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The Economic Implications of Learning by Doing

It is by now incontrovertible that increases in per capita income cannot be explained simply by increases in the capital-labor ratio. Though doubtless no economist would ever have denied the role of technological change in economic growth, its overwhelming importance relative to capital formation has perhaps only been fully realized with the important empirical studies of Abramovitz [1] and Solow [11]. These results do not directly contradict the neo-classical view of the production function as an expression of technological knowledge. All that has to be added is the obvious fact that knowledge is growing in time. Nevertheless a view of economic growth that depends so heavily on an exogenous variable, let alone one so difficult to measure as the quantity of knowledge, is hardly intellectually satisfactory. From a quantitative, empirical point of view, we are left with time as an explanatory variable. Now trend projections, however necessary they may be in practice, are basically a confession of ignorance, and, what is worse from a practical viewpoint, are not policy variables.

Further, the concept of knowledge which underlies the production function at any moment needs analysis. Knowledge has to be acquired. We are not surprised, as educators, that even students subject to the same educational experiences have different bodies of knowledge, and we may therefore be prepared to grant, as has been shown empirically (see [2], Part III), that different countries, at the same moment of time, have different production functions even apart from differences in natural resource endowment.

I would like to suggest here an endogenous theory of the changes in knowledge which underlie intertemporal and international shifts in production functions. The acquisition of knowledge is what is usually termed " learning," and we might perhaps pick up some clues from the many psychologists who have studied this phenomenon (for a convenient survey, see Hilgard [5]). I do not think that the picture of technical change as a vast and prolonged process of learning about the environment in which we operate is in any way a far-fetched analogy; exactly the same phenomenon of improvement in performance over time is involved.

Of course, psychologists are no more in agreement than economists, and there are sharp differences of opinion about the processes of learning. But one empirical generalization is so clear that all schools of thought must accept it, although they interpret it in different fashions: Learning is the product of experience. Learning can only take place through the attempt to solve a problem and therefore only takes place during activity. Even the Gestalt and other field theorists, who stress the role of insight in the solution of problems (Köhler's famous apes), have to assign a significant role to previous experiences in modifying the individual's perception.

A second generalization that can be gleaned from many of the classic learning experiments is that learning associated with repetition of essentially the same problem is subject to sharply diminishing returns. There is an equilibrium response pattern for any given stimulus, towards which the behavior of the learner tends with repetition. To have steadily increasing performance, then, implies that the stimulus situations must themselves be steadily evolving rather than merely repeating.

The role of experience in increasing productivity has not gone unobserved, though the relation has yet to be absorbed into the main corpus of economic theory. It was early observed by aeronautical engineers, particularly T. P. Wright [15], that the number of labor-hours expended in the production of an airframe (airplane body without engines) is a decreasing function of the total number of airframes of the same type previously produced. Indeed, the relation is remarkably precise; to produce the Nth airframe of a given type, counting from the inception of production, the amount of labor required is proportional to $N^{-1/3}$. This relation has become basic in the production and cost planning of the United States Air Force; for a full survey, see [3]. Hirsch (see [6] and other work cited there) has shown the existence of the same type of " learning curve " or " progress ratio," as it is variously termed, in the production of other machines, though the rate of learning is not the same as for airframes.

Verdoorn [14, pp. 433-4] has applied the principle of the learning curve to national outputs; however, under the assumption that output is increasing exponentially, current output is proportional to cumulative output, and it is the former variable that he uses to explain labor productivity. The empirical fitting was reported in [13]; the estimated progress ratio for different European countries is about ·5. (In [13], a neo-classical interpretation in terms of increasing capital-labor ratios was offered; see pp. 7-11.)

Lundberg [9, pp. 129-133] has given the name " Horndal effect " to a very similar phenomenon. The Horndal iron works in Sweden had no new investment (and therefore presumably no significant change in its methods of production) for a period of 15 years, yet productivity (output per manhour) rose on the average close to 2% per annum. We find again steadily increasing performance which can only be imputed to learning from experience.

I advance the hypothesis here that technical change in general can be ascribed to experience, that it is the very activity of production which gives rise to problems for which favorable responses are selected over time. The evidence so far cited, whether from psychological or from economic literature is, of course, only suggestive. The aim of this paper is to formulate the hypothesis more precisely and draw from it a number of economic implications. These should enable the hypothesis and its consequences to be confronted more easily with empirical evidence.

The model set forth will be very simplified in some other respects to make clearer the essential role of the major hypothesis; in particular, the possibility of capital-labor substitution is ignored. The theorems about the economic world presented here differ from those in most standard economic theories; profits are the result of technical change; in a free-enterprise system, the rate of investment will be less than the optimum; net investment and the stock of capital become subordinate concepts, with gross investment taking a leading role.

In section 1, the basic assumptions of the model are set forth. In section 2, the implications for wage earners are deduced; in section 3 those for profits, the inducement to invest, and the rate of interest. In section 4, the behavior of the entire system under steady growth with mutually consistent expectations is taken up. In section 5, the divergence between social and private returns is studied in detail for a special case (where the subjective rate of discount of future consumption is a constant). Finally, in section 6, some limitations of the model and needs for further development are noted.

1. The Model

The first question is that of choosing the economic variable which represents " experience ". The economic examples given above suggest the possibility of using cumulative output (the total of output from the beginning of time) as an index of experience, but this does not seem entirely satisfactory. If the rate of output is constant, then the stimulus to learning presented would appear to be constant, and the learning that does take place is a gradual approach to equilibrium behavior. I therefore take instead cumulative gross investment (cumulative production of capital goods) as an index of experience. Each new machine produced and put into use is capable of changing the environment in which production takes place, so that learning is taking place with continually new stimuli. This at least makes plausible the possibility of continued learning in the sense, here, of a steady rate of growth in productivity.

The second question is that of deciding where the learning enters the conditions of production. I follow here the model of Solow [12] and Johansen [7], in which technical change is completely embodied in new capital goods. At any moment of new time, the new capital goods incorporate all the knowledge then available, but once built their productive efficiency cannot be altered by subsequent learning.

To simplify the discussion we shall assume that the production process associated with any given new capital good is characterized by fixed coefficients, so that a fixed amount of labor is used and a fixed amount of output obtained. Further, it will be assumed that new capital goods are better than old ones in the strong sense that, if we compare a unit of capital goods produced at time t_1 with one produced at time $t_2 > t_1$, the first requires the co-operation of at least as much labor as the second, and produces no more product. Under this assumption, a new capital good will always be used in preference to an older one.

Let G be cumulative gross investment. A unit capital good produced when cumulative gross investment has reached G will be said to have *serial number* G. Let

$\lambda(G)$ = amount of labor used in production with a capital good of serial number G,
$\gamma(G)$ = output capacity of a capital good of serial number G,
x = total output,
L = total labor force employed.

It is assumed that $\lambda(G)$ is a non-increasing function, while $\gamma(G)$ is a non-decreasing function. Then, regardless of wages or rental value of capital goods, it always pays to use a capital good of higher serial number before one of lower serial number.

It will further be assumed that capital goods have a fixed lifetime, T. Then capital goods disappear in the same order as their serial numbers. It follows that at any moment of time, the capital goods in use will be all those with serial numbers from some G' to G, the current cumulative gross investment. Then

$$x = \int_{G'}^{G} \gamma(G)dG, \tag{1}$$

$$L = \int_{G'}^{G} \lambda(G) dG. \tag{2}$$

The magnitudes x, L, G, and G' are, of course, all functions of time, to be designated by t, and they will be written $x(t)$, $L(t)$, $G(t)$, and $G'(t)$ when necessary to point up the dependence. Then $G(t)$, in particular, is the cumulative gross investment up to time t. The assumption about the lifetime of capital goods implies that

$$G'(t) \geqq G(t - \bar{T}). \tag{3}$$

Since $G(t)$ is given at time t, we can solve for G' from (1) or (2) or the equality in (3). In a growth context, the most natural assumption is that of full employment. The labor force is regarded as a given function of time and is assumed equal to the labor employed, so that $L(t)$ is a given function. Then $G'(t)$ is obtained by solving in (2). If the result is substituted into (1), x can be written as a function of L and G, analogous to the usual production function. To write this, define

$$\begin{aligned} \Lambda(G) &= \int \lambda(G) dG, \\ \Gamma(g) &= \int \gamma(G) dG. \end{aligned} \tag{4}$$

These are to be regarded as indefinite integrals. Since $\lambda(G)$ and $\gamma(G)$ are both positive, $\Lambda(G)$ and $\Gamma(G)$ are strictly increasing and therefore have inverses, $\Lambda^{-1}(u)$ and $\Gamma^{-1}(v)$, respectively. Then (1) and (2) can be written, respectively,

$$x = \Gamma(G) - \Gamma(G'), \tag{1'}$$

$$L = \Lambda(G) - \Lambda(G'). \tag{2'}$$

Solve for G' from (2').

$$G' = \Lambda^{-1}[\Lambda(G) - L]. \tag{5}$$

Substitute (5) into (1').

$$x = \Gamma(G) - \Gamma\{\Lambda^{-1}[\Lambda(G) - L]\}, \tag{6}$$

which is thus a production function in a somewhat novel sense. Equation (6) is always valid, but under the full employment assumption we can regard L as the labor force available.

A second assumption, more suitable to a depression situation, is that in which demand for the product is the limiting factor. Then x is taken as given; G' can be derived from (1) or (1'), and employment then found from (2) or (2'). If this is less than the available labor force, we have Keynesian unemployment.

A third possibility, which, like the first, may be appropriate to a growth analysis, is that the solution (5) with L as the labor force, does not satisfy (3). In this case, there is a shortage of capital due to depreciation. There is again unemployment but now due to structural discrepancies rather than to demand deficiency.

In any case, except by accident, there is either unemployed labor or unemployed capital; there could be both in the demand deficiency case. Of course, a more neoclassical model, with substitution between capital and labor for each serial number of capital good, might permit full employment of both capital and labor, but this remains a subject for further study.

In what follows, the full-employment case will be chiefly studied. The capital shortage case, the third one, will be referred to parenthetically. In the full-employment case, the depreciation assumption no longer matters; obsolescence, which occurs for all capital goods with serial numbers below G', becomes the sole reason for the retirement of capital goods from use.

The analysis will be carried through for a special case. To a very rough approximation, the capital-output ratio has been constant, while the labor-output ratio has been declining. It is therefore assumed that

(7) $\gamma(G) = a,$

a constant, while $\lambda(G)$ is a decreasing function of G. To be specific, it will be assumed that $\lambda(G)$ has the form found in the study of learning curves for airframes.

(8) $\lambda(G) = bG^{-n},$

where $n > 0$. Then

$$\Gamma(G) = aG, \Lambda(G) = cG^{1-n}, \text{ where } c = b/(1-n) \text{ for } n \neq 1.$$

Then (6) becomes

(9) $$x = aG\left[1 - \left(1 - \frac{L}{cG^{1-n}}\right)^{1/(1-n)}\right] \text{ if } n \neq 1.$$

Equation (9) is always well defined in the relevant ranges, since from (2′),

$$L = \Lambda(G) - \Lambda(G') \leqq \Lambda(G) = cG^{1-n}.$$

When $n = 1$, $\Lambda(G) = b \log G$ (where the natural logarithm is understood), and

(10) $x = aG(1-e^{-L/b})$ if $n = 1$.

Although (9) and (10) are, in a sense, production functions, they show increasing returns to scale in the variables G and L. This is obvious in (10) where an increase in G, with L constant, increases x in the same proportion; a simultaneous increase in L will further increase x. In (9), first suppose that $n < 1$. Then a proportional increase in L and G increases L/G^{1-n} and therefore increases the expression in brackets which multiplies G. A similar argument holds if $n > 1$. It should be noted that x increases more than proportionately to scale changes in G and L in general, not merely for the special case defined by (7) and (8). This would be verified by careful examination of the behavior of (6), when it is recalled that $\lambda(G)$ is non-increasing and $\gamma(G)$ is non-decreasing, with the strict inequality holding in at least one. It is obvious intuitively, since the additional amounts of L and G are used more efficiently than the earlier ones.

The increasing returns do not, however, lead to any difficulty with distribution theory. As we shall see, both capital and labor are paid their marginal products, suitably defined. The explanation is, of course, that the private marginal productivity of capital (more strictly, of new investment) is less than the social marginal productivity since the learning effect is not compensated in the market.

The production assumptions of this section are designed to play the role assigned by Kaldor to his " technical progress function," which relates the rate of growth of output per worker to the rate of growth of capital per worker (see [8], section VIII). I prefer to think of relations between rates of growth as themselves derived from more fundamental relations between the magnitudes involved. Also, the present formulation puts more stress on gross rather than net investment as the basic agent of technical change.

Earlier, Haavelmo ([4], sections 7.1 and 7.2) had suggested a somewhat similar model. Output depended on both capital and the stock of knowledge; investment depended on output, the stock of capital, and the stock of knowledge. The stock of knowledge was either simply a function of time or, in a more sophisticated version, the consequence of investment, the educational effect of each act of investment decreasing exponentially in time.

Verdoorn [14, pp. 436-7] had also developed a similar simple model in which capital and labor needed are non-linear functions of output (since the rate of output is, approximately, a measure of cumulative output and therefore of learning) and investment a constant fraction of output. He notes that under these conditions, full employment of capital and labor simultaneously is in general impossible—a conclusion which also holds for for the present model as we have seen. However, Verdoorn draws the wrong conclusion: that the savings ratio must be fixed by some public mechanism at the uniquely determined level which would insure full employment of both factors; the correct conclusion is that one factor or the other will be unemployed. The social force of this conclusion is much less in the present model since the burden of unemployment may fall on obsolescent capital; Verdoorn assumes his capital to be homogeneous in nature.

2. Wages

Under the full employment assumption the profitability of using the capital good with serial number G' must be zero; for if it were positive it would be profitable to use capital goods with higher serial number and if it were negative capital good G' would not be used contrary to the definition of G'. Let

$$w = \text{wage rate with output as numéraire.}$$

From (1′) and (7)

$$(11) \qquad G' = G - (x/a)$$

so that

$$(12) \qquad \lambda(G') = b\left(G - \frac{x}{a}\right)^{-n}.$$

The output from capital good G' is $\gamma(G')$ while the cost of operation is $\lambda(G')w$. Hence

$$\gamma(G') = \lambda(G')w$$

or from (7) and (12)

$$(13) \qquad w = a\left(G - \frac{x}{a}\right)^{n}/b.$$

It is interesting to derive labor's share which is wL/x. From (2′) with $\Lambda(g) = cG^{1-n}$ and G' given by (11)

$$L = c\left[G^{1-n} - \left(G - \frac{x}{a}\right)^{1-n} \right],$$

for $n \neq 1$ and therefore

$$(14) \qquad wL/x = a\left[\left(\frac{G}{x} - \frac{1}{a}\right)^{n}\left(\frac{G}{x}\right)^{1-n} - \left(\frac{G}{x} - \frac{1}{a}\right)\right]/(1-n) \text{ for } n \neq 1,$$

where use has been made of the relation, $c = b/(1-n)$. It is interesting to note that labor's share is determined by the ratio G/x.

Since, however, x is determined by G and L, which, at any moment of time, are data, it is also useful to express the wage ratio, w, and labor's share, wL/x, in terms of L and G. First, G' can be found by solving for it from (2′).

$$(15) \qquad G' = \left(G^{1-n} - \frac{L}{c}\right)^{1/(1-n)} \text{ for } n \neq 1.$$

We can then use the same reasoning as above, and derive

$$(16) \qquad w = a\left(G^{1-n} - \frac{L}{c}\right)^{n/(1-n)}/b,$$

$$(17) \qquad \frac{wL}{x} = \frac{\left[\left(\frac{L}{G^{1-n}}\right)^{(1-n)/n} - \frac{1}{c}\left(\frac{L}{G^{1-n}}\right)^{1/n}\right]^{n/(1-n)}}{b\left[1 - \left(1 - \frac{L}{cG^{1-n}}\right)^{1/(1-n)}\right]}.$$

Labor's share thus depends on the ratio L/G^{1-n}; it can be shown to decrease as the ratio increases.

For completeness, I note the corresponding formulas for the case $n = 1$. In terms of G and x, we have

$$(18) \qquad w = (aG - x)/b,$$

$$(19) \qquad wL/x = \left(\frac{aG}{x} - 1\right) \log \frac{G/x}{(G/x) - (1/a)}.$$

In terms of G and L, we have

$$(20) \qquad G' = Ge^{-L/b},$$

$$(21) \qquad w = \frac{aG}{be^{L/b}},$$

$$(22) \qquad wL/x = \frac{L}{b(e^{L/b} - 1)}.$$

In this case, labor's share depends only on L, which is indeed the appropriate special case ($n=1$) of the general dependence on L/G^{1-n}.

The preceding discussion has assumed full employment. In the capital shortage case, there cannot be a competitive equilibrium with positive wage since there is necessarily unemployment. A zero wage is, however, certainly unrealistic. To complete the model, it would be necessary to add some other assumption about the behavior of wages. This case will not be considered in general; for the special case of steady growth, see Section 5.

3. Profits and Investment

The profit at time t from a unit investment made at time $v \leqq t$ is

$$\gamma[G(v)] - w(t)\ \lambda[G(v)].$$

In contemplating an investment at time v, the stream of potential profits depends upon expectations of future wages. We will suppose that looking ahead at any given moment of time each entrepreneur assumes that wages will rise exponentially from the present level. Thus the wage rate expected at time v to prevail at time t is

$$w(v)\ e^{\theta(t-v)},$$

and the profit expected at time v to be received at time t is

$$\gamma[G(v)]\ [1-W(v)\ e^{\theta(t-v)}],$$

where

(23) $$W(v) = w(v)\ \lambda[G(v)]/\gamma[G(v)],$$

the labor cost per unit output at the time the investment is made. The dependence of W on v will be made explicit only when necessary. The profitability of the investment is expected to decrease with time (if $\theta > 0$) and to reach zero at time $T^* + v$, defined by the equation

(24) $$We^{\theta T^*} = 1.$$

Thus T^* is the expected economic lifetime of the investment, provided it does not exceed the physical lifetime, T. Let

(25) $$T = \min (T, T^*).$$

Then the investor plans to derive profits only over an interval of length T, either because the investment wears out or because wages have risen to the point where it is unprofitable to operate. Since the expectation of wage rises which causes this abandonment derives from anticipated investment and the consequent technological progress, T^* represents the expected date of obsolescence. Let

$$\rho = \text{rate of interest}.$$

If the rate of interest is expected to remain constant over the future, then the discounted stream of profits over the effective lifetime, T, of the investment is

$$(26) \qquad S = \int_0^T e^{-\rho t}\, \gamma[G(v)]\, (1 - W e^{\theta t})dt,$$

or

$$(27) \qquad \frac{S}{\gamma[G(v)]} = \frac{1 - e^{-\rho T}}{\rho} + \frac{W(1 - e^{-(\rho-\theta)T})}{\theta - \rho}.$$

Let

$$(28) \qquad V = e^{-\theta T} = \max(e^{-\theta \bar{T}}, W),\ \alpha = \rho/\theta.$$

Then

$$(29) \qquad \frac{\theta S}{\gamma[G(v)]} = \frac{1 - V^{\alpha}}{\alpha} + \frac{W(1 - V^{\alpha-1})}{1-\alpha} = R(\alpha).$$

The definitions of $R(\alpha)$ for $\alpha = 0$ and $\alpha = 1$ needed to make the function continuous are:

$$R(0) = -\log V + W(1 - V^{-1}),\ R(1) = 1 - V + W \log V.$$

If all the parameters of (26), (27), or (29) are held constant, S is a function of ρ, and, equivalently, R of α. If (26) is differentiated with respect to ρ, we find

$$dS/d\rho = \int_0^T (-t)e^{-\rho t}\, \gamma[G(v)]\, (1 - W e^{\theta t})dt < 0.$$

Also

$$S < \gamma[G(v)] \int_0^T e^{-\rho t}dt = \gamma[G(v)]\, (1 - e^{-\rho T})/\rho$$
$$< \gamma[G(v)]/\rho.$$

Since obviously $S > 0$, S approaches 0 as ρ approaches infinity. Since R and α differ from S and ρ, respectively, only by positive constant factors, we conclude

$$dR/d\alpha < 0, \quad \lim_{\alpha \to +\infty} R(\alpha) = 0.$$

To examine the behavior of $R(\alpha)$ as α approaches $-\infty$, write

$$R(\alpha) = -\frac{(1/V)^{1-\alpha}}{(1-\alpha)^2}\,[(1-\alpha)V + \alpha W]\left(\frac{1-\alpha}{\alpha}\right) + \frac{1}{\alpha} + \frac{W}{1-\alpha}.$$

The last two terms approach zero. As α approaches $-\infty$, $1 - \alpha$ approaches $+\infty$. Since $1/V > 1$, the factor

$$\frac{(1/V)^{1-\alpha}}{(1-\alpha)^2}$$

approaches $+\infty$, since an exponential approaches infinity faster than any power. From (28), $V \geqq W$. If $V = W$, then the factor,

$$(1-\alpha)V - \alpha W = \alpha(W - V) + V,$$

is a positive constant; if $V > W$, then it approaches $+\infty$ as α approaches $-\infty$. Finally,

$$\frac{1-\alpha}{\alpha}$$

necessarily approaches -1. Hence,

(30) $R(\alpha)$ is a strictly decreasing function, approaching $+\infty$ as α approaches $-\infty$ and 0 as α approaches $+\infty$.

The market, however, should adjust the rate of return so that the discounted stream of profits equals the cost of investment, i.e., $S = 1$, or, from (29),

(31) $R(\alpha) = \theta/\gamma[G(\nu)]$.

Since the right-hand side of (31) is positive, (30) guarantees the existence of an α which satisfies (31). For a given θ, the equilibrium rate of return, ρ, is equal to $\alpha\,\theta$; it may indeed be negative. The rate of return is thus determined by the expected rate of increase in wages, current labor costs per unit output, and the physical lifetime of the investment. Further, if the first two are sufficiently large, the physical lifetime becomes irrelevant, since then $T^* < \bar{T}$, and $T = T^*$.

The discussion of profits and returns has not made any special assumptions as to the form of the production relations.

4. Rational Expectations in a Macroeconomic Growth Model

Assume a one-sector model so that the production relations of the entire economy are described by the model of section 1. In particular, this implies that gross investment at any moment of time is simply a diversion of goods that might otherwise be used for consumption. Output and gross investment can then be measured in the same units.

The question arises, can the expectations assumed to govern investment behavior in the preceding section actually be fulfilled? Specifically, can we have a constant relative increase of wages and a constant rate of interest which, if anticipated, will lead entrepreneurs to invest at a rate which, in conjunction with the exogenously given rate of interest to remain at the given level? Such a state of affairs is frequently referred to as " perfect foresight," but a better term is " rational expectations," a term introduced by J. Muth [10].

We study this question first for the full employment case. For this case to occur, the physical lifetime of investments must not be an effective constraint. If, in the notation of the last section, $T^* > \bar{T}$, and if wage expectations are correct, then investments will disappear through depreciation at a time when they are still yielding positive current profits. As seen in section 2, this is incompatible with competitive equilibrium and full employment. Assume therefore that

(32) $T^* \leqq \bar{T};$

then from (28), $W = V$, and from (29) and (31), the equilibrium value of ρ is determined by the equation,

(33) $$\frac{1 - W^{\alpha}}{\alpha} + \frac{W - W^{\alpha}}{1 - \alpha} = \frac{\theta}{a},$$

where, on the right-hand side, use is made of (7).

From (16), it is seen that for the wage rate to rise at a constant rate θ, it is necessary that the quantity,

$$G^{1-n} - \frac{L}{c},$$

rise at a rate $\theta(1 - n)/n$. For θ constant, it follows from (33) that a constant ρ and therefore a constant α requires that W be constant. For the specific production relations (7) and (8), (23) shows that

$$W = \frac{a\left(G^{1-n} - \frac{L}{c}\right)^{n/(1-n)}}{b} \frac{bG^{-n}}{a} = \left(1 - \frac{L}{cG^{1-n}}\right)^{n/(1-n)},$$

and therefore the constancy of W is equivalent to that of L/G^{1-n}. In combination with the preceding remark, we see that

(34) L increases at rate $\theta(1 - n)/n$, G increases at rate θ/n.

Suppose that

σ = rate of increase of the labor force,

is a given constant. Then

(35) $\theta = n\, \sigma/(1-n)$,

(36) the rate of increase of G is $\sigma/(1-n)$.

Substitution into the production function (9) yields

(37) the rate of increase of x is $\sigma/(1-n)$.

From (36) and (37), the ratio G/x is constant over time. However, the value at which it is constant is not determined by the considerations so far introduced; the savings function is needed to complete the system. Let the constant ratio be

(38) $G(t)/x(t) = \mu$.

Define

$g(t)$ = rate of gross investment at time $t = dG/dt$.

From (36), $g/G = \sigma/(1 - n)$, a constant. Then

$$(39) \qquad g/x = (g/G)(G/x) = \mu\, \sigma/(1 - n).$$

A simple assumption is that the ratio of gross saving (equals gross investment) to income (equals output) is a function of the rate of return, ρ; a special case would be the common assumption of a constant savings-to-income ratio. Then μ is a function of ρ. On the other hand, we can write W as follows, using (23) and (13):

$$(40) \qquad W = a\,\frac{\left(G - \frac{x}{a}\right)^n}{b}\;\frac{bG^{-n}}{a} = \left(1 - \frac{x}{aG}\right)^n = \left(1 - \frac{1}{a\mu}\right)^n.$$

Since θ is given by (35), (33) is a relation between W and ρ, and, by (40) between μ and ρ. We thus have two relations between μ and ρ, so they are determinate.

From (38), μ determines one relation between G and X. If the labor force, L, is given at one moment of time, the production function (9) constitutes a second such relation, and the system is completely determinate.

As in many growth models, the rates of growth of the variables in the system do not depend on savings behavior; however, their levels do.

It should be made clear that all that has been demonstrated is the existence of a solution in which all variables have constant rates of growth, correctly anticipated. The stability of the solution requires further study.

The growth rate for wages implied by the solution has one paradoxical aspect; it increases with the rate of growth of the labor force (provided $n < 1$). The explanation seems to be that under full employment, the increasing labor force permits a more rapid introduction of the newer machinery. It should also be noted that, for a constant saving ratio, g/x, an increase in σ decreases μ, from (39), from which it can be seen that wages at the initial time period would be lower. In this connection it may be noted that since G cannot decrease, it follows from (36) that σ and $1—n$ must have the same sign for the steady growth path to be possible. The most natural case, of course, is $\sigma > 0$, $n < 1$.

This solution is, however, admissible only if the condition (32), that the rate of depreciation not be too rapid, be satisfied. We can find an explicit formula for the economic lifetime, T^*, of new investment. From (24), it satisfies the condition

$$e^{-\theta T^*} = W.$$

If we use (35) and (40) and solve for T^*, we find

$$(41) \qquad T^* = \frac{-(1 - n)}{\sigma} \log \left[1 - \frac{1}{a\mu}\right]$$

and this is to be compared with $\bar{T}$; the full employment solution with rational expectations of exponentially increasing wages and constant interest is admissible if $T^* \leqq \bar{T}$.

If $T^* > \bar{T}$, then the full employment solution is inadmissible. One might ask if a constant-growth solution is possible in this case. The answer depends on assumptions about the dynamics of wages under this condition.

We retain the two conditions, that wages rise at a constant rate θ, and that the rate of interest be constant. With constant θ, the rate of interest, ρ, is determined from (31); from (29), this requires that

(42) W is constant over time.

From the definition of W, (23), and the particular form of the production relations, (7) and (8), it follows that the wage rate, w, must rise at the same rate as G^n, or

(43) G rises at a constant rate θ/n.

In the presence of continued unemployment, the most natural wage dynamics in a free market would be a decreasing, or, at best, constant wage level. But since G can never decrease, it follows from (43) that θ can never be negative. Instead of making a specific assumption about wage changes, it will be assumed that any choice of θ can be imposed, perhaps by government or union or social pressure, and it is asked what restrictions on the possible values of θ are set by the other equilibrium conditions.

In the capital shortage case, the serial number of the oldest capital good in use is determined by the physical lifetime of the good, i.e.,

$G' = G(t - \bar{T})$. From (43),

$$G(t - \bar{T}) = e^{-\theta\bar{T}/n} G.$$

Then, from (1′) and (7),

$$x = aG(1 - e^{-\theta\bar{T}/n}),$$

so that the ratio, G/x, or μ, is a constant,

$$\mu = 1/a(1 - e^{-\theta\bar{T}/n}). \tag{44}$$

From (43), $g/G = \theta/n$; hence, by the same argument as that leading to (39),

$$g/x = \theta/na(1 - e^{-\theta\bar{T}/n}). \tag{45}$$

There are three unknown constants of the growth process, θ, ρ, and W. If, as before, it is assumed that the gross savings ratio, g/x, is a function of the rate of return, ρ, then, for any given ρ, θ can be determined from (45); note that the right-hand side of (45) is a strictly increasing function of θ for $\theta \geqq 0$, so that the determination is unique, and the rate of growth is an increasing function of the gross savings ratio, contrary to the situation in the full employment case. Then W can be solved for from (31) and (29).

Thus the rate of return is a freely disposable parameter whose choice determines the rate of growth and W, which in turn determines the initial wage rate. There are, of course, some inequalities which must be satisfied to insure that the solution corresponds to the capital shortage rather than the full employment case; in particular, $W \leqq V$ and also the labor force must be sufficient to permit the expansion. From (2′), this means that the labor force must at all times be at least equal to

$$cG^{1-n} - c(G')^{1-n} = cG^{1-n}(1 - e^{-\theta(1-n)\bar{T}/n});$$

if σ is the growth rate of the labor force, we must then have (46)

(46) $\sigma \geqq \theta(1 - n)/n,$

which sets an upper bound on θ (for $n < 1$). Other constraints on ρ are implied by the conditions $\theta \geqq 0$ and $W \geqq 0$ (if it is assumed that wage rates are non-negative). The first condition sets a lower limit on g/x; it can be shown, from (45), that

(47) $g/x \geqq 1/a\bar{T};$

i.e., the gross savings ratio must be at least equal to the amount of capital goods needed to produce one unit of output over their lifetime. The constraint $W > 0$ implies an interval in which ρ must lie. The conditions under which these constraints are consistent (so that at least one solution exists for the capital shortage case) have not been investigated in detail.

5. Divergence of Private and Social Product

As has already been emphasized, the presence of learning means that an act of investment benefits future investors, but this benefit is not paid for by the market. Hence, it is to be expected that the aggregate amount of investment under the competitive model of the last section will fall short of the socially optimum level. This difference will be investigated in detail in the present section under a simple assumption as to the utility function of society. For brevity, I refer to the *competitive solution* of the last section, to be contrasted with the *optimal* solution. Full employment is assumed. It is shown that the socially optimal growth rate is the same as that under competitive conditions, but the socially optimal ratio of gross investment to output is higher than the competitive level.

Utility is taken to be a function of the stream of consumption derived from the productive mechanism. Let

$$c = \text{consumption} = \text{output} - \text{gross investment} = x - g.$$

It is in particular assumed that future consumption is discounted at a constant rate, β, so that utility is

(48) $$U = \int_0^{+\infty} e^{-\beta t}c(t)dt = \int_0^{+\infty} e^{-\beta t}x(t)dt - \int_0^{+\infty} e^{-\beta t}g(t)dt.$$

Integration by parts yields

$$\int_0^{+\infty} e^{-\beta t}g(t)dt = e^{-\beta t}G(t)\Big|_0^{+\infty} + \beta\int_0^{+\infty} e^{-\beta t}G(t)dt.$$

From (48),

(49) $$U = U_1 - \lim_{t \to +\infty} e^{-\beta t} G(t) + G(0),$$

where

(50) $$U_1 = \int_o^{+\infty} e^{-\beta t}[x(t) - \beta\, G(t)]dt.$$

The policy problem is the choice of the function $G(t)$, with $G'(t) \geqq 0$, to maximize (49), where $x(t)$ is determined by the production function (9), and

(51) $$L(t) = L_0 e^{\sigma t}.$$

The second term in (49) is necessarily non-negative. It will be shown that, for sufficiently high discount rate, β, the function $G(t)$ which maximizes U_1 also has the property that the second term in (49) is zero; hence, it also maximizes (49), since $G(0)$ is given.

Substitute (9) and (51) into (50).

$$U_1 = \int_o^{+\infty} e^{-\beta t} G(t) \left[a - \beta - a\left(1 - \frac{L_o e^{\sigma t}}{c G^{1-n}}\right)^{1/(1-n)}\right] dt.$$

Let $\bar{G}(t) = G(t)\, e^{-\sigma t/(1-n)}$.

$$U_1 = \int_o^{+\infty} e^{-\left(\beta - \frac{\sigma}{1-n}\right)t}\, \bar{G}(t)\left[a - \beta - a\left(1 - \frac{L_0}{c\bar{G}^{1-n}}\right)^{1/(1-n)}\right] dt.$$

Assume that

(52) $$\beta > \frac{\sigma}{1-n};$$

otherwise an infinite utility is attainable. Then to maximize U_1 it suffices to choose $\bar{G}(t)$ so as to maximize, for each t,

(53) $$\bar{G}\left[a - \beta - a\left(1 - \frac{L_0}{c\bar{G}^{1-n}}\right)^{1/(1-n)}\right].$$

Before actually determining the maximum, it can be noted that the maximizing value of $\bar{G}$ is independent of t and is therefore a constant. Hence, the optimum policy is

(54) $$G(t) = \bar{G}\, e^{\sigma t/(1-n)},$$

so that, from (36), the growth rate is the same as the competitive. From (52), $e^{-\beta t}G(t) \longrightarrow 0$ as $t \longrightarrow +\infty$.

To determine the optimal $\bar{G}$, it will be convenient to make a change of variables. Define

$$v = \left(1 - \frac{L_o}{c\bar{G}^{1-n}}\right)^{n/(1-n)}.$$

so that

(55) $$\bar{G} = \left[\frac{L_0}{(1 - v^{(1-n)/n})}\right]^{1/(1-n)}.$$

The analysis will be carried through primarily for the case where the output per unit capital is sufficiently high, more specifically, where

(56) $$a > \beta.$$

Let

(57) $$\gamma = 1 - \frac{\beta}{a} > 0.$$

The maximizing $\bar{G}$, or v, is unchanged by multiplying (53), the function to be maximized, by the positive quantity, $(c/L_0)^{1/(1-n)}/a$ and then substituting from (55) and (57). Thus, v maximizes

$$(1 - v^{(1-n)/n})^{-1/(1-n)} (\gamma - v^{1/n}).$$

The variable v ranges from 0 to 1. However, the second factor vanishes when $v = \gamma^n < 1$ (since $\gamma < 1$) and becomes negative for larger values of v; since the first factor is always positive, it can be assumed that $v < \gamma^n$ in searching for a maximum, and both factors are positive. Then v also maximizes the logarithm of the above function, which is

$$f(v) = -\frac{\log (1 - v^{(1-n)/n})}{1 - n} + \log (\gamma - v^{1/n}),$$

so that

$$f'(v) = \frac{v^{\frac{1}{n} - 2}}{n} \left[\frac{\gamma - v}{(1 - v^{(1-n)/n}) (\gamma - v^{1/n})}\right].$$

Clearly, with $n < 1, f'(v) > 0$ when $0 < v < \gamma$ and $f'(v) < 0$ when $\gamma < v < \gamma^n$, so that the maximum is obtained at

(58) $$v = \gamma.$$

The optimum $\bar{G}$ is determined by substituting γ for v in (55).

From (54), L/G^{1-n} is a constant over time. From the definition of v and (58), then,

$$\gamma = \left(1 - \frac{L}{cG^{1-n}}\right)^{n/(1-n)}$$

for all t along the optimal path, and, from the production function (9),

(59) $$\gamma = \left(1 - \frac{x}{aG}\right)^n \text{ for all } t \text{ along the optimal path.}$$

This optimal solution will be compared with the competitive solution of steady growth studied in the last section. From (40), we know that

(60) $$W = \left(1 - \frac{x}{aG}\right)^n \text{ for all } t \text{ along the competitive path.}$$

It will be demonstrated that $W < \gamma$; from this it follows that *the ratio G/x is less along the competitive path than along the optimal path.* Since along both paths,

$$g/x = [\sigma/(1-n)]\,(G/x),$$

it also follows that *the gross savings ratio is smaller along the competitive path than along the optimal path.*

For the particular utility function (48), the supply of capital is infinitely elastic at $\rho = \beta$; i.e., the community will take any investment with a rate of return exceeding β and will take no investment at a rate of return less than β. For an equilibrium in which some, but not all, income is saved, we must have

(61) $\rho = \beta.$

From (35), $\theta = n\sigma/(1-n)$; hence, by definition (28),

(62) $\alpha = (1-n)\beta/n\sigma.$

Since $n < 1$, it follows from (62) and the assumption (52) that (63)

(63) $\alpha > 1.$

Equation (33) then becomes the one by which W is determined. The left-hand side will be denoted as $F(W)$.

$$F'(W) = \frac{1 - W^{\alpha-1}}{1-\alpha}.$$

From (63), $F'(W) < 0$ for $0 \geqq W < 1$, the relevant range since the investment will never be profitable if $W > 1$. To demonstrate that $W < \gamma$, it suffices to show that $F(W) > F(\gamma)$ for that value of W which satisfies (33), i.e., to show that

(64) $F(\gamma) < \theta/a.$

Finally, to demonstrate (64), note that $\gamma < 1$ and $\alpha > 1$, which imply that $\gamma^{\alpha} < \gamma$, and therefore

$$(1-\alpha) - \gamma^{\alpha} + \alpha\gamma > (1-\alpha)(1-\gamma).$$

Since $\alpha > 1$, $\alpha(1-\alpha) < 0$. Dividing both sides by this magnitude yields

$$\frac{1-\gamma^{\alpha}}{\alpha} + \frac{\gamma-\gamma^{\alpha}}{1-\alpha} < \frac{1-\gamma}{\alpha} = \frac{\theta}{a}$$

where use is made of (57), (28), and (61); but from (33), the left-hand side is precisely $F(\gamma)$, so that (64) is demonstrated.

The case $a \leqq \beta$, excluded by (56), can be handled similarly; in that case the optimum v is 0. The subsequent reasoning follows in the same way so that the corresponding competitive path would have $W < 0$, which is, however, impossible.

6. Some Comments on the Model

(1) Many writers, such as Theodore Schultz, have stressed the improvement in the quality of the labor force over time as a source of increased productivity. This interpretation can be incorporated in the present model by assuming that σ, the rate of growth of the labor force, incorporates qualitative as well as quantitative increase.

(2) In this model, there is only one efficient capital-labor ratio for new investment at any moment of time. Most other models, on the contrary, have assumed that alternative capital-labor ratios are possible both before the capital good is built and after. A still more plausible model is that of Johansen [7], according to which alternative capital-labor ratios are open to the entrepreneur's choice at the time of investment but are fixed once the investment is congealed into a capital good.

(3) In this model, as in those of Solow [12] and Johansen [7], the learning takes place in effect only in the capital goods industry; no learning takes place in the use of a capital good once built. Lundberg's Horndal effect suggests that this is not realistic. The model should be extended to include this possibility.

(4) It has been assumed here that learning takes place only as a by-product of ordinary production. In fact, society has created institutions, education and research, whose purpose it is to enable learning to take place more rapidly. A fuller model would take account of these as additional variables.

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