B** premultiplied by prices

$$[p_ib_{ik}] = [W/i]\,[i/k]\,[T] = [W/k]\,[T]$$

and finally the bilinear form, pBx, for total stocks is

$$[p_i b_{ik} x_k] = [W/k]\,[T]\,[kT^{-1}] = [W]\,.$$

The dimension is pure value as expected.

Our model, therefore, is founded on the basic dimensional dependence — remaining the same in theoretical and practical computation:

$$[\mathbf{p}(\mathbf{1} - \mathbf{A})\mathbf{x}] = [\lambda]\,[\mathbf{pBx}]$$

that is

$$[WT^{-1}] = [\lambda]\,[W]$$

hence the correct dimensionality for λ must be $[T^{-1}]$, the reciprocal of time. We can turn now to analysis of λ or "the time factor".

2.3.2. *The time factor*

Time plays various roles in economics. It influences the process of reproduction from several sides. To distinguish among them appeared as hair-splitting pedantery, but was indispensable for the clear description and definition of the concepts used.

"Thrifty use of time . . . remains the first economic law in collectivist production. It even becomes a more strict law. Yet, this is essentially different from the measurement of exchange value (labor or product of labor) by labor time" [G. 89].

Besides the usual "calendar time" we have mentioned two special sorts of time: labor time and the other sort of time that has to be used thriftily: turnover time. It is important to work out in detail how to measure and balance them in the determination of λ.

The dimension of λ was $[T^{-1}]$, the reciprocal of time. This is more difficult to grasp than the dimension of $1/\lambda$, time itself. We shall try to interpret both forms.

The first interpretation is given by our former equation (23)

$$\lambda = \mathbf{p}(\mathbf{1} - \mathbf{A})\mathbf{x}/\mathbf{pBx}$$

and is well known: net product of society (profit) divided by total stocks (total capital employed). The numerical magnitude is influenced by the unit of time fixed for measuring the t_{ik} turnover times implicit in $\mathbf{B}$.

If we want to interpret equation (23) in terms of total value of production we multiply numerator and denominator by $\mathbf{px}$

$$\lambda = \frac{\mathbf{p}(\mathbf{1} - \mathbf{A})\mathbf{x}}{\mathbf{px}} \cdot \frac{\mathbf{px}}{\mathbf{pBx}}\,. \tag{25}$$

The first factor is the saving ratio (net product to be accumulated divided by total production). It is a pure number — a dimensionless ratio. The second factor

is "capital productivity", the reciprocal of the capital/output ratio. Equation (25) tells us that the growth rate, λ, is equal to the saving ratio divided by the capital/output ratio. This is the well-known formula of the Harrod–Domar type growth models. We touch upon this connection with aggregated growth models again in Part 3.

Let us turn now to the reciprocal where we insert — to facilitate interpretation — the scalar $\mathbf{pAx}$

$$1/\lambda = \frac{\mathbf{pBx}}{\mathbf{pAx}} \cdot \frac{\mathbf{pAx}}{\mathbf{p}(\mathbf{1} - \mathbf{A})\mathbf{x}}\,. \tag{26}$$

The two factors on the right side again express — in a somewhat generalized form — familiar concepts. The first factor, taking into account the definition $b_{ik} = a_{ik}t_{ik}$, is

$$\mathbf{pBx}/\mathbf{pAx} = \sum_i \sum_k p_i a_{ik} t_{ik} x_k \Big/ \sum_i \sum_k p_i a_{ik} x_k\,.$$

The product $p_i a_{ik} x_k$ is the input stream flowing from sector i to sector k multiplied by its price. Let us designate it by s_{ik}, yielding

$$\mathbf{pBx}/\mathbf{pAx} = \sum_{i,k} s_{ik} t_{ik} \Big/ \sum_{i,k} s_{ik}\,.$$

This reveals that our first factor is a weighted average of the turnover times, the weights being the corresponding input streams. Thus the first factor is simply average turnover time. It is a real average, the weights being, properly, those product flows, s_{ik}, that are tied down for the time intervals, t_{ik}.

Average turnover time, then, is inversely proportional to growth rate: if average turnover time could be cut in half, growth rate would double. The first factor has the dimension time and is measured in the same unit as used for turnover times.

The second factor of our expression above is a dimensionless ratio, converting, as it were, the time unit to a smaller one. Expressed as $\mathbf{pAx}/\mathbf{p}(\mathbf{1} - \mathbf{A})\mathbf{x}$ it could be called the input/savings ratio. It is remarkably stable in the long run, its value being around 10 in most developed countries. Thus, when t_{ik} is measured in years, this second factor will change the unit of measurement to approximately 36.5 days. Hence $1/\lambda$ shows for what multiple of the 36.5-day period the average input is tied down; and λ shows what proportion of inputs will be recovered in a period of 36.5 days.

If the average rate of profit is around 10 per cent, under the above circumstances, one-tenth of the inputs will be recovered in 36.5 days and this again amounts to an approximate average turnover time of one year. These were the orders of magnitude Marx reckoned with in his day and his assumption of a one-year period of turnover seems entirely warranted, not only as a theoretical simplification but as an everyday observation, too. The predominance of agriculture with its monotonous yearly periods and a flat average of one-year turnover time in manufacture both worked in this direction.

Economic reality has changed quite a lot since. A higher capital/output ratio, the increase in funds tied up in reproducing manpower that is still going on, have altered the overall picture quite a lot. On the other hand, there have been numerous inventions which shortened turnover periods by improving communications, saving transportation, etc. We shall return to a closer inspection of these historical trends in Part 3 in analyzing the application of the aggregate form of our model.

Here we pose another, more theoretical problem: what is the exact interdependence between turnover time and inputs, what is the relation between material and time expenditure, or to use an inexact but more intuitive wording — what is the value of time?

The labor theory of value does not attribute any "value" to things that are not produced and not reproducible by human labor. Thus if time is assigned any valuation, it can be only a reflection analogous to rent of land or other scarce factors. Most naturally, under conditions of Simple Reproduction no intrinsic value can be ascribed to time itself — hence the customary neglect of time in stagnating societies. There is no reason to expedite matters, so long as product require the old amounts of material and labor expenditure the acceleration of any process will bring no growth whatsoever. Simple Reproduction cannot be changed to Extended Reproduction by decreasing turnover times. The "take-off" from Simple to Extended Reproduction can be triggered only by some change in the matrix **A**, making its maximal eigenvalue less than 1, that is, by changing the flow coefficients themselves, improving technology, abolishing some layers of unproductive consumption.

Under Extended Reproduction timing becomes an important dimension. Then we can increase the rate of growth not only by economizing on inputs, but also by reducing turnover times. Accelerated flow in the channels on industry or commerce will therefore increase the profit rate, λ, and hence the pace of growth. If time-saving methods affect the bulk of products — and modern techniques, including more rapid transportation and communication, rationalized financial systems, etc., are geared to this purpose — then their cumulative effect on growth can surpass the influence of pure economies of material and labor. Therefore, turnover time becomes valuable, an object of economizing, and has to be used thriftily, but only under circumstances of Extended Reproduction.

How can we now compare economy of time and of material? We may answer this question from a macroeconomic standpoint, calling on our earlier formulas and investigating the effect of the two factors, average turnover time and average expenditure, on the growth rate, λ. But we may investigate the same matter from a microeconomic standpoint, too, considering only one economic transaction, a single element in our matrix $\mathbf{A} + \lambda\mathbf{B}$.

For a change let us start from the latter, considering a single "stream" in the economic metabolism. Each element of our matrix $\mathbf{A} + \lambda\mathbf{B}$ is made up of two parts. For instance, consider coal (product i) delivered for producing electricity (product k). a_{ik} will be the flow coefficient and $\lambda b_{ik} = \lambda a_{ik} t_{ik}$ the increase in coal inventory made necessary by the yearly growth rate, λ, the turnover time of this item of inventory being t_{ik}.

Our problem is to find that decrease in turnover time which is exactly counterbalanced by an increase in coal consumption, so as to leave the magnitude of the total stream and thus the growth rate unaltered.

Let us therefore assume a constant stream of coal input into electricity. Since the same relation will hold for all coefficients a_{ik}, we can generalize by dropping subscripts.

$$a + \lambda a t = \text{constant} = c.$$

Expressing a as function of t we get

$$a = \frac{c}{1 + \lambda t}.$$

Its logarithmic derivative is therefore

$$\frac{\partial a}{a} \frac{t}{\partial t} = -\frac{\lambda t}{1 + \lambda t}$$

signifying that a one per cent decrease of turnover time can be counterbalanced by a $\frac{\lambda t}{(1 + \lambda t)}$ per cent increase of material expenditure, in this example, coal consumption. The effect of a decrease of turnover time therefore will be the greater our growth rate, λ, and the longer turnover time, t, itself.

To illustrate the orders of magnitude by a numerical example, let us assume a 10 per cent annual growth rate and a three months' ($t = 0.25$) coal inventory. If the coal inventory could be decreased to a two months' inventory, then this 33 per cent decrease would be counterbalanced by an approximately $\frac{0.025}{1.025}\,0.33 \approx 0.008$ increase in coal consumption. This magnitude now is roughly equivalent to the amount yielded by the usual, everyday computation based on λ as the "rate of interest". Originally coal was tied up in inventory for three months and its cost measured by compound interest was $a(1 + \lambda t)^t = a(1.01)^{0.25}$. If now it is only tied up for two months, its cost with compound interest will be $a(1.01)^{0.16}$. The relative saving will be $\frac{a(1.01)^{0.25} - a(1.01)^{0.16}}{a(1.01)^{0.25}} = 1 - (1.01)^{-0.09} \approx 0.008$, as before.

Let us now proceed to the macroeconomic level. We are now interested in economy-wide averages. How can now these two averages move so as to counterbalance each other, leaving the rate of profit and thus the growth rate, λ, unaffected? From the reciprocal of equation (26) we have $\lambda = \frac{\mathbf{pAx}}{\mathbf{pBx}} \cdot \frac{\mathbf{p}(1 - \mathbf{A})\mathbf{x}}{\mathbf{pAx}}$. It is clear that every percentage change in the average turnover time, $\mathbf{pBx}/\mathbf{pAx}$, must affect the growth rate by the same percentage but in the opposite direction. The effect of average inputs can be handled by defining them as $\alpha = \mathbf{pAx}/\mathbf{px}$. In this case the second factor in the expression above will assume the form $1/\alpha - 1 = (1 - \alpha)/\alpha$. A one per cent change in average inputs will cause a $1/(1 - \alpha)$ per cent change in the value of λ.

Thus a one per cent increase in average input requirements must be compensated by a $1/(1-\alpha)$ per cent decrease of average turnover time. α being approximately equal to 0.9 (this is only another way of saying that $\mathbf{pAx}/\mathbf{p}(1-\mathbf{A})\mathbf{x} = \alpha/(1-\alpha)$ is generally around 10) a one percentage change in average expenditure can be offset by approximately a ten percentage change of average turnover time in the opposite direction.

Expenditure of time and of products (services, materials, labor) is therefore commensurable in the framework of the labor theory of value. The computation might be based on the "time factor", λ (the growth rate, rate of profit which is – on this level of abstraction – equivalent to the rate of interest), or on the average input coefficient α. They are connected by the symbolic equation $\alpha + \lambda t\alpha = 1$ whence

$$\frac{1+\lambda t}{\lambda t} = \frac{1}{1-\alpha} \tag{27}$$

can be derived easily. The left side shows the conversion factor from the "microeconomic" standpoint; the right side shows the same from the "macroeconomic" aspect – and both show the general tradeoff for time and product expenditure.

2.3.3. *Generalization*

We consider three ways to generalize the model. First we try to make explicit the minimal basic assumptions underlying the model. Second we show how a slight generalization of the mathematical apparatus of the model enables it to subsume non-linear and more dynamic features. Third we reinterpret the model in a probabilistic way, giving a new interpretation to the stationary states (eigenvectors).

None of the three topics is treated exhaustively. The main object is to show that there are various possibilities for further theoretical generalization and development. The most promising directions are only indicated but not thoroughly explored.

The first possibility is an axiomatic approach. By screening the assumptions leading to our model, there appear to be six that are necessary:

1. We know certain distinguishable things and these we call products, numbered from 1 to n (Identification).
2. Every product is measurable (Weighs or is countable, etc.).
3. Every product is divisible without limit (Continuity).
4. There exists a system making or creating these products by means of the same products and, at the same time and with the same activity, consuming or annihilating them. This activity of the system is called production.
5. The kth product can be produced by the system only by annihilating quantities $a_{ik} \geq 0$ of product $i (k, i = 1, 2, \ldots, n)$.
6. From the instant when product i was created to the instant when it was annihilated to produce k some time elapses. This time span is probabilistic, has an exponential density function and its expected value is $t_{ik} > 0$ $(i, k = 1, 2, \ldots, n)$. The product created but not yet consumed is called stock.

Now let us deduce our model from these 6 assumptions.

First we construct the matrix $\mathbf{A} = \{a_{ik}\}$ which is square and of order n. By assumption 5 it exists and is non-negative. Let us designate the products created in a very short, dt, time interval by $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. By assumption 2 products are measurable and by assumption 3 they can be measured in short, dt, intervals.

This vector $\mathbf{x}$ is produced by annihilating other already existing stocks of products. But in a short interval, dt, the proportion of stock i consumed to produce product k must be $\frac{dt}{t_{ik}}$. By assumption 6 we have an exponential density function, prescribing exactly this rate of mortality in the interval dt.

Thus, to make production $\mathbf{x}$ possible in the interval dt, we must have stocks enough, that is, t_{ik} times the amount used up. Hence total stocks must be $\{a_{ik}t_{ik}\}\mathbf{x}$. We will call the matrix $\{a_{ik}t_{ik}\}$ the stock matrix, $\mathbf{B}$; thus total stocks are $\mathbf{Bx}$.

Production of $\mathbf{x}$ annihilates a stock of amount $\mathbf{Ax}$. The difference between production and consumption is change of stocks. But change of stocks in the interval dt will be $\mathbf{B}\dot{\mathbf{x}}$, where $\dot{\mathbf{x}} = (dx_1/dt, dx_2/dt, \ldots, dx_n/dt)$ giving

$$\mathbf{x} - \mathbf{Ax} = \mathbf{B}\dot{\mathbf{x}}\,. \tag{28}$$

This is our model in the form of a differential equation. If it is solvable at all, we can take $\dot{\mathbf{x}} = \lambda\mathbf{x}$ which leaves us with $\mathbf{x} = (\mathbf{A} + \lambda\mathbf{B})\mathbf{x}$.

Yet, the six assumptions above are not sufficient to secure existence and uniqueness of the solution in $\mathbf{x}$ and λ; they suffice only to set up equation (28).

To secure a positive and unambiguous solution we introduced further assumptions, namely

7. $|\mathbf{A}| \leq 1$.
8. $\mathbf{A}$ is not reducible.

These are really not necessary assumptions, but they suffice to insure a unique positive solution. Because of assumption 8 not only $\mathbf{A}$ but also $\mathbf{B}$ is irreducible. Thus $(\mathbf{A} + \lambda\mathbf{B})$ must be a non-negative and irreducible, that is, Frobenius matrix with a positive eigenvector. From assumption 7 either $|\mathbf{A}| = 1$, the case of Simple Reproduction, and $\lambda = 0$, or $|\mathbf{A}| < 1$, the case of Extended Reproduction, and $\lambda > 0$.

Both assumptions 7 and 8, are easily justified by economic reality. Of the first six assumptions three need some additional comment because they are not entirely realistic.

Assumption 3. Unlimited divisibility of products. We certainly can point out quite a few instances where this assumption is wrong. Nevertheless with increase of the scale of production this assumption becomes more and more acceptable: with increase of the number of the same, individually indivisible, product production will be more and more finely subdivisible – just as the rational numbers $1, 2, \ldots, n, \ldots$ become the more divisible the greater they are.

Assumption 5. There is only one technological possibility for producing a given product. This assumption becomes more palatable if we view each sectors' technology as an average. For the future the assumption is misleading. It needs to be corrected or complemented to describe technological change correctly.

Assumption 6. Exponential density function of life spans. We have not enough facts at hand to prove or disprove this assumption. It is recommended as more realistic than the usual assumption of fixed life spans.

These eight assumptions seem to be acceptable from a theoretical and practical standpoint as long as we cannot improve them. For the time being I do not see better ones. Still let us suppose that, with increasing knowledge, we can set up better assumptions concerning technologies and life spans. Will our model be flexible enough to incorporate them? Of course it is difficult to prejudge the impact of an unknown innovation. But the mathematical apparatus of our model is quite suitable for further generalization and seems to be flexible enough to permit considerable modification. This leads us in the direction of the second generalization.

For many applications constant coefficients must somehow be made into more flexible representatives of real technical and market conditions. It is relatively simple, though not entirely satisfactory, to make the elements of the matrices **A** and **B** depend explicitly on time yielding the system of equations $\mathbf{x}_t - \mathbf{A}_t\mathbf{x}_t = \mathbf{B}_t\dot{\mathbf{x}}_t$.

The mathematical theory of the latter system is essentially analogous to that of the former one: both are linear differential equations (with constant or variable coefficients). Thus their solutions and the techniques of solving them are very similar. Certain practical computations have already been done for the latter, time-varying system based on extrapolations of observed changes in the matrices. We return to them in Part 3.

Linear operators afford a convenient general method for introducing changes in coefficients over time. Let us assume that our coefficients, a_{ik}, depend linearly on time, and on present, past and future values of the elements of $\mathbf{x}_t$ and its derivatives and integrals. We can specify our assumptions in a model entirely analogous to our fixed coefficient model, except that in place of our former matrices we use linear operators. Time shifts, differentiation and integration, being linear operations, can each be represented by a simple linear operator. Now a one-to-one correspondence can be established between linear operators and matrices. Therefore computationally, linear operators can be treated as if they were ordinary matrices. Thus, all our former tools can be reinterpreted in the world of operators. There do exist non-negative and irreducible operators and such operators still possess an unambiguous positive eigenvector and eigenvalue. Of course, the eigenvector will be a somewhat more complex phenomenon: not a simple vector of stationary proportions but a vector made up of time-functions of outputs or prices. The same dual relation will persist: there are adjoint or transposed operators ready to define dual or valuation relations, too.

The new form would encompass a very broad field of possible interrelations. But our practical experience is not broad enough to implement such a model in any realistic way. We do not know enough to write out explicitly how our coefficients ("inner proportions") depend on time, on past and future outputs, on prices, etc. And therefore I do not see much reason to enter into a detailed study of operator models. Suffice it to stress that the possibilities of theoretical generalization far exceed the data at our disposal. It is lack of pertinent information and not lack of adequate mathematical or computational tools that blocks our way.

A third way of possible generalization can reduce the rigidity of our assumptions without really affecting the fundamental mathematical form and apparatus. This is the probabilistic reinterpretation of our model already suggested by Theil [1965]. The essence of this approach is to view our coefficients as random variables. Instead of fixed magnitudes they are specified as expected values and probabilities. This certainly offers more flexibility in interpreting the fluctuations observed in real processes.

For Simple Reproduction a probabilistic interpretation does not need any additional mathematical tools. To apply it to Extended Reproduction requires a longer exposition than is warranted here where we are concerned only with general methodological directions.

We know that, for Simple Reproduction, our matrix **A** is non-negative, its maximum eigenvalue equals one and it is irreducible. Under these assumptions we can characterize the process of Simple Reproduction as a so-called ergodic Markov chain, by transforming our matrix **A** into a transition-probability matrix.

As we already know, under Simple Reproduction there exist positive left- and right-hand eigenvectors $\mathbf{p} = \mathbf{pA}$ and $x = \mathbf{A}\mathbf{x}$. Let us now denote the diagonal matrices formed of the elements of these eigenvectors by $\langle\mathbf{p}\rangle$ and $\langle\mathbf{x}\rangle$, that is,

$$\langle\mathbf{p}\rangle = \operatorname{diag}\langle p_1, p_2, \ldots, p_n\rangle = \begin{bmatrix} p_1 & & & \\ & p_2 & & \\ & & \ddots & \\ & & & p_n \end{bmatrix}$$

$$\langle\mathbf{x}\rangle = \operatorname{diag}\langle x_1, x_2, \ldots, x_n\rangle = \begin{bmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{bmatrix}$$

We form the matrices $\mathbf{C} = \langle\mathbf{p}\rangle\mathbf{A}\langle\mathbf{p}\rangle^{-1}$ and $\mathbf{D} = \langle\mathbf{x}\rangle^{-1}\mathbf{A}\langle\mathbf{x}\rangle$. Both can be interpreted as stochastic matrices with elements that represent transition probabilities. Evidently $c_{ik} \geq 0$. Note that $\sum_i c_{ik} = 1$ $(k = 1, 2, \ldots, n)$ because, if we premultiply **C** by the summing vector $\mathbf{e} = (1, 1, \ldots, 1)$, we get $\mathbf{eC} = \mathbf{e}\langle\mathbf{p}\rangle\mathbf{A}\langle\mathbf{p}\rangle^{-1} = \mathbf{pA}\langle\mathbf{p}\rangle^{-1} = \mathbf{p}\langle\mathbf{p}\rangle^{-1} = \mathbf{e}$. The same follows for **D** if we postmultiply it by **e**. Thus $d_{ik} \geq 0$ and $\sum_k d_{ik} = 1$, $(i = 1, 2, \ldots, n)$.

Hence all the column sums of **C** equal 1 and similarly all the row sums of **D** equal 1, while all elements are non-negative. Thus the formal conditions for interpreting them as stochastic matrices are satisfied.

But what does "probability of transition" really mean here — what is the economic point of it? Certainly the matrix $\mathbf{C}$ reflects cost-structures, its column k representing percentages of the necessary ingredients to produce product k. This now is the mixture in which sector k wants to buy on the market. The probability of buying from sector i is exactly c_{ik}. It should be understood in the following way: Sector k goes to market and will buy one day from one sector and another day from others. Its purchases may have an apparently irregular pattern. Some days it may not buy anything because inventories are full, to be depleted at random. Nevertheless the probabilities of spending will be allotted to the other sectors as the coefficients c_{ik} — and the real frequency of purchases, followed through, say, a year, will approach this probability.

The purchases of sector k depend not only on these probabilities but also on the amounts other sectors purchase of its products. Sector k fills its inventories to supply its customers with its product.

Let us now suppose that at a given moment $t = 0$ the sectors want to purchase $\mathbf{q}_0 = (q_{10}, q_{20}, \ldots, q_{n0})$ amounts on the market. How much additional purchase will this trigger in the next round? We do not know exactly because actual purchases fluctuate around the probabilities, but we know what the probabilities are. The expected values for the next round will be $\mathbf{q}_1 = \mathbf{C}\mathbf{q}_0$, and for the second round $\mathbf{q}_2 = \mathbf{C}\mathbf{q}_1 = \mathbf{C}^2\mathbf{q}_0$ and so on. In general $\mathbf{q}_n = \mathbf{C}\mathbf{q}_{n-1} = \mathbf{C}^n\mathbf{q}_0$.

The theory of Markov chains now can answer two important questions: do the values of $\mathbf{q}_n$ converge to some limit (in a probabilistic sense) as n increases?; and if it does, is this limit, the stationary state, independent of the initial state, $\mathbf{q}_0$?

For our simple kind of model it has been established that there is a limiting distribution* that is independent of the starting state. It can be computed by solving the equation $\mathbf{q}^* = \mathbf{C}\mathbf{q}^*$, that is $(1 - \mathbf{C})\mathbf{q}^* = 0$.

In our case this stationary state equals $\mathbf{q}^* = (p_1x_1, p_2x_2, \ldots, p_nx_n)$ — a term-by-term product of our two eigenvectors. Thus it is equal to the elements of the output vector reckoned in value prices

$$\mathbf{C}\mathbf{q}^* = \langle\mathbf{p}\rangle\mathbf{A}\langle\mathbf{p}\rangle^{-1}\begin{bmatrix} p_1x_1 \\ p_2x_2 \\ \vdots \\ p_nx_n \end{bmatrix} = \langle\mathbf{p}\rangle\,\mathbf{A}\mathbf{x} = \langle\mathbf{p}\rangle\mathbf{x} = \begin{bmatrix} p_1x_1 \\ p_2x_2 \\ \vdots \\ p_nx_n \end{bmatrix} = \mathbf{q}^*.$$

All this certainly does not yield any new numerical or quantitative results — but it furnishes new qualitative insights into the process. The three most important of these are the following:

1. We do not have to assume the constancy of the input coefficients, nor any rigidity in the structure of purchases made. We can allow their actual magnitudes to fluctuate considerably and work only with their expected values.

* See for instance Rényi [1969].

2. The stationary state is not something necessarily experienced in reality. It is just a limiting distribution around which the actual process fluctuates.

3. We do not have to assume or postulate any market forces that bring our system into equilibrium. If there exist expected values of necessary (though fluctuating) input coefficients, the system will demonstrate a behaviour which carries it toward "equilibrium".

A probabilistic formulation may serve as a very useful and realistic approach to economic systems. If we try to depict economic processes in some deterministic way we encounter two fundamental cases. Either the system so described is stable, converging to some particular — locally or globally stable — state. Or it is divergent — that is, it simply blows up or oscillates with constant or ever-increasing amplitudes.

In reality neither of the two cases happens. Proportions, outputs, prices, etc. of real economic life fluctuate with mild or strong amplitudes — but neither convergence nor divergence was ever established. Reality is more adequately described in terms of the above stochastic model, displaying the same features.

Our matrix $\mathbf{D}$ can also be interpreted as a probabilistic description of the market structure. Each element d_{ik} expresses the probability of sector i's selling its product to sector k. The stationary state, the limiting probabilities, will be the same as before: the output proportions reckoned in value prices.

It would not be difficult to interpret Extended Reproduction so that the element $a_{ik} + \lambda b_{ik}$ stands in the place of the former a_{ik}. Yet, the really interesting case where λ itself is a random variable still awaits analysis.

To sum up: the model can be modified and extended in various directions without sacrificing its general character and still maintains its roots in the labor theory of value as demonstrated in Part 1. With this in mind we close Part 2. In Part 3 we return to its original simple form and consider its application to economic reality.

Part 3

Application of the Model

Applying the model involves implementation of its matrices with factual data and drawing conclusions as to the state of the whole economy and the future path and pace of its development. We want to study this model as a tool of analysis and forecasting. If the usefulness of the model towards these ends is accepted we might consider its workability as a tool of planning, as a basis for conscious government intervention into economic processes.

Now we have to investigate our model from a practical viewpoint fo find out how faithfully it pictures everyday real economic life. Thus far we have concentrated on abstract, theoretical problems. Yet, now the tasks of forecasting and planning compel us to evaluate the model as a description not only of idealized states but of practical processes in the economy.

Now, in conjunction with these new tasks, questions of the stability and change of coefficients must be faced. It is legitimate to neglect coefficient change when describing momentary situations but not when describing real processes happening in time. Coefficients, characterizing the economic metabolism of society are subject to change with scale of production, price and quite a few other circumstances. Here we have to consider how to forecast and plan these very changes, or the consequences of ignoring them – of computing with fixed coefficients. Finally we must assess the errors that follow from inaccurately planned coefficients.

Although changing coefficients are a major problem in practice this is far from being the only one. There are parallel contradictions between economic reality and other facets of the model's abstractions and idealizations. Closedness of the model contradicts the openness of every particular country; the left- and right-hand eigenvectors may diverge from the actual price system and output proportions. These differences make interpretation of the numerical results difficult.

Two chapters will investigate these questions. The first considers general problems of application, that is, practical interpretation of the particular output proportions, planning of coefficients and error limits in computation. The second chapter discusses some variants of the open and closed models and a special sort of open model that can make use of the mathematical theory of optimal processes. Such models should prove useful in solving problems of planning.

Finally, the third chapter surveys actual applications of this and closely related models. Experience with this broad class of models is just beginning to accumulate and instances are scattered. But the results are reassuring. For the most part, the computations yield projections that are in fair agreement with observed economic reality.

3.1. Problems of Application

We begin by considering the two major unrealistic assumptions of our model: uniform expansion rate with particular output proportions and fixed coefficients. Everybody knows that economic development is not smooth and even in periods of relatively smooth growth characteristic differences in the growth rates of the particular branches of production still persist.

What guidance, then, do our equilibrium solutions give for output proportions? The interpretation is parallel to that which is given for actual and equilibrium prices. Production prices yield a uniform rate of profit. The latter are never realized in economic life, they do not "become true". We usually find an above-average profit in growing branches and a depressed rate in slow-growing ones. Classical economists considered production prices as a "center of gravity". Economists from Smith and Ricardo through Marshal looked upon them this way and so do economists of modern times. Today interest takes the place of profit and the same equilibrium may be expressed as a "no profit" situation.

The second point concerns the methods of planning or forecasting coefficient changes. They all are "first aid" solutions. We have to acknowledge changes and, at the same time, we are not ready to formulate any definitive relationship among change and the other variables of the model. The fact that we have more than one method to plan changes in coefficients indicates that there is no final, approved and universally accepted explanation of structural change. Available data are barely adequate for specifying the model at a single point of time. Much more information is needed to study coefficient change.

Finally we discuss numerical errors stemming from unrealistic theory, faulty data or ill-guessed change. How do errors in coefficients affect results of computations? How sensitive are the results to aggregation? Do the errors cancel or accumulate? These questions are investigated by methods of error-analysis and perturbation theory.

3.1.1. *Stationary state*

Stationary solutions yielding an average rate of profit and securing a uniform growth rate can be interpreted as equilibria where supply, $\mathbf{x}$, is equal to demand for flows and increments to stock: $\mathbf{Ax} + \lambda\mathbf{Bx}$.

But if we gather data for the matrices $\mathbf{A}$ and $\mathbf{B}$ for a given real state of the economy and then compute the eigenvectors – the solution will not necessarily represent an equilibrium for the economy in question. If the left-hand eigenvector

deviates from actual prices then actual prices will be apt to change, triggering some substitution or other structural changes in adaptation to the new price system. With new coefficients further changes in prices will result, bringing more changes in coefficients, new data and a different equilibrium point again. For analytical convenience we might hope that this new solution is near to the old one or at least, that after some iterations the economy will converge to its true equilibrium. But we do not understand enough to count on this. Therefore we choose to call the eigenvectors not equilibrium, but only stationary solutions, bearing in mind that they belong only to the given statistical data. There is no ground for claiming that they reflect true equilibrium proportions of the economy.

In setting up a model according to the classical approach it was legitimate to assume that the coefficients reflect "socially necessary expenditures" exactly – and that, hence, stationary and equilibrium solutions do coincide.

Why, then, do we base a planning model on the stationary state? We assume that observed coefficients do not differ significantly from coefficients that we could derive from a true equilibrium state. Observed stability of coefficients over time substantiates this assumption.

Yet one might still object that the economy never will be in a stationary state, and it is not certain it will tend toward it automatically. With given coefficients, why do we consciously try to reach those states in planning? The stationary state is desirable because in a certain sense it secures the best, "most healthy" development that can be reached with a given structure. Over the long run the growth rate, λ, connected with the stationary state is the maximal growth rate attainable by the economy.

The proof of this is not easy, the real situation is paradoxical and perplexing. This "maximal" growth rate can be surpassed at every given instant of time, but the consequence of surpassing it is to fall behind it in the longer run. The development secured by the stationary state is never the fastest at any given moment; still in totality and for the long range it is unrivalled. This dialectical antinomy – "slow and steady wins the race" or "the more haste the less speed" – is not unknown in everyday practice.

Let us start from equation (12) $\mathbf{Ax} + \lambda\mathbf{Bx} = \mathbf{x}$. We bring it into a form giving information about possible magnitudes for λ. Using the notation $(\mathbf{1} - \mathbf{A})^{-1} = \mathbf{Q}$ we transform it to $(1/\lambda - \mathbf{QB})\mathbf{x} = 0$.

This now is an eigenequation for the matrix $\mathbf{QB}$. This matrix is strictly positive as $\mathbf{Q}$ is positive and $\mathbf{B}$ is non-negative and irreducible. The stationary state is thus given by the positive eigenvector $\mathbf{x}$, belonging to the maximal, positive eigenvalue of the matrix $\mathbf{QB}$.

Yet λ is the reciprocal of the maximal eigenvalue. This implies that the reciprocal of all the other eigenvalues exceeds λ in modulus. But there is no positive eigenvector belonging to them and therefore these "greater" growth rates are not accompanied by economically meaningful output proportions. They cannot be interpreted straightforwardly as feasible output combinations, because a vector containing negative (and possibly imaginary) elements has no economic counterpart. They may, however, be construed as directions of moving off the stationary point. But these movements have to be small enough not to disturb the positivity or non-negativity of the output vector.

Thus generally there will be another eigenvalue, the reciprocal of which surpasses the stationary growth rate, and the system can move momentarily along the eigenvectors belonging to it. This implies deviation from the stationary proportions. If at a given moment, therefore, we force the system to attain its maximum possible momentary speed, then we can be certain that its movement will not point toward or be equal to the stationary proportions. We can claim that the direction securing the maximal growth rate at every given moment deviates from the stationary path.

This faster direction, however, cannot be followed for long. All the other eigenvectors contain negative elements. Moving along them, sooner or later we reach the point where one of the outputs becomes negative and we arrive at an economic impasse.

Actually this impasse makes itself felt before one of the outputs reaches zero. The system can reproduce itself and expand only so long as all outputs exceed intermediate requirements. The system can be expanded if and only if $\mathbf{x} - \mathbf{A}\mathbf{x} > 0$. When surplus becomes zero in any branch further growth of the system must stop because a closed system cannot secure inputs from outside.

The feasible output proportions, therefore, are to be found only in a sub-region, in a convex cone of totally positive vectors. The moment we reach the boundary of this cone growth must halt. We fall back to Simple Reproduction because lack of some component blocks further growth. Furthermore as we approach the boundary of the cone one or more elements of the surplus product decrease rapidly restricting thus the basis of future expansion.

Therefore if we deviate from the stationary path toward an apparently faster one we cannot follow it for long without endangering future growth. The stationary path is momentarily the slowest but it secures the fastest growth in the long run

Numerical example

Let us illustrate this paradoxical situation by a simple example. Let

$$\mathbf{A} = \begin{bmatrix} 0.2 & 0.3 \\ 0.2 & 0.3 \end{bmatrix}.$$

The feasible output proportions, that is, those $\mathbf{x}$ for which $\mathbf{x} - \mathbf{A}\mathbf{x} \geq 0$, are bounded by the straight lines $x_2 = 8/3x_1$ and $x_2 = 2/7x_1$. We may compute them from the matrix $\mathbf{1} - \mathbf{A} = \begin{bmatrix} 0.8 & -0.3 \\ -0.2 & 0.7 \end{bmatrix}$. The cone of feasible outputs is illustrated in Fig. 3.

Now let
$$\mathbf{B} = \begin{bmatrix} 0.5 & 2 \\ 0.5 & 2 \end{bmatrix}.$$

Then
$$\mathbf{Q} = \begin{bmatrix} 1.4 & 0.6 \\ 0.4 & 1.6 \end{bmatrix} \quad \text{and} \quad \mathbf{QB} = \begin{bmatrix} 1.5 & 3.5 \\ 1.5 & 3.5 \end{bmatrix}.$$

This matrix has two eigenvalues and eigenvectors, the first being

$$1/\lambda_1 = 5 \quad \text{and} \quad \mathbf{x}^{(1)} = (1, 1).$$

This is the stationary (positive) solution. The stationary path is the halfline $x_1 = x_2$, and the stationary growth rate, λ_1, is 0.2 per unit of time.

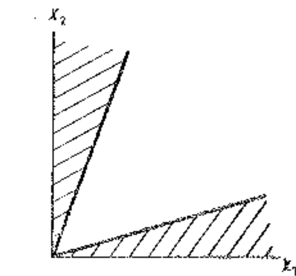

Fig. 3. Cone of feasible outputs

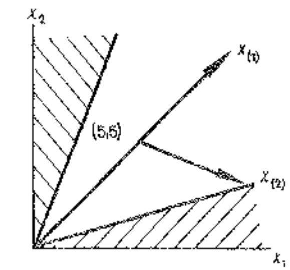

Fig. 4. The eigenvectors

The second eigenvalue and eigenvector is

$$1/\lambda_2 = 0 \quad \text{and} \quad \mathbf{x}^{(2)} = (3.5, \; -1.5).$$

This solution, then, has an infinite growth rate. The situation is illustrated in Fig. 4. The second eigenvector starts from the stationary point $x = (5, 5)$ and is directed downward, toward the boundary of feasible output proportions.

Let us assume our economy started at the stationary point $\mathbf{x} = (5, 5)$. The surplus here ready for accumulation is $(\mathbf{1} - \mathbf{A})\mathbf{x} = \begin{bmatrix} 0.8 & -0.3 \\ -0.2 & 0.7 \end{bmatrix} \begin{pmatrix} 5 \\ 5 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 2.5 \end{pmatrix}$.

If we use this surplus to grow on the stationary path, then it is just enough to secure an increase $\Delta\mathbf{x} = (1, 1)$ of 20 per cent. The necessary stocks making this increase possible are

$$\mathbf{B}\Delta\mathbf{x} = \begin{bmatrix} 0.5 & 2 \\ 0.5 & 2 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 2.5 \end{pmatrix}$$

equal to the surplus on hand.

Let us now investigate other possibilities. We cannot move exactly in the direction of the second eigenvector which would secure infinite growth. Investment being irreversible we cannot decumulate stocks of production in sector 2 in order to accumulate them in sector 1. But we might accumulate nothing in sector 2 — that is, increase production only in sector 1, and move in a horizontal direction parallel to the abscissa x_1.

The first product being less capital-intensive, we might increase capacities much more. The maximal attainable growth is $\Delta x = (5, 0)$ because $\mathbf{B}\Delta\mathbf{x} = \begin{bmatrix} 0.5 & 2.5 \\ 0.5 & 2.5 \end{bmatrix} \begin{pmatrix} 5 \\ 5 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 2.5 \end{pmatrix}$ exhausts all the surplus at hand. Instead of moving on the stationary path and reaching the point (6, 6) from point (5, 5) we have reached the non-stationary point (10, 5). This is a bigger, faster step.

But here we must ponder for a minute. To measure the length of the steps we need some price system. For the sake of simplicity let us assume the price system (1, 1), i.e. values of equal quantities. Reckoned in this price system the stationary path yields 20 per cent, the non-stationary one 50 per cent growth. If measured

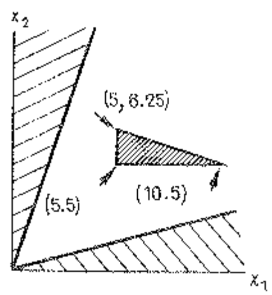

Fig. 5. The possible states

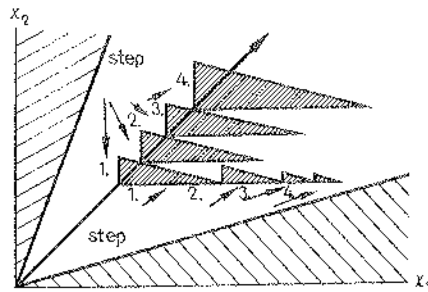

Fig. 6. The consecutive steps

in production prices, the non-stationary growth still surpasses the stationary one. In this question, then, the production price system does not orient properly. Let us add: no price system at all can orient us. Let us depict all the possible states we can reach from point (5, 5), as seen in Fig. 5.

It becomes clear that every price system will show either point (10, 5) or point (5, 6.25) as optimal and none can direct us toward the truly optimal point (6, 6) – which of course is only optimal in a long-range sense, and never optimal for a finite time horizon.

One price system neither orients properly, nor disorients, being totally neutral: the price system (0.5, 2) which is proportional to the direct capital–output ratios. This makes every point connecting the vertices (5, 6.25) and (10, 5) equally desirable. But only in two-dimensional economies can we find such a neutral price system. In general, and even in two dimensions but with a different **B** matrix, there will be no such a neutral price system. The question of reaching the stationary path cannot be solved by optimizing with any price system or objective function.

Let us now return to our original problem. How can we take the next step? From the stationary point (6, 6) we may reach the stationary point (7.2, 7.2) – again a step yielding 20 per cent growth. But this point cannot be reached from the non-stationary point (10, 5).

Anyway, this latter situation affords only (6.5, 1.5) surplus. This may look big, but surplus is only useful in certain proportions (given by the matrix **B**) and we willy-nilly begin to accumulate useless reserves. Thus we have to discard or store 4.5 units of product 1 and use only (1.5, 1.5) surplus.

By stubbornly going in the wrong direction we still may reach the point (13, 5) giving 20 per cent growth and apparently not worse than that of the stationary path. Yet in the years (or steps) to come the surplus becomes tighter and more

disproportionate. If we depict the feasible steps for four consecutive years we get the interesting Fig. 6.

From the fourth step on, the "slow" stationary path yields more of both products than the "faster" non-stationary path. It even gives more of product 1 whose augmentation was the sole purpose of deviating from the stationary proportions.

The long-range optimal stationary solution is well known as the "turnpike" or "Neumann path"* (its curious instability seems to be less well noticed).

We have still to appraise the validity or real workability of the theorem. In practice it will not be optimal to have a uniform rate of growth in every branch. Optimal growth rates of individual branches would differ in a situation where A and B were themselves functions of time. Yet the rates must differ only in accordance with the needs of changing technology.

Here we understand technological change in the broadest sense, covering all the elements of the matrices, including changes in tastes, organization, etc. Branches whose products are substituted for others should grow faster than average. Those that are becoming obsolete should grow at less than average rates. Fast growth should not be an aim in itself. If it is not in accordance with the requirements of technological change, it results in gluts and inefficiency.

Secondly, and this is also important, the optimality of the stationary path does not remove the need for inquiry into non-stationary states. The paradoxical situation exemplified above shows all too well that there is a tendency to deviate from the optimal path. The stationary path has both attractive and repulsive features. Perhaps some explanation of cycles can be built on this observation.

A theory of transients and fluctuations can be established by connecting the primal and dual side of the model. This type of research is at a preliminary stage, to be reported here. Here we consider only one particular facet, that of the fluctuations found in planned economies. Since Goldmann [1964] it is well known that quite a few planned economies show heavy investment cycles. These influence the growth process in almost all branches and resemble the business cycles of market economies.

Under planned growth and direct regulation of production the main causes of such fluctuations are hardly likely to lie in market forces, faulty financial institutions, or changing business expectations. The process of planning itself must be examined in search of the underlying mechanism. Experience shows that fluctuations permeate not only *ex post* statistical data but *ex ante* plans. Thus the plans themselves have destabilizing features. An apparent equilibrium in the plan (the balance of planned supply and planned demand) is compatible with heavy fluctuations in growth rates and may even be held responsible for the latter in some cases.

To make this more explicit: plans (the material-, financial- and other balances) may equilibrate supply and demand – or in terms of the planner: resources (domestic production, imports, etc.) and distribution (domestic consumption, exports,

*See e.g. *The Review of Economic Studies*, Jan. 1967.

etc.) — and still this procedure can result in moving off the balanced path of growth. The short-run equilibrating tendency can keep the economy on its balanced growth path only if the output proportions of the past were consistent with the Neumann path. But with any deviation from the balanced growth path, disproportionalities of output proportions will be boosted by the balancing procedure, the more so as it strives to smooth them.

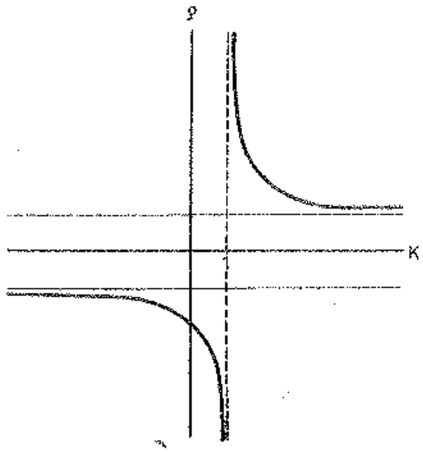

Fig. 7. The dependence of spectra

Let us see how this works. A plan for a given stretch of time is supposed to equate society's production and consumption. Outputs must be large enough to cover requirements for intermediate inputs, consumption and investment necessary for the expansion of capacity.

Let p be the plan-period (it may be one year, or a longer period), for which we are planning total outputs, denoted by the vector $\mathbf{x}_p$. With the aid of our usual notation the planners' equilibrating task can be written in the following way: given $\mathbf{x}_b$ (output vector of the present) determine $\mathbf{x}_p$ (output vector for the plan) securing equilibrium:

$$\mathbf{x}_p = \mathbf{A}\mathbf{x}_p + \mathbf{B}(\mathbf{x}_p - \mathbf{x}_b). \tag{29}$$

Assuming regularity of matrix $(1-\mathbf{A}-\mathbf{B})$ (to which assumption we will return later) we rewrite equation (29)

$$\mathbf{x}_p = -(1-\mathbf{A}-\mathbf{B})^{-1}\mathbf{B}\mathbf{x}_b. \tag{30}$$

The characteristic features of the solution can now be analyzed by inspecting the matrix $\mathbf{K} = -(1-\mathbf{A}-\mathbf{B})^{-1}\mathbf{B}$. Designating the inverse $(1-\mathbf{A})^{-1} = \mathbf{Q}$ and transforming

$$\begin{aligned} -\mathbf{K} &= (1-\mathbf{A}-\mathbf{B})^{-1}\mathbf{B} = \\ &= [(1-\mathbf{A})(1-\mathbf{QB})]^{-1}\mathbf{B} = \\ &= (1-\mathbf{QB})^{-1}\mathbf{QB}. \end{aligned}$$

If the eigenvalues of the matrix $\mathbf{QB}$ are $\varrho_1 > \ldots \geq \varrho_n$, then the eigenvalues, κ_i, of the matrix $\mathbf{K}$ will be $\kappa_i = \dfrac{\varrho_i}{\varrho_i - 1}$. The dependence between the two spectra is shown in Fig. 7.

The maximal eigenvalue of $\mathbf{QB}$ can be expected to lie in the interval $5 < \varrho_1 < 50$ (that is, the maximal growth rate, $\lambda = \dfrac{1}{\varrho_1}$, is between 2 and 20 per cent). Thus we will have an interval for κ_1 of $\dfrac{50}{49} < \kappa_1 < \dfrac{5}{4}$.

Now then κ_1, the only eigenvalue connected with a positive eigenvector (the balanced path), will certainly be dominated by all the other κ_i eigenvalues with corresponding ϱ_i eigenvalues greater than, say, 0.6. This means that we have a nondominating positive eigenvector. Thus we should expect deviations from balanced growth path to increase with every solution of equation (30) for the plan, $\mathbf{x}_p$.

Note that some ϱ_i may have values very close to 1. In this case the corresponding κ_i will be very large. If $\varrho_i = 1$, then $(1-\mathbf{QB})$ will be singular and thus $(1-\mathbf{A}-\mathbf{B})$ will be singular, too. We assumed the opposite — but there seems to be no economic basis for excluding an eigenvalue, ϱ_i, that approaches 1. In such cases balanced planning is almost impossible. Severe fluctuations will characterize planning computations themselves, and planners have to content themselves with truncated, contradictory planning balances. This phenomenon is known in practice as the "collapse" of balancing computations.

In the model of planning just considered attempts to clear the markets, i.e. to adjust requirements to supplies, may lead to increasing deviations from the balanced growth path.

Numerical example

Consider the following hypothetical but plausible economy.

Flow coefficient matrix $\mathbf{A} = \begin{bmatrix} 0.4 & 0.5 \\ 0.3 & 0.1 \end{bmatrix}$

Stock coefficient matrix $\mathbf{B} = \begin{bmatrix} 6.6 & 0.5 \\ 4.7 & 1.9 \end{bmatrix}$

Thus $-(1-\mathbf{A}-\mathbf{B}) = \begin{bmatrix} 6 & 1 \\ 5 & 1 \end{bmatrix}$

whence $-(1-\mathbf{A}-\mathbf{B})^{-1} = \begin{bmatrix} 1 & -1 \\ -5 & 6 \end{bmatrix}$.

Finally the matrix

$$\mathbf{K} = -(1-\mathbf{A}-\mathbf{B})^{-1}\mathbf{B} = \begin{bmatrix} 1.9 & -1.4 \\ -4.8 & 8.9 \end{bmatrix}.$$

The balanced growth path (rounded to 4 decimals) is (1.6365, 1.)

Thus $\begin{bmatrix} 1.9 & -1.4 \\ -4.8 & 8.9 \end{bmatrix} \begin{pmatrix} 1.6365 \\ 1 \end{pmatrix} = \begin{pmatrix} 1.70935 \\ 1.0448 \end{pmatrix}$

amounting to approximately 4.5 per cent yearly growth.

If we perturb the proportions, say start from the initial balanced growth vector rounded to one decimal $\mathbf{x}_b = (1.6, 1)$ we get a "balanced" plan $\mathbf{x}_p = (1.64, 1.22)$ and for the next period (1.408, 2.986). Even though we start very close to the balanced growth path, we soon come to a decrease in the first sector. The situation is so ill-conditioned that we do not remedy by rounding upward to $\mathbf{x}_b = (1.7, 1)$ because the plan vector will be $\mathbf{x}_p = (1.83, 0.74)$, with an instantaneous decrease in the second sector.

In spite of the very innocent appearance of the matrices, the plan computations are severely ill-conditioned. Even if we carried many more decimal places the problem would persist. The underlying cause is a second eigenvalue of the matrix **K**. This matrix has two eigenvalues, one for the balanced growth path: 1.045 and a second one, 9.755. The latter will dominate the computation. These two eigenvalues correspond to two eigenvalues of matrix **QB** of approximately 23 and 1.11. The latter is dangerously close to the ominous value, 1.

Market equilibrium — and thus planned balancing of supply and demand, production and consumption, resources and distribution — does not secure smooth growth. In planning practice we have always to start from present proportions which are never exactly on the Neumann path. The equilibrating computations will worsen the situation. There might be ways to improve it again but they will always entail idle capacity and sacrifice full employment of resources in the short-run for the sake of long-run stability. If initial proportions are off the balanced path it can be reached again only by unused capacity, increase of reserves and foreign trade changes not dictated by market forces (or plan computations imitating them).

3.1.2. *Information for change*

For long-range planning or forecasting we cannot assume constancy of the coefficients any more. However insignificant the year-to-year change of the coefficients may be, as compared to the changes in outputs, errors due to coefficient change are bound to accumulate.

How can we anticipate the development of technology? The task is well known to planners, particularly those experts who regularly draw up material balances for future years. But there is no generally accepted, intellectually or practically satisfying approach that is useful in all sectors.

In the practice of planning, future coefficients (called norms, normatives, specific indicators or ratios) are usually derived from various sources of information, experience and speculation. These are amalgamated, by intuition, conscious weighing, simple or more complex arithmetic and pondering, into the most probable guess. This domain of planning must draw on technical expertise and knowledge, general economic know-how and political common sense. Guesses for every sector require a different mixture of purely technical, organizational, economic, sociological and political knowledge — and these questions are interwoven in an ever-changing pattern. Certainly we are learning, and can continue to learn to understand coefficient change better. But even when its scientific basis is expanded, coefficient projection will always remain an art.

Coefficient projection, however difficult, is central to the planning process. Thus it is imperative to take advantage of whatever scientific methods are available in this area. Let us review the main methods now available for dealing with coefficient change.

The model itself renders it possible to organize, store and arrange our knowledge about past changes. If we know the past history of a coefficient, when and why, how much and how fast it changed, then it will be possible — even without much additional information — to assign broad limits to its future path.

In our country this factual knowledge is still scanty despite 20 years of planning practice and it is only for the last 5–10 years that we can follow through the changes in the necessary detail. To understand their movement better we need information on the age distribution of capital stocks. We lack a tabulation of the thirties, and even an earlier one would be helpful. Curiously historians rather than planners seem to be interested in this material.

As data become available, more sophisticated statistical techniques can be applied. In Hungary there are already some pioneering efforts to forecast future technological matrices systematically.* Yet even the most simple statistical analysis needs a long time-series. 15–20 years' data give a very small sample for analytical trend computations. Planning practice can only be improved on the basis of information accumulated in planning itself.

What are other sources of information for technological forecasting? There is always some possibility of "borrowing" coefficients from technologically advanced countries. However; adaptation of foreign data is not an easy job. There are always differences in nomenclature, accounting conventions and prices. Augustinovics [1969] has developed methods that are insensitive to price differences. An effort to reconcile accounting and classification for a few East-European countries is under way.

Specific technological trends observed in United States' input–output tables for 1919, 1929, 1939, 1947 and 1958 seem to be relevant for Hungary, too: for example the shift from coal to oil and natural gas, diffusion of chemical and synthetic technologies, automobilization, packaging and automation. Here United States' experience gives some basis for judging rates of diffusion and the consequences of individual changes.

Future technology is latent in current average structures and can be separated out by skilled fingers. Carter [1963] proposes the following approach. Let us assume that there are older and newer technological layers in each sector; their average is reflected by the coefficients. In the future the newer layers will have greater weights relative to older ones. Certainly there will be even "newer" technological layers in the future but these may be neglected for short-run projections.

The weights for each layer depend on investment made in the respective technologies. Therefore the life spans and age distributions of the means of production

* See Szakolczay–Vásárhelyi [1967] and Németh [1969].

particularly of machinery determine the relative importance of the layers in most branches of production. With a 30-year life span and a 10 per cent growth rate it takes 5 years to replace half of the machinery, with 3 per cent growth the same takes 11 years.* This certainly constrains the rate of change in relative weights and facilitates projections of structure up to 10 years hence.

The structure corresponding to the different technological layers can be estimated in several ways. Separate data on older and newer enterprises are sometimes available. Industry experts may prepare estimates on the basis of engineering information. This is the method most often used in present planning practice. Incremental coefficients can be computed from tabulations for two (or more) periods of time, by assuming that technological change is embodied in new investment. Carter concludes that each approach has its characteristic sources of error and recommends their joint application.

Given estimates of the structures of different layers and of technologies that are known but not yet in use, we must gauge their relative weights for the future. Kornai [1967] and his collaborators demonstrated that it is feasible albeit a formidable task to enumerate the principal alternative technologies for many individual sectors. Using linear programming he computes the optimal future mix of activities. These can serve as estimates of future coefficients.

There may be some advantage in using the closed dynamic model to consolidate separate sectoral programming models. First steps in this direction have already been taken by Ujlaky [1968] and Simon [1969].

Simon points out that the proportions of the output and price vectors of an economy-wide programming model for Hungary are fairly stable. This suggests that the stricter constraints of the closed dynamic model might not be inappropriate for long-range planning. The model also offers reasonable criteria for optimality. For reasons already stated in the previous section it makes good sense to maximize λ, the growth rate for an infinite time horizon. Programming has to use alternative criteria in order to narrow the territory of decisions. λ should be certainly one of them.

In summary, there are several possible methods of planning future coefficients.

1. Expert guesses based on engineering trends.
2. Statistical extrapolation of time series.
3. Estimates based on the experience of more developed countries.
4. Cross-sectional analysis of technological layers.
5. Computing incremental coefficients.
6. Computing optimal weights of alternative structures by sectoral programming.

All these methods use outside planning and policy information. The closed dynamic model can incorporate and process estimates of future structures but for the time being they must be introduced exogenously.

* The exact formulas for computation are given in Appendix III.

3.1.3. *Error analysis*

When we compute the model with actual statistical data the reliability of the results emerges as a numerical problem. Beyond the problems of logical consistency and appropriateness of the theory the key problem in applying the model is one of its numerical accuracy: how precisely do its parameters measure real magnitudes?

This question can be answered definitely only by experience itself. Experience still being scanty, we have recourse to sensitivity analysis for insight into what to expect from the model when it is applied. The central question is: What happens to the results of our computations if the original data are inaccurate, or are perturbed or aggregated? The tools of this analysis are mathematical error-computation and perturbation theory.

The source of errors may be in computational methods, data and inadequacy of the model itself. We are not interested in the exactness of the computational process itself nor in errors of rounding, truncating, etc. This strictly mathematical problem does not trouble us because the iterative algorithm proposed does not accumulate errors and computations can be carried out to as many decimals places as desired. The exactness of the computational process surpasses that of both the economic data and the requirements. Thus in the following we can take the exactness of computation itself for granted.

Theoretical simplifications are sources of error because it is difficult to approximate complex reality in terms of simple mathematical relations. We are driven to assume linearity, constant coefficients, etc., and to close the model. Even when we try to improve the approximation by introducing exogenous changes, we are still left with errors in planned coefficients. Further errors are endemic in the methodology of collecting and processing economic statistics. The double-entry scheme of input–output tabulations imposes a characteristic pattern on possible errors, and this must be investigated. Aggregation may also be a source of error. In practice we usually work with less than 200 sectors. Bigger tabulations are very costly from the point of view of data processing and of computation. A modern economy has tens of thousands of economic (or statistical) units. Hundreds of thousands, even millions, of products can be distinguished. An input–output tabulation must willy-nilly aggregate roughly 10^3–10^6 individual streams into each cell. Aggregation entails loss of information, but what is its effect?

All these are very broad problems and we can only to begin to study them here. Perturbation analysis of eigenvalues and eigenvectors is developed mostly in theoretical but not in computational terms. We fall back on linear approximations of errors, and this is legitimate only for minor perturbations. Furthermore it is difficult to interpret the mathematical error formulae in economic terms.

The following theorems give some basis for optimism about the workability of the closed dynamic model.

1. *Aggregating on the basis of production prices and stationary output proportions is unbiased.*

For simplicity let us designate the matrix $A + \lambda B$ by the matrix $C = \{c_{ik}\} = \{a_{ik} + \lambda b_{ik}\} = \{a_{ik} = \lambda a_{ik} t_{ik}\}$.

Assume that coefficients are already expressed in production prices. Therefore the left-hand eigenvector of the matrix C will be the summing vector $e = (1, 1, \dots, 1)$, that is, every column sum of C equals 1.

Let us now aggregate the original nxn matrix ${}_nC_n$ to an sxs matrix ${}_sC_s^*$, where of course $s < n$. We can permute columns and rows in the matrix ${}_nC_n$ until the sectors we want to aggregate become neighbors. We now aggregate

the first j_1 sectors $(j = 1, 2, \dots, j_1)$ into a common J_1 sector,
the second j_2 sectors $(j = j_1 + 1, \dots, j_2)$ into a common J_2 sector,...
... and the last j_s sectors $(j = j_{s-1} + 1, \dots, j_s)$ into a common J_s sector.

This process is equivalent to the following matrix-multiplication

$$ {}_sC_s^* = {}_sU_n\, {}_nC_n\, {}_nV_s $$

where

$$ {}_sU_n = \begin{bmatrix} e_{j_1} & 0 & \dots & 0 \\ 0 & e_{j_2} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & e_{j_s} \end{bmatrix} $$

the $e_j = (1, 1, \dots, 1)$ vectors containing as many elements (all equal to one) as there are sectors to be combined.

$$ {}_nV_s = \begin{bmatrix} v_1 & 0 & \dots & 0 \\ 0 & v_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & v_s \end{bmatrix} \quad \text{and} \quad v_1 = \frac{1}{\sum_1^{j_1} x_i} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{j_1} \end{bmatrix} $$

where the v_i vectors signify the output proportions of the subsectors in the common sector.

The following are evident

${}_1e_s^*\, {}_sU_n = {}_1e_n$	the summing vector of s elements postmultiplied by the matrix U yields a summing vector of n elements (disaggregation)
${}_1e_n\, {}_nV_s = {}_1e_s^*$	the summing vector of n elements postmultiplied by the matrix V yields a summing vector of s elements (aggregation)
${}_sx_1^* = {}_sU_n\, {}_nx_1$	the outputs are aggregated by premultiplying them by the matrix U
${}_nx_1 = {}_nV_s\, {}_sx_1^*$	the outputs are disaggregated by premultiplying them by the matrix V.

We are now ready to prove that the column sums of the aggregated matrix are equal to 1.

$$ e^*C^* = e^*UCV = eCV = eV = e^* $$

The aggregated price vector, e^*, is an eigenvector of the aggregated matrix C^*. Aggregation maintains equal column sums; therefore it does not change the maximum eigenvalue (equal to the column sums), and the left-hand eigenvector can be obtained by aggregating the original eigenvector. The dual also holds

$$ C^*x^* = UCVx^* = UCx = Ux = x^*. $$

Thus by aggregating the right-hand eigenvector we obtain the eigenvector of the aggregated matrix.

If the matrix C has unequal column sums we can compute production prices, $pC = p$, and perform the similarity transformation $c'_{ik} = p_i c_{ik}/p_k$. By converting the matrix to the production price system we can always secure the equality of column sums. Therefore aggregation on the basis of the eigenvectors, that is, the production price system and the stationary output proportions, will transform eigenvectors into eigenvectors and will leave the maximal eigenvalue of the matrix unchanged.

2. *Aggregating on the basis of faulty prices and proportions will never increase the error of the computed vectors.*

Let us assume that the vectors are not exact eigenvectors, that is,

$$ eC = e + r \qquad \text{and} \qquad Cx = x + s $$

where r and s are residual vectors. We can consider some suitable norm of these residuals as the appropriate gauge of errors.

Now of course

$$ e^*C^* = e^*UCV = e^*CV = e + r\,V = e^* + rV $$

and

$$ C^*x^* = UCVx^* = UCx = Ux + s = x^* + Us. $$

It follows, considering the non-negativity of matrices U and V and their pattern that

$$ \text{if} \quad |r| < \varepsilon e \quad \text{then} \quad |rV| < \varepsilon eV = \varepsilon e^* $$

and

$$ \text{if} \quad |s| < \delta x \quad \text{then} \quad |Us| < \delta Ux = \delta x^* $$

Thus if we gauge the magnitude of the error by ε and δ, then the gauge will not increase in consequence of the aggregation. The elements of the residuals, being of opposite signs, may cancel in the course of aggregation. According to the degree of cancellation the error can decrease quite substantially with aggregation.

3. *In a probabilistic approach there will be significant cancellation of errors with aggregation.*

Now let us assume that the residuals are expressed as random variables. Let $r = (\varrho_1, \varrho_2, \dots, \varrho_n)$ and $s = (\sigma_1, \sigma_2, \dots, \sigma_n)$ be random vectors and let us assume that their expected value is zero, $E(r) = 0$ and $E(s) = 0$. And let their dispersion (standard error) be $D(r) = D(s) = d = (d, d, \ldots, d)$.

E is a linear and distributive operator, and therefore $E(rV) = E(r)V = 0$ and $E(Us) = UE(s) = 0$. We may reason as before:

$$E(e^*C^*) = E(e^* + rV) = e^*$$

and

$$E(C^*x^*) = E(x^* + Us) = x^*$$

Therefore the expected error after aggregation will be zero in both cases.

Dispersion after aggregation can be analyzed by assuming provisionally pairwise independence of the elements of r and s. Then $D^2(rV) = d^2V^{(2)}$ where $V^{(2)} \leq V$ means the matrix whose elements are the squares of the respective elements of V, and $D^2(Us) = U^{(2)}d^2 = Ud^2$, because all the elements of the matrix U are equal to either 1 or 0. If we aggregate z streams of information, the dispersion of the price or output of the aggregated sector will be proportional to $d/z^{1/2}$. The average number of aggregated streams being between 10^3 and 10^6, this amounts to a very significant decrease in dispersion.

Yet the assumption of pairwise independence is very strong. In practice we can expect considerable correlation among errors, because of control totals. (See p. 128.) Unfortunately we do not know its magnitude. It can be expected strongly to counteract the error-cancellation based on pairwise independence. Yet to multify it the correlation coefficient would have to be +1 (perfect correlation of errors).

4. *The deviations of actual prices and outputs from the eigenvectors never affect the computation of λ significantly.*

In practice we estimate the value of λ from statistical data by forming the quotient $p(1-A)/pBx$, that is, by dividing net surplus by total stocks. This is always done on the basis of actual prices and output proportions because statistical data originate on a current price basis and reflect actual rather than stationary proportions. According to price computations now done routinely in Hungary and Czechoslovakia* actual prices may deviate as much as ±20 per cent from production prices and the same seems to be true for market economies at least when computed in the detail accessible in their input–output tabulations.

On the other hand, output proportions seem to be more exact. A ±10 per cent limit seems to be reasonable. Even lesser deviations lead to great variations in inventories. Our problem, now, is to estimate the possible influence on the estimated value of λ of respective ±20 and ±10 per cent errors.

We will know the error in λ if we know how much the bilinear form $p(A + \lambda B)x/px = pCx/px$ deviates from 1. This again amounts to the error analysis

* See for example Ganczer [1965] and Sekerka–Hejl–Kyn [1969].

of the so-called Rayleigh–Aitken quotient widely used to estimate or improve an eigenvalue when approximate eigenvectors are already known.*

The error of a product is generally the sum of the errors of the factors multiplied. Yet in the case of our quotient we are in a better situation. If one of the factors has no error the product will have none either.

Let us therefore designate the true eigenvectors by p_0 and x_0, the actual price and output systems by p and x, where the error is $dp = p - p_0$, $dx = x - x_0$. The Rayleigh–Aitken quotient can be expressed as $(p_0 + dp)\, C(x_0 + dx)/px$. Considering that $p_0C = p_0$ and $Cx_0 = x_0$ we can transform the quotient to $(p_0x_0 + p_0dx + dpx_0 + dpCdx)/px = 1 + dp(C-1)dx/px$. Thus the error of our estimate will be $\Delta = dp(C-1)dx/px$.

We now take norms element by element, that is, assume $|dp| \leq \alpha p$ and $dx| \leq \beta x$ and estimate the error:

$$|\Delta| \leq \alpha p\, |C - 1|\, \beta x/px \leq \alpha\beta\, |C - 1|\,.$$

Hence, $|C - 1|$ being at most equal to 1, we obtain $\Delta \leq \alpha\beta$. In the practical case considered above, where $\alpha = 0.2$ and $\beta = 0.1$ we can be sure that the relative error in computing λ will not exceed 0.02.

5. *Computed values of λ are much less sensitive to errors of the stock than in the flow matrix.*

We develop the exact perturbation formula and take its linear approximation. Let the perturbation of A be dA, that of B be dB. Consequently λ will change to $\lambda + d\lambda$ and the change of the eigenvectors will be dx and dp. Therefore

$$[A + dA + (\lambda + d\lambda) \quad (B + dB)] \quad (x + dx) = (x + dx).$$

We perform the multiplication

$$\begin{gathered} Ax + Adx + dAx + dAdx + \\ + \lambda Bx + \lambda Bdx + \lambda dBx + \lambda dBdx + \\ + d\lambda Bx + d\lambda Bdx + d\lambda dBx + d\lambda dBdx = x + dx\,. \end{gathered}$$

Considering that $(A + \lambda B)x = x$ and neglecting terms of higher order, we can simplify the formula to

$$\begin{gathered} Adx + dAx + \\ + \lambda Bdx + \lambda dBx + \\ + d\lambda Bx = dx\,. \end{gathered}$$

* See for example Bodewig [1962] and Wilkinson [1965]. The proof given here is based on an observation of a pupil of mine, A. Simonovits [1969].

Premultiplying the equation by p and considering that $p = p(A + \lambda B)$, we arrive at

$$pdAx + \lambda pdBx + d\lambda pBx = 0$$

whence

$$-d\lambda = p(dA + \lambda dB)x/pBx\,. \tag{31}$$

Thus the change of the maximal eigenvalue is given by a quotient of two bilinear forms* – both formed by the left- and right-hand eigenvectors.

Note two special cases. When dB and dA are respectively zero,

$$-d\lambda = pdAx/pBx \quad \text{and} \quad -d\lambda = \lambda pdBx/pBx.$$

Both formulas are well known in economic reasoning and were already used in Section 2.3.2. Savings in flows divided by total stocks yield the change of growth (or profit) rate. A one per cent change in stock requirements influences this rate by one per cent in the opposite direction. With a given level of accuracy prescribed we can allow $1/\lambda$, that is, 10–25-fold errors in B as compared with A. This is a very useful feature because it is always the stock matrix that causes difficulties of measurement.

6. *Statistical errors in the data do tend to cancel out.*

When we tabulate statistical data in the usual input–output form the row and column sums – total input and total output – are better known and more exact than the detail. If we consider these row and column sums entirely exact, there will be a special configuration of possible allocation errors.

Errors come not just in pairs but in fours. It is not possible to commit a single one. If there is an error of magnitude ε in the output of, say, the i-th sector allocated to the j-th sector, we are certain that there must be at least three more errors in the tabulation, all of the same magnitude. If the cell ij has an error $+\varepsilon$, the row sum being exact, there must be at least one cell in the same row, say, in the k-th column with an error $-\varepsilon$, balancing the first. By the same reasoning there must be a similar error in column j. If this occurred in row l then, again, the cell lk must have an error of magnitude $+\varepsilon$. Therefore the canonical pattern of a quadruple error will be:

$$\begin{array}{lcc}
 & \ldots\text{Column}\ldots & \ldots\text{Column}\ldots \\
 & j & k \\
\vdots & & \\
\text{row } i & +\varepsilon & -\varepsilon \\
\vdots & & \\
\text{row } l & -\varepsilon & +\varepsilon \\
\vdots & &
\end{array}$$

* Appendix II contains a more exact approximation, based on the resolvent.

This is of course the simplest configuration and in reality we will find more, partly overlapping error-quadrates. But all the actual errors are reducible to sums of these simple ones.

This has a peculiar effect on the flow coefficient matrix, computed from the statistical tabulation. If the flows were originally measured in the production price system p, and the outputs, x, had the proportions of the stationary state, then the error matrix, dA, will be such that $pdA = 0$ and $dAx = 0$. *Mutatis mutandis* the same is true regarding errors in the stock matrix B.

Substituting this now into our perturbation formula equation (31) we can conclude that $d\lambda = 0$.

Two remarks are in order here. First: this does not hold for the allocation error caused by mixing up stocks and flows. These latter errors are equivalent to errors in turnover time and they do not cancel. Confusion between the flow and stock account is an important source of error in practice.

Second: all this holds only when we reckon with eigenvectors (production prices, stationary state proportions). Actual prices and proportions will differ from these. But allocation errors, then, will affect λ only to the extent that the actual price system and output proportions deviate from the eigenvectors. If deviations in prices do not exceed the ± 20 per cent, and that of outputs the ± 10 per cent limits assumed above, the effect of allocation errors on λ will be very small.

3.2. Thoughts on Planning

Nowadays the "older brother" of the closed dynamic model – the open, static input–output model – is almost routinely applied to supplement traditional planning methods both for yearly (operative) and medium-range (3–5 year) plans. First let us review this application.

Traditional planning can be divided roughly into three consecutive phases. The first consists of analyzing past performance and setting main targets for the future plan. The second phase is the most time consuming: spelling out the main targets and drawing up production plans in detail. These must be accompanied by material balances, securing the necessary allocation of basic materials to producers. Finally, in the third phase all the detail is coordinated and cross-checked to make the entire system of figures consistent and to reveal and eliminate possible contradictions.

Input–output methods are mostly used in this third phase where its solid framework and double-entry book-keeping accuracy provides the perfect tool for checking and coordinating. Historically, in traditional planning methods the early "value balances of the economy" closely resembled input–output tabulations. These were also called "chessboard balances" or balances of "producing and allocating the social product". They served the same purposes, but afforded less detail. The additional information yielded by more refined input–output tabulations and the additional analytical possibilities obtainable by relatively easy mathematical manipulation of the data made input–output a welcome replacement for the chessboard balances.

Planners are now considering the possibilities of applying input–output techniques to the second phase, for drawing up detailed production plans and for help in balancing the allocation of materials. In spite of the obvious merits of the input–output approach there is no routine application yet in this phase.

Finally, an overall conviction that the most effective use of the method would be in the first phase is growing. It would be very useful for making plan targets consistent in advance and thus avoiding superfluous and inconsistent work in later detail. Yet input–output methods have only recently been considered for this phase of planning. This lag is due primarily to differences in planning and in statistical classification schemes. For statistical reporting, the standard classification basis should be reasonably stable. But until recently the planning system has had to work with government bodies, ministries, industrial and agricultural organizations, changing abruptly from year to year. The firm basis of input–output analysis being of course statistics, the transition from one nomenclature

to the other was always time consuming and full of pitfalls. This problem is becoming less important. First, with more detail in statistics the aggregation to any necessary scheme becomes viable. Second, planning is freeing itself from the devastating influence of particular and provincial interests, and is now more ready to handle the situation in scientific and statistical terms.

What is the exact relation between our dynamic model and its brother, the open, static model, approached from the problems of the traditional planners? The dynamic model resembles the static one in that it uses all the old data in the same form and frame. It is therefore backed by all the data and analysis performed with the aid of its older brother. Yet it is a closed and more dynamical version. Thus it requires additional data for the variables it considers endogenous. It calls for a stock matrix and the rows and columns reflecting expanded reproduction of manpower. Both were considered exogenous in the open, static model and although most of the additional data are already available and routinely used for analytical purposes, it is not easy to organize them into a really consistent set.

The number of workers, their skills and occupations on the one hand, and the structure and allocation of investment on the other are well known to planning bodies. Yet in the operative, yearly plans these act as exogenous, outside constraints – they can usually be not much influenced for the year to come. In normal times, that is, excluding national emergencies, the working force available is given by reasonably firm demographic accounts. To change occupational structure is a task of decades. Likewise most of the investment projects to be finished next year were already started in the past and there is not much left for the planner to decide. Only a fraction of investment will be new projects. But this investment in turn will influence output proportions only in later years. Therefore the open, static model seems to be best suited for annual planning.

In the perspective of medium and long-range plans, both factors become more flexible on the one hand and more influential on the other. If the planner changes them, they will influence planned proportions. These factors are in the last analysis responsible for the pace of development and the future proportions of the economy. Hence not only their future magnitudes but the interdependencies among their scale and composition and the pattern and movement of the whole economy have to be taken explicitly into account. These latter interdependencies can only be analyzed with a dynamic model.

We can thus characterize the dynamic model from the planner's viewpoint, as a system of data and interdependencies, based on the usual input–output tabulations but also taking into account requirements for growth of capacity and of skilled manpower and their effects on the pattern and movement of production. It considers the adaptation of the old and new productive forces and capacities to the general movement of the process of reproduction.

Symbolically, therefore, the model can be considered as a summary of all the computations performed routinely in a Planning Office. It subsumes the balancing of flow expenditures and the computations necessary to equilibrate them (as is done by the open static input–output model) and, in addition, the balancing of manpower and balancing of investments, too, systematizing their interdependence with flows and their change. Therefore it can be used to coordinate all these balances at the same time.

The dynamic model, as the open static one, reckons in monetary terms and in aggregated streams for the sake of uniformity and simplicity. This peculiarity is simultaneously its main advantage and disadvantage and decides the role it might play in the general work of longer-range planning.

First, in the early phase of planning it can give several variants, each consistent in itself, for fixing the main plan targets achievable. These are called the "main indicators" of economic development and will be enumerated later. This can secure the start of consistent detail work in different territories of the plan.

Second, the detail work finished, it can check its overall consistency. The check is a double one: first one verifies that the detailed plans do fit into a whole without contradictions. (This is usually not the case. Corrections are always necessary, but perhaps less so if the primary target-setting process was consistent.) Second, one can check whether the whole still fits into the original conception, expressed by the main indicators and targets.

Whenever a plan is drafted, the dual solution of the model will automatically yield the appropriate and specific price systems dictated by the plan's inner proportions. Theoretically, therefore, it is a model generating the necessary information for future price-planning, a subject that is dangerously neglected in present planning practice.

In the following we inquire into three subjects. The first is how the unrealistic assumptions of the model (constant coefficients, stationary state) do work out in the course of planning, how to interpret the numerical results in planning language. Then we consider planning theory and possibilities of cutting open the closed model for planning purposes. An open model, besides being practically superior in solving certain problems of planning, can be linked to the mathematical theory of optimal processes. This in turn yields new insights into and new tools of planning.

3.2.1. *Computing the plan*

What are the numbers we can obtain from the model for practical planning? Depending on the detail and nomenclature represented in the model, it can yield all the "main indicators" that figure as plan targets and chief analytical data of a medium or long-range plan, except those which are expressed in natural units – because, as mentioned, the model only reckons in monetary terms.

These main indicators now are usually the following:

The national income and social product and their growth rates,

the share of the individual branches in national income and social product and the change of these shares,

the division of national income between consumption and accumulation,

the allocation of accumulation + depreciation = investment among individual branches,

the composition of consumption and its change,

the allocation and skill composition of manpower.

Some additional indicators are readily expressible as functions of the former ones, say, the relation of industrial to agricultural output, or the ratio of heavy to light industry, etc.

One major advantage of this type of model is the simultanous generation of all these measures. There is a fruitless and longwinded debate about where to start medium and long-range planning. Is it future consumption patterns and tastes that should serve as a base? Or perhaps extrapolation of the achievable growth of national income to be divided between consumption and accumulation? Or should production possibilities dominate our thoughts as we begin to draft the plan? All these approaches are justifiable to a certain extent but none is entirely satisfactory. A comprehensive model makes it possible to consider the whole complex of interdependent questions. It can insure consistent solutions for alternative policy decisions – giving rise to plan-variants useful in evaluating the decisions themselves.

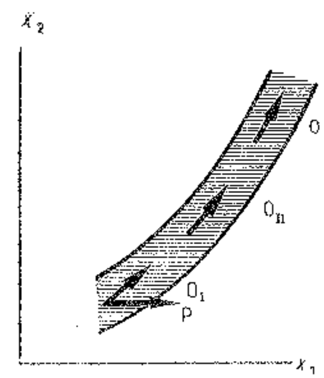

Fig. 8. Plan computations

Yet we can only compute stationary solutions and assume fixed coefficients. How can we use, then, this model for planning?

We have already proven that stationary proportions are the best ones, securing optimal growth, if coefficients remain fixed. Yet coefficients change. In planning practice it is usual to separate changes into two categories. The changes in the first category are out of reach of the planner: he can forecast them more or less accurately but not influence them directly. The second can be influenced and will serve as policy variables. It is patterns of foreign trade, consumption habits and certain technological trends (say, trends toward increased use of natural gas, etc.) that usually is considered as a policy variable and can be influenced with some success.

If it is not an easy job, it is at least feasible to plan future flow coefficient matrices (the stock coefficient matrix being fairly stable over long periods). With a present and a future matrix in hand we are ready to compute two "optimal solutions". The situation is pictured in Fig. 8.

We distinguish the present actual development path, P, the present "optimal" one, O_I, and the future "optimal", O_{II}, usually all different from each other. It is natural to seek the best transformation path, connecting the optimal states for each year. It necessarily will be found in the "tube" outlined in the neighborhood of the stationary states. The more data we have for the intermediate plan periods the more exact we can make the picture and the more narrowly we can delimit the corridor of reliable transformation paths. In any case, we can exclude a great region from consideration. This excluded region may be economically feasible but it is uninteresting, being far from the stationary states.

With this kind of interpretation our very approximate assumptions can be used in looking for near-optimal transformation paths generated by planned changes of coefficients. The model does not itself yield a rough and ready plan, but only guide-posts of a future evolutionary path. Yet these guide-posts, these "optimal" stationary states, should not be considered truly optimal for several reasons. First, all data, even planned data, have errors. These render all the computed results rough approximations. Second, we have no guarantee concerning the coincidence of the real equilibrium position with the stationary solution. Thus all results will be only tentative.

At the beginning of planning we do not need anything more exact. We want only tentatively to search the territory where the economy will be over the next 5–25 years. At the start we have only to boil down all the apparent growth possibilities to a reasonable corridor or "tube" around the estimated optimal trajectory. Once this is marked out, the subsequent detailed planning work (which should not be constrained too tightly) can be done with greater safety, without risking serious inconsistencies of balances. The problems encountered in the third, coordinating phase of planning are most often reducible to contradictory initial hypotheses that led to conflicts in the individual parts of the detail work. A certain consistency in the first phase is needed. At the same time, a too narrow and rigid basis would make subsequent detail work pointless and unrewarding.

In the third, coordinating, phase the model – or rather the necessary double-entry book-keeping tabulation – can show what is left of the original consistency. This is no great improvement over what is already routinely done in the framework of the open static model. Still, an additional check is possible. The finished and aggregated plans already contain all the necessary coefficients. By computing stationary solutions again for the final coefficients, we can analyze how far and why the planned proportions deviate from the theoretically optimal ones. Furthermore the dual solution, the projected price system, might indicate soundness of the whole structure or spot trouble.

One of the difficult problems encountered in planning practice is the coordination of the investment plan with the production plan. The production must permit delivery of all the necessary investment goods to support the increases of output in all sectors. If investment allocated to a certain sector is changed, then the planned increases of its production should be changed accordingly.

The latter interdependence can be secured by traditional methods. But planning practice does not yet consider the influence of a change in investment on the technological structure of the sector affected.

Given the importance of investment rates for coefficient change it seems only reasonable that plans for the two be coordinated. Carter [1969] suggests a method to overcome this difficulty. Given a knowledge of flow coefficients for newer techniques, it is always possible to take explicit account of the effects of growth and changeover investment on the A matrix. The information necessary about the new technology is best provided by technical experts and planners themselves. In the absence of direct information, incremental coefficients can be used as crude estimates of new technology. These tell what the new technology must have been to produce observed changes in coefficients with known rates of new investment.

Foreign trade presents further problems for planners. Hungary having a relatively high proportion of trade with both planned and market economies, foreign trade is one of the major problems of planning. The first approach to this problem is to handle foreign trade as any other sector in the closed system, imports being its output, resulting from export inputs. One might even subdivide the sector according to various foreign markets – a subdivision that, considering bilateral agreements, is not entirely unrealistic. Since there are very close substitutes between domestic and foreign production of the same commodity, the input structure of foreign trade is easier to modify than that of an ordinary industrial sector. The most practical course is to assume various possible export structures to represent various foreign trade policy decisions, and then to analyze the stationary solution and growth rates resulting from them. The existence of foreign trade therefore gives a new degree of freedom to the model, or rather cuts it open. Further analysis of the possibilities of such an open dynamic model will be taken up in the next two sections.

3.2.2. *Opening the model*

The central task of every economy – thus we started to set up the model – is to allocate society's labor, manpower, to particular activities. Some of these are not really economic activities and should not be part of the analysis of reproduction. A certain amount of value is separated from the reproduction process and consumed by these activities. They are best treated as a "final demand" in an open model.

For instance science and the arts are creative but not repeatable and therefore not reproductive activities. They are not production processes and although they yield a product, this is usually only enjoyed but not consumed, used but not used up.

At the other extreme we have defense and other wasteful yet, for the time being, apparently unavoidable activities. Expenditures for these activities are decided on the basis of other, exogenous (moral, aesthetic, political) considerations. These decisions are not entirely independent of the state of the economy, because they do depend on the order of magnitude of the surplus.

Really, the choice between the open and closed model is somewhat arbitrary. The logic of the open system makes exogenous factors decisive. As independent variables they become the objectives of the economic process.

Of course the closed model does not eliminate these extra-economic activities; it is not a "consumptionless" model. It certainly can include a government sector (or an assortment of government sectors: defense, police, administration, justice, health, etc.) and government budgeting shows us that it is possible to anticipate their structure and cost.

In practical work – as opposed to pure theory – there are some questions of analysis and planning that can be handled more readily by the open model. As mentioned, the impact of foreign-trade-policy decisions on the total economic process can be better appraised by the open model. Whenever we are interested in the outcome of some decision (using the common *ceteris paribus* method of analysis) we might use an open model because it takes the form of a question: How does a particular element, separated out for purposes of analysis, influence the economic process? It is therefore the model suitable for analysing the impact of policy decisions, be it a new structure of foreign trade, increase in defense spending or a new health-care program.

It is not difficult to write out the open dynamic model in mathematical terms. One has only to bear in mind that the surplus on hand must cover not only capacity increase but some outside consumption, final bill of goods, too. Using differential equations:

$$(1 - A)x_t = B\dot{x}_t + y_t. \tag{32}$$

The technique of solution is analogous for difference and differential equation systems. It depends on whether we can specify y_t as a function of time (as the time path of final outputs) or describe it in terms of its derivatives $\dot{y}_t, \ddot{y}_t, \ldots, \overset{(n)}{y}_t, \ldots$ for a given instant of time t. Let us first study the second from making use of the differential operator $\mathbf{D}$. This operator is linear and commutative for all matrices, that is $A\mathbf{D} = \mathbf{D}A$. Thus the above equation can be expressed in operator calculus as

$$(1 - A)x_t = B\mathbf{D}x_t + y_t \tag{33}$$

and solved for x_t

$$x_t = (1 - A - B\mathbf{D})^{-1} y_t. \tag{34}$$

If the inverse exists, this expression can be written as an infinite geometric series. Denoting $(1 - A)^{-1} = Q$, we have

$$x_t = (Q + QB\mathbf{D}Q + \ldots + Q(B\mathbf{D}Q)^n + \ldots)y_t \tag{35}$$

that is

$$x_t = Qy_t + QBQ\dot{y}_t + \ldots + Q(BQ)^n \overset{(n)}{y}_t. \tag{36}$$

We assume that x_t remains finite for every t, that is, we postulate the convergence of the series. Yet we know that the maximal eigenvalue of BQ equals $1/\lambda$, the reciprocal of the stationary growth rate. The series will converge if

$$\mathbf{D}\overset{(n)}{y}_t < \lambda \overset{(n)}{y}_t \quad \text{from some finite } n \text{ and } t.$$

This assumption can be interpreted, somewhat loosely, as a constraint on the growth of the final bill of goods. In the long run its growth rate cannot exceed the growth rate of the system itself. That is, expenditures on those non-productive or non-reproductive activities should grow almost never faster than the economy itself.

In terms of the time path of y_t, we can write out the usual textbook solution of our system:

$$x = e^{B^{-1}(1-A)t}\{x_0 - \int_0^t e^{B^{-1}(1-A)\tau} B^{-1} y_\tau \, d\tau\} \tag{37}$$

x_0 being the state of the system at time $t = 0$. The form of the solution will again constrain the possible growth of y_t. Yet the presence of the matrix $B^{-1}(1 - A)$ alerts us to further theoretical problems. First, B itself can be singular in practice. If, say, two sectors have the same capital structure, then B, having two columns equal, will be singular, and if the capital structures are very similar, B will be severely ill-conditioned. Furthermore, $B^{-1}(1 - A)$, if it exists at all, will have the economically meaningful growth rate, λ, as its eigenvalue of minimal modulus. Actual computation, then, will be dominated by other eigenvalues, and therefore be clumsy and inexact. In the final chapter we review one way to circumvent this problem in practical work.

Yet, before turning to practical computation, let us consider the application of the mathematical theory of optimal processes to the open model.

3.2.3. *Optimal processes*

The open model can be transformed, or rather interpreted, from a slightly different viewpoint. Here the mathematical theory of optimal processes gives new insights.* Its generic model is also an "open" one, and the behavior of the system will be again determined by exogenous variables. The first change is simply one of interpretation: we do not consider those outside variables an object in themselves but only as means to attain some other object, namely optimal development of the economy. The final bill of goods is therefore transformed into a variable load on the economy, serving to steer or control it toward previously determined targets.

The practical application of this model will need quite an amount of further investigations both in theoretical and practical-statistical respect. What can be outlined below is only an abstract theoretical description of a very simplified control model. Some problems needing future investigation will be pointed out. Still it is important to include this approach here because it seems to be a very powerful one to apply to the problems of optimal planning.

Thus far the mathematical technique of linear programing, originally conceived for the choice of optimal technology, has been our major tool of optimization. However, the burden of computing really long-range economy-wide systems may be forbidding.

Another mathematical technique, that of the control-system engineer, based on the Pontryagin-principle of optimal processes, will be applied here. This technique simplifies the computational problem. Besides, it gives some further theoretical insight into the working of a complex dynamic economic system.

First of all optimality, the criterion of choice, needs a strict and clear definition. Economists have taken two approaches. One tries to maximize the output of the economy (total production or final bill of goods or export surplus, etc.) and considers the time horizon of the plan, T years or periods which may extend — at least theoretically — to infinity, as predetermined. The other approach considers

* See Pontryagin–Boltyanskii–Gamkrelidze–Mishchenko [1962].

the transformation of economic proportions (corresponding to a given level and way of living or a given level and structure of industry, etc.) as a prescribed task and seeks to minimize the time necessary to accomplish it.

Planning practice usually follows the first approach although the first economy-wide plan, Lenin's GOELRO for electrification of the Soviet Union, was clearly conceived with the second approach in mind. But both approaches must be applied with skill and careful judgement to avoid pitfalls.

Maximization of output with an infinite time horizon is often inconclusive because we cannot foresee all the important changes in technology, taste or habits far ahead. A finite time horizon mitigates the difficulty of anticipation but, by neglecting posterity, it may throw the economy out of its balance (this will be illustrated below).

Minimization of time involves a subjective element in the selection of the economy's specific goals. This choice can only be sidestepped for developing countries, if they accept a leading country's standards as a guide. Thus both approaches present severe economic and moral problems and the different results should be carefully weighed. They may mutually complement each other.

In the literature of control-system theory, both approaches are well known. Output maximization is called a "fixed time-free endpoint", and time minimization a "fixed endpoint-time optimal path" problem.

Now, the essential elements of the control problem are the following. First consider the *system* itself, to be controlled. Here we accept the dynamic model as a statistically implementable and well-behaving representation of economic reality. The system's *state* at time k is given by its gross production vector x_k and its *output* may be production itself or some related variable such as personal consumption or net accumulation. Second we choose a set of *controls* which operate on the system forcing it to deviate from the normal dynamic course determined by its flow and stock coefficients.

The main control in the planner's hand is decision about future investment. A control instrument, however, should be a device to be used optionally. Thus, ideally, it will be an economic *reserve* which may be used or not used at will to achieve the best performance of the system. In advocating a reserve we are already at variance with the common-sense planner. He usually deplores unused resources and assumes that operating at full capacity and investing whatever there is to invest must be a necessary condition of optimality.

Our control instrument, then, is unused surplus, reserve, something that may be invested to obtain new productive capacity in the system, but may be left idle if this serves our final aim better. It can be stand-by capacity or strategic raw materials or any other stock which may be withheld temporarily from productive use. The very generality of this concept does not allow us to set once for all limits on the tolerable amount, but in real life reserves certainly are limited somehow in every case. Mathematical theory starts simply by assuming some limits to, or constraints on them and does not worry about data. The constraints may themselves be interdependent. Sometimes they are set by political necessities: say, limits to idle manpower permissible. If no other constraint is operational the prevailing

amount of surplus is still given and one can either use up all of it, part of it, or none.

Consider the closed dynamic system given in the form of difference-inequalities

$$B(x_{k+1} - x_k) \leq (1 - A)x_k \tag{38}$$

with investment limited by the surplus product on hand. Now we introduce unused surplus as slack variables, y_k. These will transform equation (38) into equalities and at the same time may also be considered as our *control* variables:

$$B(x_{k+1} - x_k) = (1 - A)x_k - y_k \tag{39}$$

and

$$b \geq y_k \geq 0.$$

Here b stands for some limit to tolerable "waste". Now we suppose that B is regular and has an inverse, thus

$$x_{k+1} = [B^{-1}(1 - A) + 1]x_k - B^{-1}y_k\,. \tag{40}$$

Denoting $B^{-1}(1 - A) + 1 = D$ as a one-step transformation matrix we write

$$x_{k+1} = Dx_k - B^{-1}y_k\,. \tag{41}$$

Let the state of the system be given at $k = 0$ by its total production vector, x_0. We wish to maximize output T periods later, x_T, measured by some objective function cx_T

maximize cx_T with cx_0, T given

subject to $x_{k+1} = Dx_k - B^{-1}y_k \qquad k = 0, 1, \ldots, T-1$

and $\qquad 0 \leq y_k \leq b.$

All this assumes, of course, a lot: B might be singular or we might not find a $y_k \leq b$ which transforms equation (38) into equalities. These are matters for further investigation and the ensuing difficulties might or might not be solved in practical cases. For the time being we assume here that our system *is* controllable.

The solution to our problem can now be worked out in the following way. Suppose we already know the optimal sequence of the control variables, y_k^* $(k = 0, 1, \ldots, T-1)$ yielding the optimal sequence of the states, $x_k^*(k = 1, \ldots, T)$. This sequence can be optimal if and only if any possible perturbation of the final state $x_T^* + x_T'$ (which can be reached by giving suitable perturbations, y_k', to the control variables) does not increase the value of the objective function. The sequence x^* is optimal if and only if $c(x_T^* + x_T') \leq cx_T^*$ for any possible x_T'. Thus

$$cx_T' \leq 0 \tag{42}$$

emerges as the necessary and sufficient condition of optimality. We will now transform this condition into an equivalent condition concerning the control variables.

Perturbations of the states and those of the control variables are connected by equation (41) in the following manner:

$$x'_{k+1} = Dx'_k - B^{-1} y'_k . \tag{43}$$

We now introduce an *auxiliary price vector sequence* formed by the following recursive prescription:

$$p_T = c \tag{44}$$

and

$$p_k = p_{k+1} D .$$

Multiplying both sides of equation (43) by p_{k+1} and summing from 0 to $T-1$, we have then

$$\textstyle\sum_0^{T-1} p_{k+1} x'_{k+1} = \sum_0^{T-1} p_{k+1} Dx'_k - \sum_0^{T-1} p_{k+1} B^{-1} y'_k . \tag{45}$$

The first term of the right side can be transformed by equation (43) as follows:

$$\textstyle\sum_0^{T-1} p_{k+1} Dx'_k = \sum_0^{T-1} p_k x'_k .$$

But the same expression appears on the left side of equation (45), too, except for the time subscripts. Considering $x'_0 = 0$ (because x_0 is given and thus cannot be perturbed) and $p_T = c$, we may simplify equation (45) to

$$cx'_T = - \textstyle\sum_0^{T-1} p_{k+1} B^{-1} y'_k . \tag{46}$$

Thus, our former condition for optimality, equation (42) can be written in the equivalent form of

$$\textstyle\sum_0^{T-1} p_{k+1} B^{-1} y'_k \geq 0 . \tag{47}$$

This is already a condition on the perturbations of the control variables.

We now show that, at an optimum, each element of the control variable must be at the limit of its possible range.

The reasoning runs as follows. The perturbations of y_k^* may take any values within the limits set by $0 \leq y_k \leq b$. We now fix our attention on one element of a given period's control variable, say $(y'_k)_i$, and consider the perturbations of all the other elements and all the other periods to be zero.

For what value of $(y'_k)_i$ will equation (47) be true? If $(p_{k+1}B^{-1})_i > 0$, then equation (47) can be true only if $(y_k^*)_i \geq 0$. But if the i-th element of the optimal control variable of the k-th period, $(y'_k)_i$, were not on the lower limit of its possible range, then we might perturb it in a negative direction; thus an $(y'_k)_i$ could be chosen less than zero and equation (47) would not hold. Therefore if $(p_{k+1}B^{-1})_i > 0$, then $(y_k^*)_i$ *must* be at its minimal value where only positive perturbations are possible.

If however $(p_{k+1}B^{-1})_i < 0$, then $(y_k^*)_i$ must be at its maximal value. Otherwise a positive perturbation, $(y'_k)_i > 0$, would be possible and equation (47) could be again violated.

In summary, if:

(a) $(p_{k+1} B^{-1})_i > 0$, then $(y_k^*)_i = 0$, that is, at the minimum of its range,

(b) $(p_{k+1} B^{-1})_i < 0$, then $(y_k^*)_i = b_i$, that is, at the maximum of its range,

(c) $(p_{k+1} B^{-1})_i = 0$, then $(y_k^*)_i$ is undetermined, thus optional within its range.

After the recursive sequence of the auxiliary price vector is computed, the optimal control variable may be determined element by element, period by period, according to the respective signs of $(p_{k+1} B^{-1})_i$.

Numerical example

To show the salient features of the procedure we present a 2-sector illustrative model.

We begin with the following assumed flow and stock matrices:

$$A = \begin{bmatrix} 0.4 & 0.3 \\ 0.5 & 0.3 \end{bmatrix} \qquad B = \begin{bmatrix} 2.0 & 1.0 \\ 1.0 & 1.0 \end{bmatrix} .$$

We invert B

$$B^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

and compute the one-step transformation matrix

$$D = B^{-1}(1 - A) + 1 = \begin{bmatrix} 2.1 & -1.0 \\ -1.6 & 2.7 \end{bmatrix} .$$

It is easy to check that this dynamic system when left to itself ($y = 0$) has a unique "turnpike path", producing a 10 per cent growth rate. Starting from $x_0 = (100, 100)$

$$x_1 = Dx_0 = \begin{bmatrix} 2.1 & -1.0 \\ -1.6 & 2.7 \end{bmatrix} \begin{pmatrix} 100 \\ 100 \end{pmatrix} = \begin{pmatrix} 110 \\ 110 \end{pmatrix}$$

$$x_2 = Dx_1 = \begin{bmatrix} 2.1 & -1.0 \\ -1.6 & 2.7 \end{bmatrix} \begin{pmatrix} 110 \\ 110 \end{pmatrix} = \begin{pmatrix} 121 \\ 121 \end{pmatrix} .$$

Now common-sense and some turnpike-theorems seem to suggest that travelling on the turnpike *is* the output-maximizing conduct. But it is not.

Let us assume we have to maximize x_2 with $c = (1, 1)$ — that is, the price of every unit produced is 1 dollar. Our former turnpike output then was worth 242 dollars.

Let us further assume we tolerate 1 per cent "waste" on total production, that is, our constraints on the control variable are $0 \leq y_k \leq 0.01\, x_k$.

We first compute the auxiliary prices:

$$p_2 = c = (1, 1)$$

$$p_1 = p_2 D = (1, 1)\begin{bmatrix} 2.1 & -1.0 \\ -1.6 & 2.7 \end{bmatrix} = (0.5, 1.7).$$

Then check the signs of pB^{-1}

$$p_2 B^{-1} = (1, 1)\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = (0, +)$$

$$p_1 B^{-1} = (0.5, 1.7)\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = (-, +).$$

Thus in the first period we have one negative sign in the first element. Therefore all elements of the control vector may be taken to be zero except its first element in the first period where it should be at its maximum: $0.01 \cdot 100 = 1$.

We are now ready to compute the new, controlled, path:

State	Transformation		Control		End state
x_k	Dx_k	$-$	$B^{-1}y_k$	$=$	x_{k+1}
$\begin{pmatrix} 100 \\ 100 \end{pmatrix}$	$\begin{bmatrix} 2.1 & -1.0 \\ -1.6 & 2.7 \end{bmatrix}\begin{pmatrix} 100 \\ 100 \end{pmatrix}$	$-$	$\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$=$	$\begin{pmatrix} 109 \\ 111 \end{pmatrix}$
$\begin{pmatrix} 109 \\ 111 \end{pmatrix}$	$\begin{bmatrix} 2.1 & -1.0 \\ -1.6 & 2.7 \end{bmatrix}\begin{pmatrix} 109 \\ 111 \end{pmatrix}$	$-$	0	$=$	$\begin{pmatrix} 117.9 \\ 125.3 \end{pmatrix}.$

Thus we end up with 243.2 dollars' worth of output, that is, 1.2 dollars of additional output, in spite of investing 1 dollar less.

Our example not only substantiates the ease of computation but reveals some shortcomings of the approach, too. Greater speed of development is attained by going off balance – by abandoning the turnpike path, the only path which maximizes growth for an infinite time horizon. This is the symptom mentioned earlier: finite time maximization neglects posterity and may throw the economy into a cycle whence it is not easy to recover.

Similar procedures can be used to solve certain extensions of the problem just discussed.

Instead of x_T, total production, other variables may be maximized, say, final consumption or surplus or any other related items. If, for instance, we wished to maximize surplus, that is $(1 - A)x_T = s_T$, the procedure would be the same except for equation (44) where $p_T = c(1 - A)$ is the proper starting point for the new auxiliary price sequence.

Instead of variables for the final, T, period we may maximize across the whole time path. The cumulated output may be maximized by max $\sum_1^T cx_k$, moreover

our objective function may have different parameters for each period: $c_k(k = 1, 2, \ldots, T)$, that is, we may seek max $\sum_{k=1}^{T} c_k x_k$. In the latter case the procedure is the same except that equation (44) must be again changed to $p_k = p_{k+1}D + x_k$.

A linear loss function for unused surplus ($y_k > 0$), or changing $b_k(k = 0, 1, \ldots, T - 1)$ constraints on slack, might also be introduced.

Finally, instead of fixed coefficients in the A and B matrices we may introduce technological change, specifying a different A_k and B_k separately for each and every period. There is even some hope of solving the problem for systems where the change of coefficients depends linearly on previous investment, or where A and B are more complicated linear operators embodying leads and lags.

In spite of all this, further research is needed before embarking on practical model building and solving. For the model to be realistic quite a few side-constraints need to be established. For example, disinvestment in fixed capital cannot exceed the amount of normal wear and tear and investment in inventory is not totally reversible either. All these and related problems need assessment both from the economic and the computational viewpoint.

We now turn to the second problem which is to guide our system to a desired state in minimum time. In posing the problem we change to the continuous version of the system. Minimization of time is easy when time can be considered continuous. Then we do not have to bother – as in the discrete case – about overshooting our objective in T periods when it cannot be reached in $T - 1$ periods. Otherwise our model and its assumptions about controllability remain the same.

A time-optimal path connects two given states x_0 and x_1 and can be traversed in a shorter time than any other possible connecting path, subject to the constraints given by the structure of the system itself and the limitations on the control variables.

Let us now consider the closed dynamic system in the form of differential inequalities and apply reasoning analogous to that used in the first problem.

$$B\dot{x} \leq (1 - A)x \tag{48}$$

(here $\dot{x}$ stands for dx/dt and we omitted the time subscripts).

At a given moment say, $t = 0$, the state of the economy is given by the production vector $x_{t=0} = x_0$. We wish to lead the system into another state, x_1, in the shortest possible time, T, subject to the limitation given by equation (48).

For "steering" our system we introduce a piecewise smooth, bounded and non-negative "slack" vector, $y \geq 0$. Now our problem is to determine that time-path of y which brings our system from x_0 to x_1 in minimum time, subject to

$$B\dot{x} = (1 - A)x - y. \tag{49}$$

We introduce again an auxiliary price vector by the prescription

$$\dot{p}B = -p(1 - A). \tag{50}$$

Now we can form the so-called Hamiltonian function

$$F(p, x, y) = 1 + p(1 - A)x - py\,. \tag{51}$$

Our Hamiltonian has the following partial derivatives:

$$\frac{\partial F}{\partial p} = (1 - A)x - y = B\dot{x} \tag{52}$$

$$\frac{\partial F}{\partial x} = p(1 - A) = -\dot{p}B\,.$$

We are now ready to write our problem as follows.
Find that time path of y which minimizes

$$J(y) = \int_{x_0}^{x_1} [F(p, x, y) - pB\dot{x}]\, dt. \tag{53}$$

The expression in brackets is identically 1. [Considering equations (51) and (52).] Thus $\int_{x_0}^{x_1} dt$ is equal to the transition time from x_0 to x_1, and this is exactly what we have to minimize.

The solution is given by the following *Theorem*:

Only that time path of y may be considered optimal which extremizes the scalar product py for all t.

Proof. Let us suppose, as before, we have arrived at the optimal solution, y^*, yielding the time-optimal trajectory, x^*, leading from x_0 to x_1 in minimum time. Optimality again requires that for perturbations in y^* and corresponding changes in x^* and $\dot{x}^*$ our functional

$$J(y^*) \leq J(y^* + y')\,. \tag{54}$$

We will transform this condition of optimality again into a more explicit formula.

The change of the functional $\Delta J = J(y^* + y') - J(y^*)$ can be expressed by taking differentials. This is feasible because we supposed the differentiability in x and y, thus after neglecting terms of higher order

$$\Delta J = \int_{x_0}^{x_1} \left[\frac{\partial F}{\partial x} x' + \frac{\partial F}{\partial y} y' - pB\dot{x}'\right] dt\,. \tag{55}$$

Integrating $pB\dot{x}'$, by parts

$$\int_{x_0}^{x_1} pB\dot{x}'\, dt = [pBx']_{x_0}^{x_1} - \int_{x_0}^{x_1} \dot{p}Bx'\, dt\,.$$

The expression in parentheses vanishes because $x_0' = x_1' = 0$, that is, the perturbation of the trajectory must not alter its two end points.

Thus equation (55) may be transformed into

$$\Delta J = \int_{x_0}^{x_1} \left[\left(\frac{\partial F}{\partial x} + \dot{p}B\right) x' + \frac{\partial F}{\partial y} y'\right] dt\,. \tag{56}$$

Here again the first member is identically zero because of equation (52), leaving us with

$$\Delta J = \int_{x_0}^{x_1} \frac{\partial F}{\partial y} y'\, dt \leq 0 \tag{57}$$

as the condition of optimality.

Passing to the limit $y' \to 0$ and considering y' to be of optional sign, this leads to the condition of optimality

$$\frac{\partial F}{\partial y} = 0 \text{ for every } t \text{ from } x_0 \text{ to } x_1\,. \tag{58}$$

Thus our Hamiltonian equation (51) must be extremal as a function of y on the optimal trajectory. Now y enters equation (51) only in the scalar product $-py$, hence we conclude that whenever

(a) $p_i > 0$ then y_i must be at the minimum allowed by its constraints
(b) $p_i < 0$ then y_i must be at the maximum
(c) $p_i = 0$ then y_i is undetermined.

The parallel with the first problem is, indeed, straightforward.

This second, continuous, solution is rather inconvenient when it comes to actual computation. All we could prove was: there *exists* a price vector (as a function of time) which may serve as a means to find the optimal control. But we usually have not enough data to find it: all we know is the prescription of equation (50). This is a linear differential equation which lacks the boundary conditions. Still the general theorem has a certain advantage in properly stating qualitative characteristics of the optimal control variable.

Both formulations indicate that no matter how we constrain (or how reality constrains) the control variables, they invariably turn out to be on the limit of their range when the system performance is optimal. Thus optimality and swinging to extremes seem to be connected.

Of course one could avoid economic interpretation maintaining that this connection is only a feature of the mathematical model and not of economic reality itself. But it is not easy to deny that economic systems are prone to fluctuations. Perhaps economic cycles are a kind of flutter phenomenon induced by the strains of optimization.

Let us relate this finding to the result of our numerical example. We know that the turnpike path is unique and time-optimal in the sense that we can go from one point to another on it in minimum time. We also know that on this path the control must be zero, $y \equiv 0$. This entails $p_i > 0$ for all i and all t. But there is only one entirely positive solution to $\dot{p}B = p(1 - A)$ for all t: the well-known "production price" vector, yielding an average rate of profit after investment: $p(1 - A) = \lambda pB$.

How well now movements around this balanced growth path and equilibrium price may be described by control theory formulation of the dynamic model needs study. Empirical research might show that the spectra of flutter frequencies (computed from the flow and stock matrices) and those frequencies found in the spectral analysis of economic time series are alike. But this is only a conjecture not yet substantiated.

In summary: an interesting theoretical possibility emerges with the Pontryagin theory. It yields promising new insights into both theoretical and practical problems of planning. Yet our theoretical and factual knowledge is insufficient to assess its real usefulness. This is, then, one of the main domains of future research.

3.3. Practical Computations

The model that we have been discussing is hardly new. Economists in many different parts of the world have implemented variants of this system, some on a very limited experimental scale, and some on a more ambitious level. In this final chapter we review computations of the open and closed form of the model. This may help in assessing the potential value of the approach to long-range planning. In this survey we exclude the many instances where the model serves only as the inner core of programming procedures. The latter, although interesting in itself, does not help us to appraise how well the model itself reflects reality.

We have relatively more factual knowledge about the aggregated form of the model, equivalent to the Harrod–Domar model. Its performance in the light of our general knowledge about growth rates, saving ratios and capital–output ratios will be analyzed in Section 3.3.1. The main question to be answered is: how well does this model explain the variety of growth rates experienced in modern history?

We have less material concerning the more detailed forms of the model. No definitive assessment is possible, and we shall give more attention to special questions of application than to numerical results.

Yet experience prompts us to claim, very cautiously, that the implementation of the model is within reach, its solutions are well-conditioned, computation is smooth and easy to handle, and – finally – numerical results are in good agreement with statistical facts and readily interpretable.

3.3.1. *The aggregated form*

Let us inquire into the explanation of secular growth rates given by the one-sector model. As already pointed out in Section 2.3.2 the one-sector form of our model is:

growth rate = saving ratio/capital–output ratio.

In this aggregated form the model is equivalent to the formula presented by Harrod [1936] and later by Domar [1946]. The main points were already implicit in Kalecki [1935] and developed very early in Feldmann's two-sector models ([1927] and [1928]).

All these models are of the relatively early theoretical vintage of the twenties when eastern economists, and of the thirties and forties when western scholars, became interested in growth. Yet reliable data to test the models and implement their use outside the classroom became available only after the second World War. At that time reconstructing historical time series became fashionable, and this made it possible to anchor theories of growth on a firmer basis.

This is certainly not the place to enumerate and review the data for many countries in detail. The following quotation summarizes some findings of a study where the secular relation of national wealth and national income were analyzed in detail.*

"When investigating the capital/output ratio for shorter periods (shorter than the length of the business cycle) we find heavy fluctuations . . . These fluctuations can be explained by the movement of the business cycle. The wealth once invested can be decumulated, if at all, only to the extent of its wear and tear (its scrappage) and thus it cannot follow closely the cyclical fluctuations of output. It is meaningless to compare the increase in wealth and in output over the short run because, quite often, we find opposite changes, too: increase in wealth may be accompanied by decreasing output and *vice versa*. The marginal quotient that relates these differences is therefore totally unreliable. (This phenomenon will manifest itself naturally with greater force in particular branches — industry, agriculture etc.).

In spite of heavy short-range fluctuations the capital/output ratio, measured at peak periods, is manifestly stable. In the time-stretch investigated it changed only very slowly. There were no discernible international differences in its absolute magnitude nor in its direction and speed of change. Changes in technology and the great differences in productivity among countries apparently do not influence the behavior of the capital/output ratio very much.

The relation of national wealth (as the sum of tangible and reproducible values) to national income shows a characteristic secular movement in the course of historical development. This wealth/output ratio will increase in the beginning, but this increase decelerates. The ratio reaches a peak and then starts to decrease slowly again. Its whole motion is constrained between tight boundaries. At its zenith — which is established after an approximately half century of industrialization, and can be dated around the turn of the century for western countries — the ratio does not surpass its average by as much as 50 per cent. At its maximum it is not greater than 4–4.5 (which means that 4–4.5 years' national income is accumulated as tangible wealth) and the ratio has not sunk below 2.5–3 nowadays."

If we consider national income as output, define savings ratio as the part of national income accumulated (as is common in western literature) and investigate capital requirements of producing national income, then we arrive at an awkward impasse. Let us assume, as a first approximation, a constant savings ratio. Then, given the course of actual capital/output ratios, the developed countries should have been growing at a decreasing rate until the turn of the century, thereafter experiencing accelerated growth. This clearly contradicts historical facts. On the contrary, almost all western countries had rapid (6–12 per cent) growth in the first phase (up to the first World War or the recession years of the thirties). In the second phase, growth was slower (2–3 per cent).

Thus observed growth rates could be explained by the model only if we dropped the assumption of a constant savings ratio. We would have to assume quite a high

* Bródy–Rácz [1966].

savings ratio. Table 3 shows what the savings ratio would have had to be to produce the observed growth rates with observed capital/output ratios.

Table 3

Typical savings ratios

Year	Typical annual growth rate	Typical capital/output ratio	Resulting savings ratio computed
1850	10 per cent	3	30 per cent
1890	10 per cent	4	40 per cent
1910	10 per cent	4.5	45 per cent
1930	3 per cent	3	9 per cent
1970	3 per cent	2.5	7.5 per cent

The computed savings ratios contradict whatever vague and sparse statistical evidence we have about actual savings ratios. Accumulation out of national income will generally surpass the 10 per cent limit in developed or developing countries, and can surpass 25 per cent for a couple of years at most.

Rostow [1960], analyzing the different phases of the growth process, calls the first phase "sustained growth after the take-off" or "the drive to maturity". What we described as "an approximately half century of industrialization" he highlights as follows: "Historically, it would appear that something like sixty years was required to move society from the beginning of take-off to maturity."

Rostow considers "a rise in the rate of productive investment from, say, 5 per cent or less to over 10 per cent of national income" a condition *sine qua non* for this growth phase and generalizes it by saying: "After take-off there follows a long interval of sustained if fluctuating progress . . . Some 10–20 per cent of the national income is steadily invested, permitting output regularly to outstrip the increase in population."

Thus, our first period, industrialization, with its extensive growth and high growth rate corresponds to his "drive to maturity"; our second phase, essentially intensive and with lower growth rate, corresponds to his "maturity". In neither phase are observed proportions compatible with expectations based on the one-sector model.

The actual savings ratio is not sufficient in the first phase to induce the growth observed; in the second phase saving is overabundant and could apparently trigger a much higher growth rate, if the one-sector model were correct.

In addition two broad observations are left unexplained by the one-sector model. First, those countries that grow fastest are not those where the savings ratio is high or where the capital/output ratio is low.* Second, the model is plainly inade-

* See, for example, Adelman [1966].

quate for analyzing "reconstruction periods". These periods – after wars, great recessions and other national emergencies – are sometimes followed by 20–25 per cent growth rates for several years. They begin with a great slump in national income. Any increase of the savings ratio is out of the question: usually savings will be under average, consumption requiring a greater share of the decreased national income. Yet, just after the emergency, outputs increase very rapidly in spite of the absolutely and relatively low amounts of savings and accumulation. Then when times normalize and the savings ratio assumes its former, higher rate, the fast growth rate slackens.

All these contradictions are more apparent than real, and can be eliminated by modifying the specification of the model and reinterpreting its dynamics. The Keynesian school – leading to the Harrod–Domar interpretation – abhorred "double counting" and reckoned in "national income" and "value added" terms rather than in "severely double counted", "social product" or "gross domestic output" terms. Yet, it is these double counted categories that give us the necessary clue. Growth rates of national income and of gross domestic output, GDO, will certainly coincide in the simple dynamic model. There seems to be no clear historical trend for GDO to increase faster or slower than national income. Savings ratios on a national income and a GDO base will therefore move in parallel. Rostow's 10–20 per cent savings ratio from national income will simply be transformed to a 7–14 per cent savings ratio based on GDO.

Yet the capital/output ratio is modified by the broader interpretation. As already pointed out, we have to consider capital intensity of manpower, too. We must reckon with labor's stock as well as its flow aspect. In the Harrod–Domar model labor plays a peculiar role. It only consumes part of the national income, as a hungry mouth, preventing it from being accumulated. It is like a net loss to the system, unproductive waste. To increase the part of national income consumed by the population has no positive impact on production. The system could work better with a savings ratio equal to 1. Actually if the number, skill, knowledge, quality of manpower are not increased (all these requiring massive amounts of investment), production will not grow, however high the savings ratio. This is really a lesson well learned by socialist and developing countries. And it is mostly human investment (or its lack) that will determine the growth rate. It is larger than any other investment for growth, and acts on growth with a very long lag period.

Let us now consider the three phases, reconstruction (annual growth rate 20–25 per cent), industrialization (growth rate 6–12 per cent) and intensive growth (growth rate 2–4 per cent).

In a reconstruction period it is characteristic that manpower, including skilled and highly qualified manpower, is obtainable in abundance; the labor market is glutted by would-be workers, educated and trained earlier. If productive capacities are more or less intact, all accumulation can go into filling depleted inventories. This growth, then, does not need much accumulation or investment at all, except for easing bottlenecks or replacing obsolete technology. With wages depressed even the oldest machines may still yield a profit.

Once idle capacities and manpower capacities are absorbed, every increase in output, every additional worker and every addition to this skill has to be paid for by society. Growth in "non-reconstruction" periods will therefore be much slower.

How can now the characteristic difference between "industrialization" and "maturity" be explained?

First of all we must be prudent in interpreting differences in characteristic growth rates. Some portion of the difference is undoubtedly due to statistical bias. The proportion of statistical coverage increases with industrialization, as more activities come under market control and later, efficiency of regulation and statistical collection improves. Thus there is an upward bias to estimates of the growth rate in the period of rapid industrialization. This in turn exaggerates the difference between observed growth rates for the industrialization and the mature period. Nevertheless it cannot explain away the entire difference.

The remaining difference can be accounted for by increased requirements for reproducing skilled manpower. An industrialized, technically developed country requires a labor force that is more valuable and needs much more expenditure for rearing, education and training. To quote Rostow: "Those in manufacture, construction and transport – including skilled workers – rose about in proportion to the total rise in the working force. But semi-skilled workers increased more than twice as rapidly as the working force as a whole; professional people and office workers three times as rapidly as the working force as a whole. The era of the professional technician, and of the skilled and semi-skilled worker had come; and this trend in structure of the working force has proved virtually universal to all post-maturity societies." Increases in human capital/output ratios may well counterbalance any tendency for the fixed capital/output ratio to decline. The total capital/output ratio, including human capital, may be therefore much higher in a mature economy than in a developing one.

Let us look into orders of magnitude, to see whether increases in human capital requirements have been significant enough to explain the differences in growth rates at various phases of development.

Section 2.1.3 noted that the capital/output ratios in reproducing manpower are equal to at least half of raising time. If therefore the average age of beginning work is 10 years, then the capital/output ratio is about 5. That is, 5 years' national income is tied up in youth not yet in the working force, that is, in the "gestation period" of investment. The assumption that children enter the work force at age 10 is quite realistic in societies before take-off. The picture may be complicated by international migration of human capital. Before 1930 the stream of immigrants to the United States endowed it generously with the human capital necessary for rapid growth. Transfers of human capital from poor to wealthy countries still persist and their value may well exceed transfers of conventional capital.

In a mature, industrialized economy the average age of entering employment is 16–18 years and is still increasing. With improvements in living standards, improvements in the system of education and health-care (dictated not necessarily by humanitarian considerations but by sheer needs of modern production processes and overall efficiency) the human capital/output ratio has at least doubled. Probably human capital has increased even faster, considering the increase in consumer durables, shortening of the working week and day, increases in social funds – security, health, recreation, etc. Let us stick to a conservative guess of 10–15 years' national income invested.

This capital/output ratio makes the traditional one – reproducible tangible wealth representing 2–3 years' national income – seem to dwindle. Thus the total capital/output ratio which might be around 7–8 years' national income (say, 3 for equipment and 5 for human investment) at the beginning of industrialization will increase to 12–17 years (say, 2 for equipment and 15 for human investment) in maturity. This difference, then, clearly entails a slow-down of the growth rate attainable by a mature economy, reducing it, at least in order of magnitude, to half of that of the industrialization period.

This process has a lot of side-effects, phenomena observed long ago by sociologists and economists. The increasing share of the service sector in GDO has long been noticed. Many of these services really serve the reproduction process of manpower. Rising educational and health expenditures incurred by the state are often mentioned. The new economic power of youth is well known, and does not deserve lengthy comments. They are a market of their own, creating their own music, art and theatre, garments and fashion, and perhaps even their science and universities in the future. All this is related to the increasing importance of this "gestation period", the "goods in process" part of manpower.

The above gauging of orders of magnitude needs to be supplemented with more precise, more detailed estimates. But the rough estimates suffice to explain the turning point from rapid industrialization to slow maturing. The main levers for increasing the growth pace in maturity will not be found in increasing the traditional savings ratio, or improving the technology of manufacturing or transport, but only in making the system of education more efficient, by avoiding senseless waste of time and teaching of irrelevant knowledge, by shortening and reorganizing curricula and by connecting learning with doing.

Here is where the socialist countries have an advantage as compared to the private-ownership countries. Where there is no difference in the ownership of schools and of enterprises, their "production" processes can be better coordinated. Whether they do indeed effect this coordination remains to be seen.

In certain partial questions the planned economies have already recognized their growing duties and made plans and established institutions for housing, health, family care, education, culture, etc. Still, coordination and integration of all these processes have hardly begun.

Human capital does increase continuously. But there is a turning point where the growth rate changes suddenly, within the stretch of a couple of years or at most a decade. Explanation of this abrupt change is tied to the problem of a closer coordination of schooling and production.

The turning point comes when the influx of agricultural population into industry slackens and finally stops. Before the turning point youth already trained at an early age in agriculture (trained mostly on the job) comes to industry (usually into mining or construction) and acquires the necessary additional skill by further on-the-job training. There is little waste of time, the worker is productive while learning – and slowly educates and works himself into higher and higher jobs.

This inner migration (from agriculture to industry and within industry toward higher qualification) is almost totally discontinued after the end of industrialization. Industrial, that is, "mature" societies tend to train each individual for his very specialized role as machinist, draftsman, teacher or whatever else. The long gestation period impedes higher growth rates.

There is no way out: if we want more growth, we have to integrate education and production, learning and doing, securing at the same time the mobility of population towards higher qualification.

3.3.2. *The closed model*

The first closed-model computations were made at the Harvard Economic Research Project [1954] in the early fifties. They, and Tsukui's early work [1965] were never published. These pioneering computations established the feasibility of statistical implementation and computation of the closed dynamic model.

A later simplified version* of the model assumed that every commodity i has a uniform life span t_i, independent of its destination. Hence the matrix **B** was derived from the flow matrix **A** by multiplying each row of **A** by an estimated average life span. If we define **T** as a diagonal matrix made up to the t_i life spans, $\mathbf{T} = \langle t_1, t_2, \ldots, t_n \rangle$, then $\mathbf{B} = \mathbf{TA}$ and the simplified primal model becomes $\mathbf{x} = (1 + \lambda\mathbf{T})\,\mathbf{Ax}$.

This simple system requires less data for implementation and is very easy to solve. Besides the matrix **A** it requires only informed guesses about life spans. It was computed for 7×7 order **A** matrices roughly descriptive of the American economy in 1947 and 1958, displayed in Table 4.

The guesses about the appropriate life spans were

1. Agriculture, food, textiles	1 year
2. Chemicals, plastics, rubber, metals	2.5 years
3. Machinery, fabricated metal products	15 years
4. Construction, cement, glass	50 years
5. Fuel, electric utilities	0.1 year
6. Transportation, services, undistributed	0.1 year

The above guesses were based on the following reasoning.

Sectors 1 and 2 produce most of the inventories, including those of households. The results of the computation are not sensitive to varying these life spans, between 0.5 and 3 years.

Sectors 3 and 4 are indeed the most important, and they are taken from Domar [1957]. Moderate changes of these magnitudes will affect the growth rate proportionately – this is one of the sensitive spots of the model.

* See Bródy [1965].

Table 4

Flow coefficients for the American Economy 1947 and 1958
(rounded to 4 digits and multiplied by 10^4)

	1	2	3	4	5	6	7
1947							
1. Agriculture, food, textiles	3924	400	195	798	98	839	2335
2. Chemicals, plastics, rubber, metals	298	3758	1876	857	238	218	258
3. Machinery, fabricated metal products	245	303	2586	1176	373	561	20
4. Construction, cement, glass	128	219	147	684	462	541	1
5. Fuel, electric utilities	122	380	66	217	3420	212	353
6. Transportation, services, undistributed	1237	998	751	1650	895	1825	6090
7. Households	5094	2112	3721	3342	1561	3338	399
1958							
1. Agriculture, food, textiles	4014	319	168	662	149	619	2778
2. Chemicals, plastics, rubber, metals	294	3024	1420	716	191	226	177
3. Machinery, fabricated metal products	232	493	2521	1300	487	576	22
4. Construction, cement, glass	112	240	141	913	546	403	6
5. Fuel, electric utilities	152	497	101	247	3311	346	571
6. Transportation, services, undistributed	1244	1479	1112	2081	1070	2142	6117
7. Households	3269	1837	2451	3809	526	4511	430

Source: Preliminary estimates in the files of the Harvard Economic Research Project 1965.

Sectors 5 and 6 produce mostly non-durables and here the life span has almost no effect on the growth rate.

For the seventh sector, households, a twenty-year guess seems to be appropriate. However, the total payroll, on which the coefficients of the last row are based, covers the needs of Extended as well as Simple Reproduction. In other words, the wage rate is above subsistence and includes expenditures for education and qualitative improvement of the labor force. Expenditures on capital account are included in the flow coefficients because they are not distinguished in the statistics. Accordingly $t_7 = 0$.

The computation started with actual gross domestic outputs in 1947 and 1958 and followed the iterative procedure formulated in Section 2.2.2. The results are summarized in Table 5.

The first approximation of λ, based on actual prices and output proportions, differs from the theoretical rate by only 0.046 for 1947 and 0.01 for 1958. The observed average growth rate of GDO was 3.56 per cent between 1947 and 1958. This is less than the computed rate, probably because 1947 was a year of upswing while 1958 was a recession year.

For 1947, computed outputs differed from observed by as much as ten per cent. Construction, cement and glass output fell short of their optimum — and subsequent fast development of this sector verifies this. The lag might have been related to post-war readjustments. Computed output of chemicals, plastics, rubber and metals was well above actual, but the sector's growth rate did not accelerate later. Here crude aggregation might have been the reason: a lot of information is buried when we add fast-growing chemicals and plastics to metals which were losing relative importance.

Table 5

Actual and computed outputs for the American Economy
1947 and 1958

For the simplified growth model
(Billions of 1958 $)

	1947		1958	
	Actual	"Optimal"	Actual	"Optimal"
1. Agriculture, food, textiles	139.3	135.4	192.9	199.2
2. Chemicals, plastics, rubber, metals	53.0	61.0	62.0	66.0
3. Machinery, fabricated metal products	78.1	68.4	104.5	105.1
4. Construction, cement, glass	49.4	61.2	81.0	84.1
5. Fuel, electric utilities	28.2	26.8	55.3	56.2
6. Transportation, services, undistributed	210.0	206.4	335.5	319.8
7. Households	210.4	209.2	293.0	294.9

Table 6

Growth rates

Computed for the simplified growth model
(per cent per year)

	Iteration 1	2	3	4	5	6
1947	3.876	3.906	3.920	3.922	3.922	3.922
1958	3.779	3.764	3.779	3.776	3.778	3.778

The 1958 results were closer to actual with an average error of 2–3 per cent and no error exceeding 5 per cent except in chemicals. Indeed, both computations suggest that the economy actually runs close to its "Neumann-path", its "turnpike". Thus the stationary state has explanatory power.

When we find deviations between actual and "optimal" output, which should we trust? It might be that the computation is entirely sound and that real economic proportions are inefficient. How, then, can the deviations be allocated between "faults of the model" and "faults of reality"? This question needs several years' work in a well-known economy and must be left to the future.

The successive values for λ in the course of iterations are displayed in Table 6.

Note that in 1958, the recession year, the computed growth rate is lower, and actual proportions are closer to optimal ones than in 1947. In spite of this, the iterations cause the computed growth rate to fluctuate and the oscillation can be traced in the iterated output vectors, too. On the other hand, the 1947 data give a straightforward, non-oscillating convergence.

The same model, but with a complete **B** matrix, was computed for Hungary, 1962. Table 7 gives the stock and flow matrices.

Table 7

Flow and stock coefficients for Hungary, 1962
(rounded to 2 digits and multiplied by 10^2)

	A matrix 1	2	3	4	5	B matrix 1	2	3	4	5
1. Industry	41	12	33	84	40	51	130	575	81	74
2. Agriculture	6	35	2	8	12	3	51	1	2	1
3. Other productive	5	3	7	7	11	0	0	0	0	0
4. Foreign trade	10	2	2	0	2	20	14	67	22	1
5. Households	21	27	32	19	20	2	3	3	1	500

Both matrices are crude approximations, heavily aggregated. Yet, the results are quite in accord with outside knowledge about the performance for our economy. They are displayed in Table 8.

The outcome of computations (considering GDO as of unit amount, that is, giving sectoral shares in GDO) was:

Table 8

Actual and computed outputs for Hungary, 1962
(in percentages of total output)

	Actual	"Optimal"
1. Industry	46.4	45.2
2. Agriculture	10.5	11.4
3. Other productive	7.0	6.8
4. Foreign trade	9.0	6.4
5. Households	27.1	30.2

The deviations are quite as expected, indicating labor shortage, over-industrialization, underdevelopment of agriculture, and an inflated foreign-trade turnover. Nevertheless the deviations from the turnpike path are negligible and exceed 10 per cent only for Households and Foreign Trade, the very sectors where data are the least reliable (particularly in column 5 of matrix **B**).

The convergence in λ was rapid and non-oscillating beginning with 5.33 and ending with 5.35. The observed growth rate for 1962 was slightly over 5 per cent and the average for the period 1960–1965 was about the same.

Convergence to four digits in the eigenvector took only 7 iterations, although the actual (initial) price system was out of proportions. The dual computation took 9 iterations. Its results are displayed in Table 9.

The results make sense. Manpower and foreign currency are underpriced. The second fact is perhaps responsible for the excessive foreign trade, resulting in an enormous drive to import. The first may well be responsible for the labor shortage

Table 9

Price indices for Hungary, 1962

	Production price/ actual price
1. Industry	0.84
2. Agriculture	0.85
3. Other productive	1.05
4. Foreign trade	1.11
5. Manpower	1.14

The only result that runs counter to our understanding of the economy is that industrial and agricultural products are overpriced in the same proportion. Our belief that agriculture was underpriced in relation to industry, was based on actual wage rates. It may well be that, if wages were raised to the production price level, the situation would have changed. Considering not only direct but total wage inputs agriculture is less wage-intensive than industry.

Anyhow, with such crude data the results are surprisingly close to the facts. This should certainly justify more detailed and refined tabulations, particularly for stocks of the Household sector, a territory that is universally neglected.

Let us turn now to other contemporary work. The most detailed study was undertaken for Japan.

The original computation of Tsukui* was followed by a study of the balanced growth path for the Japanese economy for the seven years 1960 to 1966. Variants of the model embraced 10 to 12 sectors. The first variant projected linear growth of consumption and did not constrain disposal activities when it started to reach the turnpike path. This model discontinued some activities quite suddenly before it reached the balanced path. Thus it predicted cutbacks for the first two years in certain sectors and then, for later years, had to supply additional investment for the capacities discarded in the beginning. No wonder its time path was slightly inferior to actual growth experienced in Japan.

The second variant was essentially a closed dynamic model. It assumed exponential growth of consumption and fixed coefficients. The third model, besides minor differences in treating foreign trade, incorporated technological change.

* Its theoretical outlines were given in Tsukui [1965].

Variants 2 and 3 gave a turnpike path quite close to the actual one. The third model was not significantly better than the second: Tsukui–Murakami–Tokoyama [1969] write:

"We can first note that specification of the objective function is immaterial. A solution to the second model – along with the solutions to other models which we cannot introduce here – shows that any efficient path can be approximated by the turnpike during almost all of the planning periods . . .

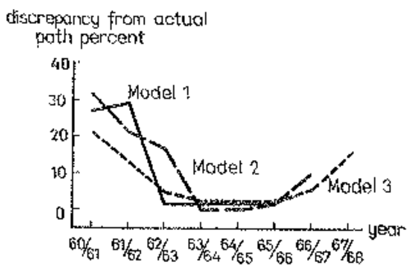

Fig. 9. Turnpike and actual path for Japan

Let us now compare our solutions with the actual path of the Japanese Economy. Knowing that the Japanese Economy seems relatively unstable, we are naturally led to conjecture that the actual path will be rather divergent from, than convergent towards the turnpike. But, surprisingly enough, such a tendency of divergence is not revealed, as we may observe in Fig. 1.* The actual path of the Japanese Economy clings to the turnpike, so to speak.

"There is a gap between the actual path and the planned path, of course. For example, a gap between the actual path and the planned path is widening in the transportation equipment industry. We seem still to underestimate an unmistakable trend of mass motorization in Japan. Chemical industry may possibly be another example. In other industries, however, we may be able to detect a tendency toward the turnpike.

In Model 1, the planned path is generally lower than the actual path because of the "waste" incurred in the first two years of adjustment, but, from 1963 on, the actual path and the planned path generally remain parallel, which means that a gap between the two sets of output ratios is not widening. In Model 2, the actual path and the planned path are generally narrowing their gap especially at the later stage of the planning period. In Model 3, the two paths often remain parallel, though the planned path generally gets ahead of the actual path in the last two years. Our evidence for a coincidence of the actual path and the planned path may not be a strong one in itself. But, if we recall the relative instability of the Japanese economy, our evidence will become more than impressive. We may suggest a conclusion that the actual path of the Japanese economy is surprisingly close to the turnpike, or is not so far from the turnpike as is expected from its interindustrial structure."

The Japanese computation is all the more conclusive because these were the years when Japan had very rapid growth unrivalled in Western economies. The growth rate was close to 10 per cent and a revolutionary structural change was under way. Though it has fixed coefficients, this closed model seems to work even in tough situations.

* Displayed here as Fig. 9.

3.3.3. *The open model*

In recent years, two versions of the open dynamic input–output model have been implemented in considerable detail in the United States. The earlier is that of Clopper Almon, now becoming a tool of practical business forecasting. More recently Leontief has published a reformulation of the open dynamic model [1969] he first published in 1953.

Let us discuss Leontief's computation first. His formulation shows how similar static and dynamic, open and closed models really are.

He starts with the difference equations of the open dynamic model:

$$\begin{aligned}
(1-A+B)x_t \quad &- Bx_{t+1} = y_t\\
(1-A+B)x_{t+1} &- Bx_{t+2} = y_{t+1}\\
&\vdots\\
(1-A+B)x_N \quad &- Bx_{N+1} = y_{N+1}\\
&\vdots
\end{aligned}$$

and so on to infinity.

We can now write up the corresponding infinite matrix. It is doubly infinite because the reproduction process has no beginning and no end, but it is easier to grasp the essence of this matrix, if we write out only 3 equations and then supply the missing ones mentally. Designating $G = (1 - A + B)$, the first three equations can be written

$$\begin{bmatrix} G & -B & 0 \\ 0 & G & -B \\ 0 & 0 & G \end{bmatrix} \begin{pmatrix} x_t \\ x_{t+1} \\ x_{t+2} \end{pmatrix} = \begin{pmatrix} y_t \\ y_{t+1} \\ y_{t+2} \end{pmatrix}.$$

The third equation is left truncated, the matrix $-B$ missing (and its multiplier, the vector x_{t+3}, missing, too). What we have in reality is a doubly infinite band matrix, with G in its diagonal and $-B$ above. But the method of handling it and solving the system should be clear from the truncated form.

It is easy to check by direct computation that, if the matrix G has an inverse, G^{-1}, then the form below is a solution for x of the truncated system:

$$\begin{pmatrix} x_t \\ x_{t+1} \\ x_{t+2} \end{pmatrix} = \begin{bmatrix} G^{-1} & G^{-1}BG^{-1} & (G^{-1}B)^2G^{-1} \\ 0 & G^{-1} & G^{-1}BG^{-1} \\ 0 & 0 & G^{-1} \end{bmatrix} \begin{pmatrix} y_t \\ y_{t+1} \\ y_{t+2} \end{pmatrix}.$$

The rule for forming the inverse of the infinite band matrix should be clear from this, too. The inverse has G^{-1} in its diagonal, and every successive band above the diagonal is premultiplied by $G^{-1}B$ again and again.

The infinite matrix will have a convergent inverse only if $G = (1 - A + B)$ is regular, and $G^{-1}B = (1 - A + B)^{-1}B = (1 + QB)^{-1}QB$ has all its eigenvalues within the unit circle. Otherwise its powers will not converge. Now if QB has the eigenvalues, ϱ_i, then $G^{-1}B$ will have the eigenvalues $\varrho_i/(1+\varrho_i)$ and these will usually be less than 1, except when $\varrho_i \leq -0.5$. Yet the latter difficulty can always be remedied by increasing the time unit used for measuring B and thus decreasing B itself. Increasing the time unit will decrease the modulus of all the eigenvalues, and hence ϱ_i can be made greater than -0.5 if it is negative. This secures the regularity of $G = 1 - A + B$, too. That is, $(1 - A)$ being regular, one can decrease B to a minor perturbation by changing the time unit, then nothing endangers the regularity of G.

Now if regularity and convergence are granted, one can prove that every row and every column of the infinite Dynamic Inverse adds up to $(1 - A)^{-1} = Q$. As noted above, a typical row of the infinite inverse, starting with the diagonal term:

$$G^{-1} + G^{-1}BG^{-1} + (G^{-1}B)^2G^{-1} + \ldots + (G^{-1}B)^nG^{-1} + \ldots =$$

$$= \sum_0^\infty [(1 - A + B)^{-1}B]^n (1 - A + B)^{-1} =$$

$$= [1 - (1 - A + B)^{-1}B]^{-1}(1 - A + B)^{-1} =$$

$$= [(1 - A + B) - B]^{-1} = (1 - A)^{-1} = Q\,.$$

Thus the infinite form of the Dynamic Inverse is a generalization, or rather a disaggregation of the Open Static Inverse. If the destabilizing features mentioned above do not manifest themselves, then we can consider the Static Inverse an aggregated form of the Dynamic one. The Static Inverse answers the question: How big must the total outputs be to make the production of a given final bill of goods, y, possible? The Dynamic Inverse answers the same question, only in more detail. It dates the total outputs and tells, how much of the total amount already given by the Static Inverse has to be produced now, how much of it had to be produced last year, and how much of it two years ago – and so on. The analytical power of such a matrix is immense. It specifies not only the magnitude but also the timing of the streams required to deliver some end product.

Leontief computed the Dynamic Inverse for the United States at 59 sector detail both for 1947 and 1958 technology, and also performed a third computation with changing technology. (He had given time subscripts to the A and B matrices and interpolated the formulas for the years between 1947 and 1958.)

In spite of considerable technological change in the decade covered by the computation, the shapes of the time paths of the necessary outputs do not differ much from those based on a fixed technology. This indicates a fairly stable pattern of timing and suggests a certain regularity of lags.

Almon [1957] made a long-range forecast using the Open Dynamic Model. Intended as a guide and framework for practical business decisions, it was first computed in 1964 and covers a full decade of the American future, to 1975.

Almon's model is cut open only with respect to foreign trade and government expenditures. Since its objective is practical forecasting, he uses the best available estimates for future flow and capital coefficients and an ingeniously adapted version of the Houthakker–Taylor method of projecting household expenditures. It is an impressive structure, taking full advantage of modern computing techniques.

His experience indicates that, whenever there is enough information to specify and implement the non-linear parts of the model, it can be done. The model incorporates and embraces additional information with ease. Understandably better-than-linear approximations are first sought for the consumer expenditures and labor productivity. More recently he has introduced non-linear investment functions in place of fixed capital coefficients.*

Almon's method of solution deserves attention. As already indicated, when we solve the difference equation system $B\Delta x = (1 - A)x - y$ in the usual way, with the base year outputs given, that is, by solving it for Δx, the computation $\Delta x = B^{-1}(1 - A)x - B^{-1}y$ will be ill-conditioned.

Almon has devised an iterative algorithm that solves the model for x. That is, he starts from $(1 - A)x_{j+1} = B\Delta x_j + y$. This amounts then to the iteration $x_{j+1} = QB\Delta x_j + y$.

This approach will certainly converge, QB being well-conditioned and positive. Its dominant eigenvalue is the reciprocal of the growth rate, which in turn has a positive eigenvector, the balanced growth path. Thus the computation will approach "the turnpike" – a more sophisticated turnpike than our stationary proportions because the model reckons with non-linearities and changing technology.

Almon's forecasted overall growth rate is slightly above 5 per cent. This is at variance with my – admittedly very crude – computations reviewed in Section 3.3.2 which indicate long-range growth below 4 per cent. The difference may be due to the fact that human capital is neglected, or rather not entirely included in Almon's model. Anyhow, it is up to the facts to decide this question. The official figures up to now are in favor of Almon's forecast. But there are five more years to go.

* See Almon [1969].

Summary

It is generally taken for granted that Marxian economics and recent achievements in mathematical economics with large-scale digital computation are worlds apart. Certainly this need not be the case. This study formulated Marxian theories of value and reproduction so as to reveal basic similarities in the essential logic of both approaches. From the synthesis, a practical model emerged which is mathematically solvable, computationally well-conditioned and statistically easily implemented. This model builds upon quantities and relations implicit in traditional methods of volume and price planning already in use. Of necessity, the theoretical scope of the work was limited and theories of currency, interest and rent were not treated.

*

1. Setting up of the Model

Value theory and reproduction theory are dual reflections of society's economic metabolism. They can be comprised in two systems of equations, two models. These models are tied by a close interdependence or symmetry called duality. Duality stems from the fact that both equation systems are based on the same coefficients. These coefficients reflect the structural interdependence of the whole economic process, its basic proportions. Value theory and reproduction theory can be developed in parallel by the dual interpretation of the single central structure.

1.1. *Simple Reproduction*

The central task of every economy — whatever its specific institutional form — is to allocate society's live and embodied labor to particular functions or areas of employment.

Recording all the a_{ik} "input coefficients",* or — to use Marx's expression — "the quantitative rules and proportions" to which the division of labor gives rise, we set up the matrix $A = \{a_{ik}\}$. This matrix characterizes the productive

* The input of product i used to produce one unit of product k. We only postulate that such coefficients exist under given circumstances, at a given time and place, and that they are statistically measurable with the necessary precision. No question of change or stability enters the discussion until the third part of the work.

processes of the complete and closed system. (The matrix includes household coefficients, the necessary inputs for reproducing labor power, too.)

With the aid of this matrix we set up a system of equations, which allocates labor power among the different branches of production. To solve this problem in the abstract does not require a notion of value. This "primitive" model gives an exact criterion for reproduction to be "simple", that is, non-expanding:

Simple Reproduction is possible if and only if the maximal characteristic value of our matrix is equal to 1, $|\mathbf{A}| = 1$. If $|\mathbf{A}| < 1$ it is possible to expand the production process, if $|\mathbf{A}| > 1$ reproduction ceases to be complete and only diminishing, restricted reproduction is possible.

As the dual of this allocation problem we set up the equation system which determines the values of products according to the classical labor theory of value.

In terms of the production vector, $\mathbf{x}$, and value (or price) vector, $\mathbf{p}$ (given by two eigenvalue equations $\mathbf{Ax} = \lambda\mathbf{x}$ and $\mathbf{pA} = \lambda\mathbf{p}$), we can formulate our fundamental theorem in the following dual fashion:

Given the non-negative and irreducible matrix $\mathbf{A}$ characterizing a closed and complete system of production

(a) if we can find a positive production vector $\mathbf{x}$ [price vector $\mathbf{p}$] for which $\mathbf{Ax} = \mathbf{x}$ [$\mathbf{pA} = \mathbf{p}$], then Simple Reproduction is possible in this system,

(b) if we can find a positive $\mathbf{x}$ [$\mathbf{p}$] vector, for which $\mathbf{Ax} < \mathbf{x}$ [$\mathbf{pA} < \mathbf{p}$], then Extended Reproduction is possible. That is, the $\mathbf{x} - \mathbf{Ax} > 0$ surplus product [$\mathbf{p} - \mathbf{pA} > 0$ surplus value] may be used to increase production or may be withdrawn from the economy without jeopardizing Simple Reproduction,

(c) if neither (a) nor (b) is fulfilled, then only Restricted or Incomplete Reproduction is possible.

The theorem leads to a proof that the matrix $(1 - A)$ is regular under conditions of Simple or Extended Reproduction.

The matrix $\mathbf{A}$ is irreducible because every product needs labor input and every product serves – directly or indirectly – the fulfilment of human wants.

The model of Simple Reproduction describes the historical situation of simple commodity production, which Marx considered typical before the advent of capitalism. Little or no surplus value was created. If a greater amount of surplus value is created and not extended to increase production but consumed unproductively, the above theorems must be modified. In this case the last element of the value vector no longer measures the value of labor power but the value created by it. If we disaggregate our model by separating "paid" and "unpaid" labor and productive and unproductive consumption, a "rate, of exploitation" is determined.

1.2. *Extended Reproduction*

The Marxian system of production prices values products according to the total (that is, direct + indirect) amount of investment tied up in their production.

The equation system is analogous to that Section in 1.1 but based on the matrix $\mathbf{A} + \lambda\mathbf{B}$.*

The "law of value" of simple commodity production says that exchange on the market is regulated by the proportions of total labor necessary to produce commodities. This "law" is different under capitalism: exchange is regulated by the proportions of the total investment tied up in the production of different commodities.

Analysis of the mathematical conditions for Extended Reproduction shows that a unique positive price system and positive average rate of profit always exist. In this sense the Marxian definition is unambiguous and workable.

The dual system gives stationary proportions of production which may be increased by a common factor, λ, the growth rate. λ is the dual of the average rate of profit.

Although this dual model of reproduction is deficient in many respects, it clarifies the Marxian equilibrium growth conditions in a state of constant "organic composition of capital".

Analysis of turnover time points up an error in the common practice of dividing inputs into "flow" and "stock" inputs. Flow and stock are only two aspects of the same economic transaction. Every transaction is a "flow" and a "stock" at the same time. They are connected by turnover time. By extension of this notion the mathematical structure of matrix B is analogous to that of matrix A and thus both must be non-negative and irreducible. It also becomes clear that the dimension of the average rate of profit (growth rate), λ, must be the reciprocal of time $[T^{-1}]$. The exact relation and difference between turnover time and life span are cleared up. Finally, we postulate probabilistic durability instead of fixed life spans and extend the concept of conservation of value. We can assume that every product imparts its value in one lump sum at the end of its probabilistic life span. Under probabilistic conditions turnover time and life span become equal.

1.3. *Related models*

The systems of interdependence which Marx investigated with simple mathematical tools are shown to be mathematically equivalent to well-known modern linear models of the economy. Close formal resemblances are shown to exist among four major well-known and widely used models: the von Neumann model, the theory of games, the Leontief model, and the linear programming model. These and the model set up above can all be transformed into a common mathematical form. This is clear, even though they have on the surface strikingly different orientations: deterministic-causal, teleological-optimizing, equilibrating. It demonstrates the

* Here λ is the average rate of profits and $B = \{b_{ik}\}$ the matrix of capital coefficients. As in the case of the flow input coefficients we postulate only that capital coefficients exist and are measurable.

validity of von Neumann's important remark: "... if one has really technically penetrated a subject, things that previously seemed in complete contrast, might be purely mathematical transformations of each other."

"Closed" and "open" models in this family are compared and their relative merits, principal fields of application and possibilities of joint exploitation are analyzed.

Each of the models is characterized by the essential property of duality. It is this dual viewpoint which makes ordering, measuring and controlling of great systems possible and manageable.

2. Discussion of the Model

Part 2 explores some further features of the mathematical model which stem from its particular logical structure and form.

2.1. *Three types of price systems*

The function of prices is considered first. Under circumstances of Simple Reproduction it is value prices which orient properly in choosing production techniques. Under Extended Reproduction the same task is performed by production prices.

A proof based on an inclusion-theorem for eigenvalues is offered. Using this mathematical theorem, it is possible to answer the question of "worse or better" without presupposing any special utility or preference function. The more complex and intricate questions "how much worse or better?" and "better for whom?" are sidestepped.

It is also proved that the so-called "two-channel" price system (cost + mark-up after investment + mark-up after wages) is a totally consistent system of production prices where the mark-up after wages serves as interest on human capital.

The two-channel price applied in socialist economics is actually a mixture of value and production prices. Because of the absolute and relative increase in human capital, the properly orienting system of prices tends in the future toward value prices.

The historical evolution of the particular types of prices are related to the historically changing needs and forms of the economy.

2.2. *Circularity*

The Marxian development of labor theory has been severely criticized because of its "circularity". The proper answer to this does not consist in denying it but in clarifying the role, scope, appropriateness and consistency of certain "circular" definitions used for defining value or prices.

The eigenvalue equation $Ax = x$ is itself a circular definition in x, and yet its analytical value is clear. Our mathematically formulated definitions of value prices, production prices and two-channel prices are eigenvalue equations. Each of these definitions is unambiguous and analytically straightforward.

With the system more explicitly formulated some issues concerning the value of simple and skilled labor, and the value of joint products are cleared up.

"The problem of transforming values into production prices" provides the framework for setting up an iterative process for computing the stationary states. It converges quickly. Another major advantage of this procedure is that it begins every iteration with the same initial data, thus minimizing problems of rounding and error-accumulation.

2.3. *Miscellanea*

Dimensionality problems connected with the model are taken up next. Detailed consideration is given to λ, the average rate of profit or growth rate. Its exact dependence on time spans is explained.

The model allows us also to relate economy of inputs to economy of time: 1 per cent increase of material inputs is equivalent to a $\frac{1+\lambda t}{\lambda t}$ percent decrease of turnover time on a microeconomic level. (Here t stands for turnover time.) On a macroeconomic level the corresponding percentage is $\frac{1}{1-\alpha}$ (here α stands for average share of inputs $\alpha = \frac{pAx}{px}$).

Three directions for generalizing our model are explored: The axiomatic approach is investigated first: 6 axioms set up the fully-fledged model and 2 economically self-evident propositions (sufficient but not necessary) secure the existence and uniqueness of solutions.

Second, alternative mathematical forms are surveyed. Linear differential equation systems with a time parameter and, finally, linear operator equations are considered. It is not lack of tools but lack of information which constrains us to the use of the linear model with constant (observed or planned) coefficients.

Third, the most promising direction of generalization involves a probabilistic approach. Here the equivalence of our model with a Markovian chain process is proven. This suggests that coefficient information be interpreted in a new way: each is the expected value of a probability distribution.

These modifications and generalizations of the model do not change its essential character. Its connections with labour theory and classical economics are preserved.

3. Application of the Model

Before using our model as a tool for realistic analysis, forecasting and planning (that is, as a description not only of abstract states but of real processes), we must reconsider some of its basic features. In particular the "stability" of coefficients, the "closedness" of the model and the interpretation of "stationary" states need further attention.

3.1. *Problems of application*

Reasons are presented for using "stationary" or "equilibrium" states as benchmarks for planning. The turnpike-theorem is introduced with a common-sense interpretation, and the instability of this equilibrium path is demonstrated. Thus any deviation from the equilibrium path leads ultimately into an impasse.

With changes in technology the "optimal path" changes, too. There are several methods for reckoning with these changes: expert guesswork, statistical extrapolation, borrowing data from more developed countries, and statistical estimation of "technological layers". The most efficient method seems to be programming within the sectors to achieve an optimal mix of technologies.

Constant coefficients, or ill-planned ones, and unavoidable aggregation are sources of errors in applied work. Theoretical sensitivity analysis proves the following important and optimistic theorems:

1. Aggregating on the basis of production prices and optimal production proportions is error-free.
2. Aggregating on the basis of faulty prices and proportions will never increase their faults.
3. A probabilistic approach leads us to expect significant cancellation of errors with aggregation.
4. The deviation of actual prices and proportions from the theoretical eigenvectors never affects the computation of λ significantly.
5. The estimation of the capital matrix may be less exact than that of the flow matrix by an order of magnitude without endangering the exactness of the results.
6. The mutual allocation errors of statistics do cancel each other without seriously affecting the computation.

3.2. *Thoughts on planning*

Our model specifies all the major categories used in long-range planning. It gives numerical values to social product, national income, accumulation, wage fund, investment, consumption, etc., and computes their structure, interrelations and variations at any given level of detail.

Furthermore it deals simultaneously and consistently with the many estimates and computations which are traditionally carried out separately and independently in the first stages of drafting a national economic plan. Besides coordinating the various facets of planning, the present approach makes it possible to turn out consistent first variants in a relatively short time.

In planning, the assumption of constant coefficients and a stationary state must be interpreted realistically. After planning the changes in coefficients, we want to know the consequent changes in the optimal path. A good plan will follow the optimal trajectory based on the planned coefficients.

At the same time the dual solutions can be computed for price planning, too.

Some problems of planning require opening the model. Certain non-productive or non-reproductive activities must not be neglected in the course of planning. Inputs for these activities may be treated as final consumption. Mathematically the open model is not especially difficult to solve, but the solution remains a function of final consumption.

The theory of optimal processes, developed by Pontryagin *et al.*, is most useful in dealing with such problems. Final consumption can be treated as a variable load on the economy, as a control-variable. As a means of control it offers systematic solutions to two deep problems of planning: It chooses the time-optimal transformation of the economy from one given state into the state sought for and (in another version) it chooses the transformation trajectory which maximizes output.

3.3. *Practical computations*

A totally aggregated version of our model (comparable with the Harrod–Domar growth model but taking human capital into account, too) explains the great differences in historical growth rates: 20–25 per cent for recovery periods, 6–12 per cent for industrialization and 2–4 per cent for a mature economy. The dominant force constraining development is the reproduction of human resources, skilled and highly qualified manpower.

A rough computation shows that in a mature economy human capital amounts to approximately 7–15 times the national income, whereas all the tangible investment never exceeds 3–4 years' national income. For long-range planning this aspect of development is thus of utmost importance.

Finally, some actual computations made by W. Leontief, C. Almon, J. Tsukui, A. Carter and myself are presented. Although the experience thus far is not sufficient to demonstrate the appropriateness of the model definitively, the evidence is unequivocally favorable.

Appendix I

Definitions and theorems

A vector $x \neq 0$ is called an *eigenvector* of the square matrix A, if there is a number, α, called the *eigenvalue*, belonging to it, such that $Ax = \alpha x$.

Every element of the eigenvector is multiplied by a common factor, the eigenvalue, and thus multiplication by the matrix does not change the proportions of the elements of the eigenvector. It follows that eigenvalues and eigenvectors can be computed from the *eigenequation* $(A - \alpha I)x = 0$.

We solve this homogenous equation by setting $\det(A - \alpha I) = 0$.

The determinant, expanded in powers of α, yields a polynomial of degree n. It has n, not necessarily distinct, roots and these are the eigenvalues of A.

We are interested in the non-negative eigenvectors of a positive matrix, $A > 0$. Therefore we set $x \geq 0$ and survey those positive values of α that satisfy $Ax \geq \alpha x$ and are maximal for various $x \geq 0$.

We can now prove the following:

1. *α has a maximum for some x*

Because of $A\varrho x = \varrho Ax$ we need to consider only the closed and bounded simplex set: $x_i \geq 0$; $\Sigma_i x_i = 1$. The maximal value of α is a continuous and bounded function on this simplex, because $0 < \alpha < M$, where M is the maximal row sum of A. Therefore it will have a maximum somewhere.

2. *The maximum value of α will occur at a point where $Ax = \alpha x$*

If α were at a maximum when $Ax \geq \alpha x$, then the vector $c = Ax - \alpha x \geq 0$ would have to have at least one positive element. In that case $Ac = A(Ax - \alpha x) > 0$, and hence $A(Ax) > \alpha Ax$. If the latter is true α cannot be maximal. Therefore, $Ax = \alpha x$ is a necessary condition for a maximum. Thus the maximal value of α will be an eigenvalue and the corresponding vector, x, where this maximum occurs, will be an eigenvector.

3. *The eigenvector, x, is strictly positive*

$Ax > 0$ otherwise implies $Ax \geq \alpha x$. Yet according to 2, this is excluded. It follows that $x > 0$.

4. *x is determined unambiguously*

Let us suppose the contrary: two vectors, x_1 and x_2, belong to the same eigenvalue, α. We can then choose ϱ so that $c = x_1 - \varrho x_2 \geq 0$ with at least one element of c equal to zero. Then $Ac = Ax_1 - \varrho Ax_2 = \alpha(x_1 - \varrho x_2) = \alpha c$. Therefore c would have to be another eigenvector belonging to α. But according to 3, none of the elements of an eigenvector can be zero. Therefore only one eigenvector can belong to α.

5. *The matrix* A *has no non-negative eigenvector other than that belonging to the maximal eigenvalue* α

According to 4, there will be just one eigenvector x, belonging to the eigenvalue α. But let us assume that there exists another positive eigenvalue, β. Since α is maximal $0 < \beta < \alpha$. Assume that to β belongs a non-negative eigenvector, $y \gtrsim 0$, that is, $Ay = \beta y$. Now we can choose ϱ so as to render $c = x - \varrho y > 0$. Then $Ac = A(x - \varrho y) = \alpha x - \beta\varrho y \geq \alpha(x - \varrho y) = \alpha c$, and this cannot be since α is maximal. Thus no positive eigenvalue, other than α, can have a non-negative eigenvector. Furthermore a negative or complex eigenvalue cannot belong to a non-negative eigenvector ($A > 0$ and $x \geq 0$ implies $Ax > 0$). Therefore the eigenvector x belonging to the maximal eigenvalue α is the only non-negative eigenvector of A.

6. *If we increase any element of the matrix* A, *its maximal eigenvalue* α *will increase and if we decrease any element, the eigenvalue will decrease*

Let the matrix obtained by increasing A be given by $B \gtrsim A$ where for at least one element $b_{ik} > a_{ik}$. Now if $Ax = \alpha x$, then $Bx \gtrsim \alpha x$, with strict inequality holding for at least one element. But then, according to theorem 2, α cannot be the maximal eigenvalue of B. The maximal eigenvalue of B must therefore be greater. If we exchange the roles of A and B we can prove the reverse.

7. *Inclusion of the maximal eigenvalue*

Let $A > 0$ and x be positive and optimal. Now forming the quotients $q_i = (Ax)_i/x_i$ $(i = 1, \ldots, n)$. There are two possible cases:
(a) $q_i = \alpha$ for every $i = 1, \ldots, n$.
(b) $\min q_i < \alpha < \max q_i$ $(i = 1, \ldots, n)$.

The consequence of (a) is trivial: $Ax = \alpha x$ and $x > 0$. According to theorem 5, the quotient is equal to the maximal eigenvalue.

For case (b) we increase the elements of A and construct a matrix B with $\max q_i$ as its maximal eigenvalue. Thus $Bx = \max q_i x$, From theorem 6 it follows

that this eigenvalue must be strictly greater than α. By decreasing some elements of A we can establish the opposite inequality.

We now generalize theorems 1–7 valid for $A > 0$ to apply to all non-negative, primitive, irreducible matrices.

A matrix A is *reducible* if by permuting its rows and columns we can bring it into the form

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where A_{11} and A_{22} are quadratic.

If A is irreducible and its maximal eigenvalue has the multiplicity 1 (is a single root of the eigenequation), then we call it *primitive*. If a matrix is primitive, there is a unique eigenvector belonging to its maximal eigenvalue, and if its maximal eigenvalue has a unique eigenvector, then the matrix must be primitive.

8. *Let* $A \geq 0$. *The matrix* A *raised to some positive power* m *will be positive,* $A^m > 0$, *if and only if* A *is irreducible and primitive*

a) If A is reducible its powers remain reducible.

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}^m = \begin{bmatrix} A_{11}^m & A_{12}^{(m)} \\ 0 & A_{22}^m \end{bmatrix}$$

where $A_{12}^{(m)} = A_{11}A_{12}^{(m-1)} + A_{12}A^{m-1}$.

Thus A must be irreducible to yield $A^m > 0$.

(b) If A is primitive, A^m is primitive too, because raising the matrix to powers does not alter the multiplicity of the maximal eigenvalue.

If A is irreducible and primitive, its powers remain irreducible and primitive.

(c) Every positive matrix, $A > 0$, is primitive, because it is irreducible and, according to theorem 4, its maximal eigenvalue is of multiplicity 1.

(d) If $A^m > 0$, then from 8(a) A must be irreducible. If A were not primitive, then A^m would not be primitive because the multiplicity of its maximal eigenvalue is not changed by raising it to any positive power.
Therefore if A^m is irreducible and primitive, A must be irreducible and primitive too.

9. *Generalization*

Any primitive, irreducible and non-negative matrix, A, can be raised to some power, m, so that $A^m > 0$. This matrix, A^m, being positive, theorems 1–7 apply to it.

A and A^m have the same eigenvectors $A^2x = AAx = \alpha Ax = \alpha^2 x$ and so forth and the eigenvalues of A^m are respective powers of the eigenvalues of A.

Thus theorems 1–7 apply to every primitive, irreducible and non-negative matrix, A.

Appendix II

The resolvent

We call the inverse $(\lambda 1 - A)^{-1}$ the resolvent of the equation $\lambda p = Ap + v$. If the eigenvalues of the matrix A are $\alpha_1, \alpha_2, \ldots, \alpha_n$, then the matrix $(\lambda 1 - A)$ will have the eigenvalues $\lambda - \alpha_1, \lambda - \alpha_2, \ldots, \lambda - \alpha_n$. This matrix $(\lambda 1 - A)$ is therefore regular (and has an inverse: the resolvent) except if λ coincides with one of the eigenvalues of A. In all the cases considered below λ will always be greater than the maximal eigenvalue of A; thus the resolvent exists and its power series converges:

$$Q_\lambda = (\lambda 1 - A)^{-1} = 1\lambda^{-1} + A\lambda^{-2} + A^2\lambda^{-3} + \ldots + A^n\lambda^{-(n+1)} + \ldots$$

We will need the following formula:

$$AQ_\lambda = \lambda Q_\lambda - 1. \tag{1}$$

This can be verified by multiplying the power series above by A.

$$Q_\lambda < \frac{1}{\lambda} Q_1 \text{ if } \lambda > 1, \text{ and } Q_\lambda > \frac{1}{\lambda} Q_1 \qquad \text{if } \lambda < 1. \tag{2}$$

Both inequalities follow from comparing the series above with Q_1/λ

$$Q_1/\lambda = 1\lambda^{-1} + A\lambda^{-1} + A^2\lambda^{-1} + \ldots + A^n\lambda^{-1} + \ldots$$

The first terms are identical, the following ones of Q_1/λ exceed those of Q_λ if $\lambda > 1$ and are smaller if $\lambda < 1$.

$$Q_1 - \frac{\lambda - 1}{\lambda} Q_1^2 < Q_\lambda \qquad \text{if } \lambda > 1$$

and (3)

$$Q_1 - \frac{\lambda - 1}{\lambda} Q_1^2 = Q_1 + \frac{1 - \lambda}{\lambda} Q_1^2 > Q_\lambda \qquad \text{if } \lambda < 1.$$

This can be checked by inspecting

$$1 = Q_\lambda(\lambda 1 - A) = Q_\lambda(1 - A) + (\lambda - 1)Q_\lambda$$

postmultiplying by Q_1

$$Q_1 = Q_\lambda + (\lambda - 1)Q_\lambda Q_1.$$

From 2 it now follows that:

$$Q_1 < Q_\lambda + \frac{\lambda - 1}{\lambda} Q_1^2 \qquad \text{if } \lambda > 1$$

and

$$Q_1 > Q_\lambda + \frac{\lambda - 1}{\lambda} Q_1^2 \qquad \text{if } \lambda < 1.$$

Generalization of the formula of Section 1.1.2

Let $\mathbf{A} = \begin{bmatrix} A, & c \\ v, & 0 \end{bmatrix}$

If now $vQ_\lambda c = \lambda$, then the maximal eigenvalue of $\mathbf{A}$ equals λ, the right-hand eigenvector is given by $\mathbf{x} = (Q_\lambda c, 1)$, and the left-hand eigenvector by $\mathbf{p} = (vQ_\lambda, 1)$.

Proof: Considering 1 and $vQ_\lambda c = \lambda$

$$\mathbf{Ax} = \begin{bmatrix} A, & c \\ v, & 0 \end{bmatrix} \begin{pmatrix} Q_\lambda c \\ 1 \end{pmatrix} = \begin{pmatrix} AQ_\lambda c + c \\ vQ_\lambda c \end{pmatrix} = \begin{pmatrix} \lambda Q_\lambda c \\ \lambda \end{pmatrix} = \lambda \mathbf{x}$$

and

$$\mathbf{pA} = (vQ_\lambda, 1) \begin{bmatrix} A, & c \\ v, & 0 \end{bmatrix} = (\lambda v Q_\lambda, \lambda) = \lambda \mathbf{p}.$$

The last two formulae were applied in Section 1.1.2 only for the special case $\lambda = 1$.

Error analysis of the eigenvalue

The error limits given in Section 3.1.3 can be narrowed. *The saving or diseconomy measured by the eigenvectors will always be greater than its actual magnitude.*

Let us start from the matrix $\mathbf{A}$, assuming that its maximal eigenvalue equals 1. (It can characterize Simple or Extended Reproduction.) In this case, then, $vQ_1 c = 1$.

Now let one of the activity vectors change from c to $c + dc$. (This need not necessarily be the household sector. $\mathbf{A}$ could be partitioned so as to single out any other sector.) Thus $\mathbf{A}$ changes to

$$A^* = \begin{bmatrix} A, & c + dc \\ v, & 0 \end{bmatrix}.$$

The expression $pdc = vQ_1 dc$ is the linear approximation of the change in the eigenvalue and coincides with everyday practical assessment of change in money terms. Let its magnitude be $vQ_1 dc = \pm\, \varepsilon$ ($-$ for saving and $+$ for diseconomy). We assume that the final product, $pc = vQ_1 c$ is of unit magnitude.

The true change in the eigenvalue of matrix $\mathbf{A}$ will be less than its linear approximation $\pm\, \varepsilon$. $\mathbf{A}^*$ will have a maximal eigenvalue greater than $1 - \varepsilon$, or in the case of diseconomy less than $1 + \varepsilon$.

Let λ be the true new eigenvalue. This implies $vQ_\lambda(c + dc) = \lambda$. According to equation (2)

$$\lambda = vQ_\lambda(c + dc) < \frac{1}{\lambda} vQ_1(c + dc) = \frac{1}{\lambda}(1 + \varepsilon) \quad \text{if } \lambda > 1$$

and

$$\lambda = vQ_\lambda(c + dc) > \frac{1}{\lambda} vQ_1(c + dc) = \frac{1}{\lambda}(1 - \varepsilon) \quad \text{if } \lambda < 1\,.$$

Hence it follows that

$$\lambda^2 < 1 + \varepsilon \qquad \text{if } \lambda > 1$$

$$\text{or} \qquad \lambda^2 > 1 - \varepsilon \qquad \text{if } \lambda < 1.$$

Thus *a fortiori*

$$\lambda < 1 + \varepsilon \qquad \text{if } \lambda > 1$$

$$\text{or} \qquad \lambda > 1 - \varepsilon \qquad \text{if } \lambda < 1.$$

Estimation for the eigenvector of a perturbed matrix

It is customary to assume full information about the spectral and modal matrix.* But the resolvent theory shows that for this purpose only the knowledge of the inverse is required.

Again let us assume $\mathbf{A}^* = \begin{bmatrix} A, & c + dc \\ v, & 0 \end{bmatrix}$.

How will the perturbation dc affect the eigenvector, $\mathbf{Ax} = \mathbf{x}$?

Assume that it increases the eigenvalue of $\mathbf{A}^*$, then $\lambda > 1$. From equations (2) and (3) we know that

$$Q_1 - \frac{\lambda - 1}{\lambda} Q_1^2 < Q_\lambda < \frac{1}{\lambda} Q_1\ .$$

The new eigenvector is given by $Q_\lambda(c + dc)$. Now we normalize its last element by setting it equal to 1. We can thus deduce

$$\left(Q_1 - \frac{\lambda - 1}{\lambda} Q_1^2\right)(c + cd) < Q_\lambda(c + dc) < \frac{1}{\lambda} Q_1(c + dc) < Q_1(c + dc).$$

* "To improve an arbitrary eigenvector all eigenvalues and vectors of A must therefore be known". (Bodewig *op. cit.* p. 334 and analogously Wilkinson *op. cit.* pp. 69–70.)

Since $Q_1c = x$, therefore

$$dx < Q_1 dc$$

and

$$dx > Q_1 dc - \frac{\lambda - 1}{\lambda} Q_1^2(c + dc).$$

If the change in λ resulting from dc is sufficiently small, then $(\lambda - 1)$ will be small too and therefore

$$dx \sim Q_1 dc = Q dc\,.$$

Thus a linear approximation of the change of the eigenvector can be computed by premultiplying the perturbation by the inverse, Q. Note the analogy with the open static model. Yet the content of the inverse here is slightly different.

Appendix III

Turnover time and life span

The purpose of this Appendix is to explore the question raised in Section 1.2.1. We start from the pioneering work of Domar [1957].

We neglect changes in technology and therefore obsolescence. We also neglect investment cycles. Random life spans will replace later Domar's fixed ones.

In the following list of variables lower case letters stand for flows, upper case letters for stocks:

Investment	b
Net investment	n
Accumulation	f
Gross fixed capital	B
Net fixed capital	N
Depreciation	k
Scrappage	s
Life span (fixed or expected)	T
Growth multiplier	$1 + \lambda = e^r$
where r is the exponential growth rate	
Time	t or τ.

1. *Domar's findings*

Domar assumed a fixed life span of T years for capital goods. At the end of it the investment good is scrapped and has zero salvage value. He reasons as follows:

Investment flows increase exponentially.

$$b_t = e^{rt} b_0. \tag{1}$$

To simplify matters we assume unit intensity of investment at time $t = 0$, thus $b_0 = 1$. Hence

$$b_t = e^{rt}. \tag{1'}$$

The gross stock of fixed capital at t is equal to the sum of investment flows over the interval $[t - T, t]$. Therefore

$$B_t = \int_{t-T}^{t} e^{r\tau} d\tau = \left[\frac{1}{r} e^{r\tau}\right]_{t-T}^{t} = e^{rt}\,\frac{1 - e^{-rT}}{r} = b_t\,\frac{1 - e^{-rT}}{r}\,. \tag{2}$$

Assuming straight line depreciation:

$$k_t = \frac{B_t}{T} = b_t \frac{1 - e^{-rT}}{rT} = e^{rt} \frac{1 - e^{-rT}}{rT}. \tag{3}$$

At time t we scrap everything that was invested T years ago:

$$s_t = b_{t-T} = e^{rt-T}. \tag{4}$$

If the economy is growing, depreciation exceeds scrappage:

$$\frac{k_t}{s_t} = \frac{e^{rt} - e^{r(t-T)}}{e^{r(t-T)}rT} = \frac{e^{rT} - 1}{rT} = \frac{rT + \frac{(rT)^2}{2!} + \ldots + \frac{(rT)^n}{n!} + \ldots}{rT} =$$

$$= 1 + \frac{rT}{2!} + \ldots + \frac{(rT)^{n-1}}{n!} + \ldots > 1. \tag{5}$$

The excess of depreciation over scrappage can be invested. To this extent accumulation can be financed from a source other than surplus, growth itself. This source varies directly with r and T.

2. *Turnover time of net capital stock*

The turnover time of gross fixed capital is, of course, T years. The rate of profit and the growth rate are usually reckoned on capital net of depreciation. We can compute net capital in two ways.

(a) *In terms of life spans*

With straight line depreciation equipment u years old has lost the fraction u/T of its value. The fraction $1 - u/T = (T - u)/T$ of its original value remains. The remaining value of capital invested in the interval $[t - T, t]$ is

$$N_t = \int_{t-T}^{t} \frac{\tau - t + T}{T} e^{r\tau} d\tau = \frac{1}{T} \int_{t-T}^{t} e^{r\tau}(\tau - t + T)\, dt =$$

$$= \frac{1}{T}\left[\frac{1}{r} e^{r\tau}(T - t) + \frac{1}{r}\left(\tau - \frac{1}{r}\right) e^{r\tau}\right]_{t-T}^{t} =$$

$$= \frac{1}{T}\left[e^{r\tau} \frac{T - t + \tau - \frac{1}{r}}{r}\right]_{t-T}^{t} =$$

$$= \frac{1}{T} e^{rt} \frac{T - \frac{1}{r}}{r} + \frac{1}{T} e^{r(t-T)} \frac{1}{r^2} =$$

$$= e^{rt} \frac{rT - 1 + e^{-rT}}{r^2 T}. \tag{6}$$

(b) *In terms of depreciation*

Alternatively we might arrive at remaining value by subtracting total depreciation from total investment. In this case it is more convenient to sum over the interval $[-\infty, t]$. Considering equations (1′) and (3):

$$N_t = \int_{-\infty}^{t} e^{r\tau}\left(1 - \frac{1 - e^{-rT}}{rT}\right) d\tau = \left| e^{r\tau}\left(\frac{1}{r} - \frac{1 - e^{-rT}}{r^2 T}\right)\right|_{-\infty}^{t} = e^{rt} \frac{rT - 1 + e^{-rT}}{r^2 T}.$$

This is identical with equation (6).

Turnover time is the ratio of net fixed capital depreciation. From equations (6) and (3) with a little manipulation:

$$\frac{N_t}{k_t} = T\left(\frac{1}{1 - e^{-rT}} - \frac{1}{rT}\right). \tag{7}$$

The term in parentheses connects life span with turnover time. Designating it by γ

$$\gamma = \left(\frac{1}{1 - e^{-rT}} - \frac{1}{rT}\right) = \left(\frac{e^{rT}}{e^{rT} - 1} - \frac{1}{rT}\right).$$

The numerical value of γ is always between 1/2 and 1, as can be verified by passing to the limits 0 and ∞

$$\lim_{rT \to \infty} \gamma = \left(\frac{1}{1 - e^{-rT}} - \frac{1}{rT}\right) = 1$$

and

$$\lim_{rT \to 0} \gamma \left(\frac{e^{rT}}{e^{rT} - 1} - \frac{1}{rT}\right) = \lim_{rT \to 0} \frac{1 + rT + \frac{(rT)^2}{2!} + \ldots}{rT + \frac{(rT)^2}{2!} + \ldots} - \frac{1}{rT} =$$

$$= \lim_{rT \to 0} \frac{rT + (rT)^2 + \frac{(rT)^3}{2!} + \ldots - rT - \frac{(rT)^2}{2!} - \ldots}{(rT)^2 + \frac{(rT)^3}{2!} + \ldots} =$$

$$= \lim_{rT \to 0} \frac{\frac{(rT)^2}{2} + \frac{(rT)^3}{3} + \dots}{(rT)^2 + \frac{(rT)^3}{2!} + \dots} = \frac{1}{2}.$$

For Simple Reproduction, $r = 0$ and therefore the turnover period is exactly half of the life span. Turnover time will be the closer to life span the greater the life span and the growth rate.

With fixed life spans, accumulation based on gross and net capital formation differ. Normally we define accumulation as investment — investment minus depreciation. Therefore from equations (1′) and (3):

$$n_t^1 = b_t - k_t = e^{rt}\left(1 - \frac{1 - e^{-rT}}{rT}\right) = e^{rt}\,\frac{rT - 1 + e^{-rT}}{rT}. \tag{8}$$

This is identical to the growth of net fixed capital computed from equation (6):

$$\frac{\partial N_t}{\partial t} = e^{rt}\,\frac{rT - 1 + e^{-rT}}{rT}.$$

On the other hand we can define accumulation as the difference between investment and scrappage. Then from equations (1′) and (4):

$$n_t^2 = b_t - s_t = e^{rT}(1 - e^{-rT}). \tag{9}$$

This is identical to the growth of gross fixed capital computed from equati on (2):

$$\frac{\partial B_t}{\partial t} = e^{rt}(1 - e^{-rT}).$$

This contradiction can be eliminated by introducing random life spans.

3. *Distribution of life spans*

Life tables of productive equipment tell us that life spans are not really fixed more than they are for human beings. We assume an exponential distribution of life spans. This distribution is easy to handle and we have no knowledge that another is more realistic. It is equivalent to assuming a stable probability of survival. This assumption is analogous to the linearization of input-output relations, and can serve as a first approximation.

The probability of a random life span being equal to t is $1/T \exp(-t/T)$. T is now interpreted as expected (average) life span. The probability of survival at age t will be $\exp(-t/T)$.

4. *Turnover time with random life spans*

We can now modify all our former equations, substituting exponential density functions and computing expected values:

$$b_t = e^{rt} \tag{1*}$$

$$B_t = \int_{-\infty}^{t} e^{r\tau} e^{-\frac{1}{T}(t-\tau)}\, d\tau = \int_{-\infty}^{t} e^{\left(r+\frac{1}{T}\right)\tau} e^{-\frac{1}{T}t}\, d\tau =$$

$$= \left[\frac{1}{r + \frac{1}{T}}\, e^{\left(r+\frac{1}{T}\right)\tau} e^{-\frac{1}{T}t}\right]_{-\infty}^{t} = e^{rt}\,\frac{T}{rT+1} = b_t\,\frac{T}{rT+1}. \tag{2*}$$

$$k_t = \frac{B_t}{T} = e^{rt}\,\frac{1}{rT+1} \tag{3*}$$

$$s_t = \int_{-\infty}^{t} e^{r\tau}\,\frac{1}{T}\, e^{-\frac{1}{T}(t-\tau)}\, d\tau = \frac{1}{T}\int_{-\infty}^{t} e^{\left(r+\frac{1}{T}\right)\tau} e^{-\frac{1}{T}t}\, d\tau = e^{rt}\,\frac{1}{rT+1}. \tag{4*}$$

In equations (3*) and (4*) depreciation and scrappage became identical because the exponential density function is exactly $1/T$ times its integral (that is, the distribution function). Thus

$$\frac{k_t}{s_t} = 1. \tag{5*}$$

We cannot compute net stock in terms of life spans as before because the life spans of some capital goods exceed the expected life spans. Accumulated depreciation thus can go "into the red", that is, exceed original book values. But we can compute net stock by subtracting total depreciation from total investment:

$$N_t = \int_{-\infty}^{t} e^{r\tau}\left(1 - \frac{1}{rT+1}\right) d\tau = \int_{-\infty}^{t} e^{r\tau}\,\frac{rT}{rT+1}\, d\tau =$$

$$= \left[\frac{T}{rT+1}\, e^{r\tau}\right]_{-\infty}^{t} = e^{rt}\,\frac{T}{rT+1}. \tag{6*}$$

The gross and net values of capital stock are now identical because scrappage always equals depreciation. From another viewpoint: fixed stocks maintain their original value until the moment of their scrappage, and then suddenly transfer it to the product. Turnover time will now be the same whether we reckon with gross or net stocks. Equations (6*) and (3*) both lead to

$$\frac{B_t}{k_t} = \frac{N_t}{k_t} = T. \tag{7*}$$

Thus expected life span and turnover time are equal and now turnover time is independent of the rate of growth of investment. This allows us to unify the mathematical theory of turnover.

Net investment based on depreciation or scrappage are now identical, and accumulation can be defined unambiguously. Equations (1*) and (3*) on the one hand and (4*) on the other lead to

$$n_t = b_t - k_t = b_t - s_t = e^{rt}\left(1 - \frac{1}{rT+1}\right) = e^{rt}\frac{rT}{rT+1}. \qquad (8^*), (9^*)$$

Growth of gross and net stocks are identical. Equations 6(*) and (2*) both lead to

$$\frac{\partial N_t}{\partial t} = \frac{\partial B_t}{\partial t} = e^{rt}\frac{rT}{rT+1}.$$

5. *Age distribution of capital stock*

Consider capital with a (fixed or expected) life span of T years invested exponentially with growth rate r. We assume now that investment is of unit magnitude at time t, that is, $e^{rt} = 1$. We give parallel formulae for fixed and expected life spans.

Fixed life span $= T$		*Expected* life span $= T$
	The value of gross fixed capital stock at time t	
$\frac{1 - e^{-rT}}{r}$		$\frac{T}{rT+1}$
	Value of stocks of age k	
e^{-rk}		$e^{-\left(r+\frac{1}{T}\right)k}$
if $0 < k < T$		if $0 \leq k < \infty$
and zero, if $k \geq T$.		
	Value of stocks of age group $(0, m)$	
$\frac{1 - e^{-rm}}{r}$		$\frac{T}{rT+1}\left(1 - e^{-\left(r+\frac{1}{T}\right)m}\right)$
	The share of the age group $(0, m)$	
$\frac{1 - e^{-rm}}{1 - e^{-rT}}$		$1 - e^{-\left(r+\frac{1}{T}\right)m}$
	The average age of stocks	
$\frac{1}{r} - \frac{1}{r(e^{rT} - 1)}$		$\frac{T}{rT+1}$

Numerical example

Let us assume $T = 20$ years and $r = 0.05$.

The share of capital stocks less than 10 years old is

$\frac{1 - e^{-0.5}}{1 - e^{-1}} \approx 62\,\%$		$1 - e^{-1} \approx 63\%$
Half of the stocks is older than 7.6 years		older than 7 years
The average age of stocks is 8.3 years		10 years

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